Division of space by Voronoi graphs, percolation within percolation and application to the models of porous membranes

A thesis submitted to the

University of Manchester Institute of Science and Technology

for the degree of Doctor of Philosophy

Kittisak Nui Tiyapan

BEng, BSc, BEng, MSc

1st September, 2004

No portion of the work referred to in this thesis has been submitted in support of an application for another degree or qualification of this or any other university, or other institute of learning.

K N Tiyapan

To my father and my mother Niwat and Somjit Tiyapan (*Nivat, Somcitṫa Tịyȧbanḋha)*,
my chemistry and sword teacher Siddhiponr Songsataya (*Sidhibaur Drongsatẏa*), and
my dad and mum Allister and Patricia Johnstone.
To Emeritus Professor David John Bell and
Emeritus Professor Graham Arthur Davies,
both of whom had started this project,
one of whom had powered it.
To these people, then, is my dedication of this thesis,
but the thesis may dedicate itself to whomever it wishes.

Abstract

Statistics for Voronoi Tessellations are obtained from numerical simulation. These are, for example, the number of edges, faces, and vertices per cell, and the cell's surface area and volume. Statistics are obtained for the 2-d section of 3-d Voronoi networks and for compressed VT. For networks of from two to ten dimensions are given the number of vertices per cell and the ratio between the volume and the hypercubic volume. The similarity between a percolation curve and an s-curve is discussed. Percolation is applied to the study of traffic congestion. In filtration problems sometimes tiny particles come together to form clusters which block the pores. This is due to the Van der Waals force which acts to pull them together and bind them. It is proposed here that such phenomena may be represented as a percolation within percolation, or more precisely a *continuum* percolation within a *network* percolation. Here channelling occurred in filters causes the concentration of the suspension to fluctuate. Pores whose internal concentration exceeds the continuum percolation threshold become blocked. This blockage may not be wholly random but depend on the quantum-mechanical wavefunction of the particles when their wave-particle duality are taken into account, as described by the de Broglie formula $\lambda = h/p$. These blocked pores then act to percolate the filter as a network percolation problem. Algorithms and programs for doing the simulation and compiling the thesis are given, and so are translation of three seminal papers by Voronoi (1908 and 1909) and another one by Dirichlet (1850).

Table of contents

a. List of figures vii
b. List of tables ix
c. List of theorems x
d. List of algorithms x
e. List of definitions xi
f. List of assumptions xi
g. Acknowledgements xii
h. Notation xiv
1. Preface and introductions 1
  1.1. Preface 1
  1.2. A general introduction 1
  1.3. Mathematics 5
  1.4. Geometry 6
  1.5. Physics 16
  1.6. Cosmological structure 19
  1.7. Filtration 20
  1.8. Statistics 23
  1.9. Poisson process 24
  1.10. Phase transition 24
  1.11. Random processes 25
  1.12. Structures in nature 30
  1.13. Computational geometry 32
  1.14. Geometric algorithms 38
  B.5. TEXnicalities 321
2. Division of space 41
  2.1. Stereographic projection 42
  2.2. Covering lattices 45
  2.3. Viscous fingers 47
  2.4. Crystals, quasicrystals, and polycrystals 48
  2.5. Random lines and percolation 51
  2.6. Convex hull 52
  2.7. From convex hull to the Voronoi tessellation 53
  6.10. The first part, suspended particles 190
3. Voronoi tessellation 65
  3.1. Quadratic equations 68
  3.2. Quadratic forms 68
  3.3. Voronoi algorithms 69
  3.6. Voronoi section 74
  3.5. Voronoi statistics 71
  3.6. Voronoi section 74
  3.7. Number of vertices and edges 75
  3.8. Faces in different dimensions 87
  3.9. Beam intersection study 93
  3.10. Voronoi of a Voronoi 97
  3.11. Transformations of a Voronoi 98
  3.12. Compressed Voronoi 99
  3.13. Voronoi tessellation in higher dimensions 109
4. Percolation 113
  4.1. S-curves and the percolative phenomena 119
  4.2. Voronoi percolation in two dimensions 119
  4.3. Voronoi percolation in three dimensions 121
  4.4. Percolation of 2-dimensional Voronoi sections 123
  4.5. Network percolation 126
  4.6. Percolation statistics in literature 126

4.7. Percolation of n-gons in continuum 127
4.8. Polygon percolation threshold 132
4.9. 2-homohedral tilings 136
4.10. Cosmology 151
4.11. CCTV, forest fire, the navy and porcupines 151
4.12. Fractals 154
5. Porous media 157
5.1. Zeolites 157
5.2. Crystalisation 158
5.3. Fluid flow within networks 159
5.4. Material science 159
5.5. Forces between particles 161
5.6. Arbitrarily shaped particles 162
5.7. Non Poisson number distributions of particles 164
6. Filtering membranes 167
6.1. Separation processes 176
6.2. Dead-end filtration 177
6.3. The centre of gravity 177
6.4. Molecular dynamics 181
6.5. Problem definition and algorithms 183
6.6. Simplified algorithm for filtration 184
6.7. Filtering problem when physical blockage is prominent 185
6.8. Percolative filtering with very small particles 185
6.9. Percolation within percolation 189
6.10. The first part, suspended particles 190
6.11. The second part, flow through the cells 195
7. Percolation in traffic modelling 201
7.1. Percolation of road networks 204
7.2. Graphs theory and its applications 209
8. Conclusion 211
8.1. Suggestions for future work 211
9. Bibliography 213
9.1. Publication and writings, Kittisak Nui Tiyapan 219
A. Programs 221
A.1. Object location 221
A.2. Network percolation, two dimensions 222
A.3. Network percolation, three dimensions 225
A.4. Network percolation, 2–d section 230
A.5. Continuum percolation of n-gons 233
A.6. Tilings 235
A.7. Covering lattices 246
A.8. Covering contour 247
A.9. Number of vertices 248
A.10. Vertices per cell and cell ratio 248
B.5. TeXnicalities 321
A.12. Language macros for TeX 256
A.13. Program for the dissertation 263
A.14. Program for the thesis 265
A.15. Degeneracy test and other programs 267
A.16. Face statistics in n dimensions 268
A.17. Beam intersection program 270
A.18. Number of vertices in high dimensions 271
A.19. Example batch program, simulation and data extraction 271
A.20. Voronoi operator of various order 271
A.21. Voronoi data structure for filtering membrane study 272
A.22. Centroid process on 2-d VT 274
A.23. Centroid process on 3-d VT 275
A.24. Defining irregular objects 279

A.25. Investigating the distribution observed from within the network 281
A.26. Miscellaneous functions 282
A.27. Percolated traffic networks 284
A.28. Volume, surface area, cell- and face perimeters 290
A.29. Volume in higher dimensions 296
A.30. Regular lattices in three dimensions 297
A.31. Effects of channelling 301
A.32. Stereographic projection 306
A.33. Percolation of regular polygons in two dimensions 308
B. Terminology and other resources 312
B.1. Abbreviation 316
B.2. Biographies 316
B.3. Computation and softwares 319
B.4. Internet resources 321
B.5. TEXnicalities 321
B.6. Voronoi statistics of earlier simulations 322
C. Publications and submissions of papers 327
C.1. Critical probability of 2-d tessellation 327
C.2. Abstracts of books I wrote 329
D. Translation 334
D.1. G. L. Dirichlet, 1848 334
D.2. G. F. Voronoi, 1908 (I) 341
D.3. G. F. Voronoi, 1908 (II) 381
D.4. G. F. Voronoi, 1909 421
E. Index 471

List of figures

1.1. Kissing number versus dimension 6
1.2. Icosidodecahedron and small ditriagonal icosidodecahedron 11
1.3. Great ditrigonal icosidodecahedron and great dodecahedron 11
1.4. Truncated tetrahedron and octahemioctahedron 12
1.5. Tetrahemihexahedron and truncated octahedron 12
1.6. Truncated cube and rhombicuboctahedron 12
1.7. Pentagrammic polyhedra 13
1.8. Temperature of the universe 18
1.9. Voronoi graph with bridge-shaped beta distribution 25
1.10. Voronoi graph with J-shaped beta distribution 25
1.11. Voronoi graph with U-shaped Beta distribution 26
1.12. Voronoi graphs of various random p.d.f.'s 26
1.13. Point process $r(\theta)$, where both r and θ are independent random generators 28
1.14. Point process x, y in a circular domain, where x and y are independent random generators 29
1.15. Point process x, y in a square domain, where x and y are independent random generators . 29
1.16. Additional generators for the incremental method 38
2.1. Stereographic projection of a cube 42
2.2. Rotated cube 43
2.3. Developmental stages of programmes for the stereographic projection 44
2.4. Covering lattice, stone pavement, or galaxies in the forming 46
2.5. The area of multiply covered cells 47
2.6. Face number of non isometric crystal forms 49
2.7. The 71 Voronoi cells in an aggregation 53
2.8. Four groups of tiles 60
2.9. Percolation of networks related to the square lattice 61
2.10. Percolation of networks related to the honeycomb lattice 62
2.11. Percolation of networks related to the basket pattern, kagome lattice 62
3.1. Rooms with doors puzzle 65
3.2. Departure and arrival rooms 66
3.3. Voronoi from degenerative data 66
3.4. Shifted hexagonal lattices 66
3.5. Hexagonal, Kagome, and the covering of the latter 67

3.6. The next covering lattice 67
3.7. Voronoi graphs of covering lattices 67
3.8. The minimum of the average normalised edge length 72
3.9. The maximum of the average normalised edge length 72
3.10. The mean of the average normalised edge length 72
3.11. Mean, maximum, and mean of the cell-average normalised edge length 73
3.12. The variance of the expected values of the normalised length of edges of a cell 73
3.13. Number of vertices per cell 74
3.14. Ratio of cells in the original domain 74
3.15. Sectioning by a plane 74
3.16. Abundance of elements 76
3.17. The electrical resistivity and conductivity of elements 76
3.18. Space covered by Voronoi 77
3.19. Number of vertices versus number of cells 77
3.20. The CPU time in seconds 78
3.21. Characteristic length versus the number of cells 78
3.22. Characteristic area versus the number of cells 78
3.23. Distribution of the number of edges per face 79
3.24. Statistics of a 231 cells Voronoi structure 82
3.25. The distributions of the number of faces and edges per cell 82
3.26. v_n/c_n in Voronoi networks of various dimensions 83
3.27. CPU time in creating the Voronoi networks 83
3.28. Distribution of vertices in high dimensions 84
3.29. Distribution of vertices in a 6-d Voronoi structure 85
3.30. Number of cells having a particular number of vertices 86
3.31. Counting the valences 87
3.32. Distribution of v_c in two dimensions 88
3.33. Distribution of vertices in 3-d Voronoi 89
3.34. Distribution of vertices in 4-d Voronoi, 100 nuclei 90
3.35. Distribution of vertices in 4-d Voronoi, 300 nuclei points 92
3.36. Distribution of vertices in 5-d Voronoi, 200 nuclei points 92
3.37. Intersection by a line 93
3.38. Line intersection in a Voronoi with Rayleigh distribution 96
3.39. Two Voronoi operators applied in succession 97
3.40. The Voronoi operator applied six times 97
3.41. The number of cells and vertices when recursively applying the Voronoi operator 98
3.42. The effects of axis compression 99
3.43. Mean and standard deviation of surface area of cells under compression 99
3.44. Some hidden characteristics revealed 100
3.45. Volume of compressed Voronoi cells 100
3.46. Cell perimeter 101
3.47. Face perimeter 102
3.48. The cell perimeter and the face perimeter 103
3.49. Area, volume and perimeter; past results 106
3.50. Statistics of face areas under compression 107
3.51. Number of generators required for ten inner cells 110
3.52. Mean and standard deviation of the ratio Voronoi per defining cube 110
3.53. The ratio V_p/V_h, dimension from 1 to 6 111
4.1. predicted value of the heat capacity of monoatomic solids according to Debye 119
4.2. Section by the plane (0.01, 0.5, 0.5, -0.5) 124
4.3. Section by the plane (0.5, −0.5, 0.5, 0.01) 124
4.4. Section by the plane (−0.7, −0.3, 1, 0.01) 125
4.5. Regular polygon 128
4.6. Star 128
4.7. Tilting 128
4.8. When two n-gons practise the Ayudhya sword together 130
4.9. n-gons 131
4.10. Finding the threshold area ratio of pentagons 132

4.11. Percolation of n-gons ... 134
4.12. Thirty-nine 2-homohedral tilings ... 137
4.13. Sample trend of the biggest, the second biggest, and the smallest cluster ... 142
4.14. Basic units of the thirty-nine 2-homohedral tilings ... 142
4.15. Range of observatory towers ... 152
4.16. Tortuous boundaries ... 153
4.17. Porcupine diagram superimposed on a map ... 153
5.1. Schematic diagram of a separator ... 160
5.2. Coulomb force *vs* distance in log-log scale ... 162
5.3. *Kit* the existential object ... 163
5.4. Distributions by weight and by number ... 164
5.5. Spherical object on a sieve ... 166
6.1. Centroid of the Voronoi tessellation ... 178
6.2. Evolution of nuclei towards centroids ... 178
6.3. Evolution of nuclei towards centroids in three dimensions ... 180
6.4. Change in size distribution as a result of a nuclei adjustment process ... 180
6.5. Lennard-Jones potential ... 182
6.6. Force corresponding to the Lennard-Jones potential ... 182
6.7. Force balance with one particle fixed ... 187
6.8. The packing density calculation of the closest-packed densities ... 188
6.9. Quantum-mechanical effect in filtration ... 189
6.10. Percolation within percolation ... 190
6.11. Percolation of the cubic close-packed lattice ... 192
6.12. The eight areas defined by the eight types of unit blocks ... 195
6.13. Intersection of convex hull faces and the horizontal plane ... 197
6.14. Correcting the effect of rounding or discretisation ... 197
6.15. After increasing the precision ... 197
6.16. `Spying` the neighbours ... 198
6.17. Sections of bonds and cells ... 199
7.1. Traffic status, $p_c > 0.5$ ... 201
7.2. Traffic status, $p_c < 0.5$ ... 202
7.4. Proposed traffic control in real time. ... 202
7.4. Proposed traffic control in real time. ... 202
7.5. Manchester road networks ... 204
7.6. Plots of the largest clusters, Manchester ... 205
7.7. Road networks of Amsterdam ... 205
7.8. Plots of the largest clusters, Amsterdam ... 206
7.9. Road networks of Brussels ... 207
7.10. Plots of the largest clusters, Brussels ... 207
7.11. Road networks of Freiburg ... 208
7.12. Plots of the largest clusters, Freiburg ... 208

List of tables

1.1. Three types of function ... 5
1.2. Perimeter per unit area of n-gons ... 8
1.3. Some important properties of regular solids ... 9
1.4. Surface area per volume of regular solids ... 9
1.5. The number of vertices, edges and faces of some polyhedra ... 10
1.6. Archimedean solids ... 15
1.7. Origin of membrane processes ... 18
1.8. Methods of preparation of membranes ... 19
1.9. Probability density functions ... 26
2.1. Symmetries of crystals ... 48
3.1. Moments of 3-d Poisson-Voronoi tessellation, *cf* van de Weygaert (1994) ... 69
3.2. Moments of 2-d section of 3-d Voronoi, *cf* van de Weygaert (1994) ... 70
3.3. Statistics of 3-d VT, van de Weygaert (1994) ... 70
3.4. Statistics of the planar section of the 3-d Voronoi tessellation, van de Weygaert (1994) ... 71
3.5. Statistics for the line section of the 3-d Voronoi tessellation, van de Weygaert (1994) ... 71

3.6. Statistics of a 3-d Voronoi network 74
3.7. Neighbour statistics 76
3.8. Three dimensions, statistics 78
3.9. Results from 527 Voronoi cells 82
3.10. Statistics of the number of vertices in high dimensions 83
3.11. Two-dimensional Voronoi, 100 nuclei 87
3.12. Three-dimensional Voronoi, 50 nuclei 88
3.13. Four-dimensional Voronoi, 100 nuclei 90
3.14. Four-dimensional Voronoi, 300 nuclei 92
3.15. Five-dimensional Voronoi, 200 nuclei 92
3.16. Intersection by a line, statistics 94
3.17. Rayleigh distribution, line intersection statistics 96
3.18. Mean of the cell surface area of compressed Voronoi 99
3.19. Standard deviation of the cell surface area of compressed Voronoi 99
3.20. Numerical values of the mean of the compressed cell volume 101
3.21. Numerical values of the standard deviation of compressed cell volume 101
3.22. Mean cell perimeter, numerical values 102
3.23. Mean face perimeter, numerical values 103
3.24. Standard deviation of the face perimeter, numerical values 103
3.25. Cell and face perimeters, numerical values 104
3.26. Numerical statistics of the area and perimeter of face 106
3.27. The numerical values of face area 108
3.28. Volume of d-hypercubes whose dimension is ℓ 109
3.29. Voronoi volume in higher dimensions 110
4.1. Percolation thresholds, Stauffer and Aharony (1998) 116
4.2. Percolation probabilities, *cf* Sahimi (1994) 117
4.3. Critical exponents and fractal dimensions, *cf* Sahimi (1994) 117
4.4. Critical probabilities from Frisch *et al* (1961) 126
4.5. Critical probabilities from Dean (1963) 126
4.6. Critical probabilities and connective constants, Shante and Kirkpatrick (1971) 127
4.7. Percolation statistics of the 2-homohedral tilings 145
5.1. Particle size definitions 159
6.1. Statistics of the modified Voronoi cells, Jafferali (1995) 173
6.2. Comparative Voronoi statistics, Jafferali (1995) 173
6.3. More comparative Voronoi statistics, Jafferali (1995) 174

List of theorems

1.1. Relationship between α and r 10
1.1[1]. Relationship between A, V and r_f 10
1.2. Average number of edge per polygon 31
3.1. Starting and ending nodes 65
3.1[1]. Path along two dimensional Voronoi network 66
3.1[2]. Path along three dimensional Voronoi network 66
3.1[3]. Path along a two dimensional face of a three dimensional Voronoi network 66
3.2. Vertices and edges, Euler 75
3.3. Vertices in three dimensions, Tiyapan 75
3.4. Edges in three dimensions, Tiyapan 75
4.1. Segments always intersect 128
4.2. Aggressive party 129
4.3. Finding overlap 130
6.1. The centre of gravity 179

List of algorithms

1.1. Perimeter per unit area of regular polygons 8
1.2. Area per volume of regular polyhedra 9
1.3. Thinning algorithm 28
1.4. Triangulation edge length distribution 30

1.5. Intersection between three planes 33
1.6. Point inside a polygon 33
1.7. Euclidean minimum spanning tree 34
1.8. Rotation in three dimensions 34
1.9. Ray and triangle intersection 35
1.10. Convex hull in three dimensions 36
1.11. Convex hull in two dimensions 36
1.12. Quick hull algorithm 37
1.13. Quaternary incremental method 37
2.1. Generalised Johnson-Mehl model, Miles (1972) 41
2.2. Drawing stereograms from crystal models 44
2.3. Graham's scan 52
2.4. Quickhull in two dimensions 52
2.5. Convex hull of convex hulls 52
2.6. Tilings in two dimensions 60
3.1. Voronoi 2-d algorithm, Green and Sibson (1978) 69
3.2. Valence relations 86
3.3. Volume, area and perimeter algorithms 105
4.1. Ant in a labyrinth 116
4.2. Network percolation in 2-d 119
4.3. Managing the Voronoi data in three dimensions 121
4.4. Voronoi percolation in three dimensions 122
4.5. Plane section of Voronoi in three dimensions 123
4.6. Continuum percolation of regular polygons 130
4.7. Threshold area ratio 133
5.1. Fracture by attrition, Lim *et al* (1999) 158
5.2. Arbitrarily shaped particles 163
6.1. Voronoi construction in three dimensions, *cf* Tanemura *et al* (1983) 167
6.2. Find the largest inscribed circle 168
6.3. Interaction between particles and surface pores 168
6.4. Interaction between each internal particle and the pores 169
6.5. Particle cake formation, *cf* Jackson (1994) 169
6.6. Flux decline across the membrane model 170
6.7. The random walk of particles during removal by backflushing 171
6.8. Non-woven fibre simulation, *cf* Wilkinson *et al*, 1986 171
6.9. Voronoi construction in 3-d 172
6.10. Find faces of a polyhedron 172
6.11. Nuclei points of the modified point process, *cf* Jafferali (1995) 173
6.12. Asymmetric Voronoi tessellation, *cf* Jafferali (1995) 173
6.13. Pore property evaluation, *cf* Jafferali (1995) 174
6.14. Inscribed circle of faces 175
6.15. Inscribed sphere of a polyhedron 175
6.16. Verlet algorithm 181
6.17. Filtration in Voronoi tessellation 183
6.18. Voronoi data structure for the study of membrane filters 183
6.19. Percolation by tiny particles due to attrition in membranes, preparing the structure 191
6.20. Percolation by tiny particles due to attrition in membranes, percolation simulation 191
9.1. Beneath-Beyond algorithm 319

List of definitions

3.1. Volume of a d-simplex 109
4.1. Cell's aggressiveness 129
4.2. Threshold area ratio 132

List of assumptions

3.1. Volume of a unit cube 109
3.2. Volume of d-simplices 109

Acknowledgements

There are no less than two Emeritus Professor's to whom this present work is indebted. They are David John Bell and Graham Arthur Davies, both of whom I thank for the supervision and also the latter for the maintenance fund throughout the course of the project. I thank Simon Hood and Zaiem Bakker for the help in computing matters, and Stuart Holmes *et al* for suggesting zeolites. I thank Walter Korchinsky for the lab demonstrating jobs, Ted Roberts for the extended computer access and for giving me a visitor's status for a few weeks in September 2003 to conclude my thesis for submission, and Ian Rosindale for extending the computer access for me. These people mentioned, unless stated otherwise, are all titled Dr. I thank the graduate school administrator Muriel Shingler, Rodney Holt at the helpdesk of UMIST's ISD and also Steven Coppersthwaite-Lewis who looks after the computers at the Chemical Engineering Department. I thank Peter Senior who is an excellent teacher, and Professor Ian Metcalfe, both for letting me help with the demonstrations.

For the help with my everyday's work I thank all the staffs at the John Rylands Library at Deansgate, the John Rylands University Library and the Joule Library, especially but not only those at the issuing desks whom I see regularly. For encouragement I thank Carlos Cervantes, Dirk Engelberg, Clara Entwistle and Yann Frouin. For inspiration I thank mathematicians like Descartes, Dirichlet, Euler, and Voronoi.

Many of the ideas and solutions described herein came to me while I was on an excursion away from Manchester. I thank those people who had taken me to those far corners, away from the madding crowd, places as unique and distant from one another as Penzance, Portree, Llandudno, Nagazaki and Nice. These people are in particular the drivers of National Express's coaches, train operators in Europe and Japan, pilots, and also Aoife O'Higgins who organised several trips for us.

I thank Sister Eileen and all the staffs at the St. Gabriel Hall of Residence who have made my stay there during the summer of 2002 a memorable experience. Special thank to Yann Frouin my flatmate at St. Gabriel for his encouraging words in times of turmoils, 'Believe in your ideas moreover they are beautifull'. [sic] I thank Carlos Manuel Cervantes Lara, Daniel Scamman, Ewan of the Spitalfields Market's poi-swinging and blaung-spinning workshops, Carlos Harrison for encouragement and accommodation.

I also would like to thank PCA 4819, one of the two policemen who helped me once in time of trouble. My interest is rather eclectic. But it may be true what Debussy says, perhaps *nothing that has happened is ever useless.* How could I possibly forget to thank our security staffs, in particular Richard and Graham. Because of them UMIST is made safe for all.

Above all I thank God for letting me be and for everything.

No writings can be too long
nor tunes futile in any song
where to someone thanks we say
and in our heart for them we pray.

Yet at times we forget
and amidst our busy tasks let
all the pains in and here reside.
We could have let God us abide.

O, gentle reader, never judge!
We last no longer while we grudge!
Search again, and you shall find so much
there is nothing meaningless as such.

Kit Tiyapan
Manchester, September 2003.

The above poem was written on 24^{th} September 2003 . By the end of the month I have already given a copy of it to many people, and many more in the form of emails to friends in the UK, the US, Japan and Thailand. This is how I come to realise that I have missed out quite a few of the names of those people who have helped. If I had to rid my writing of all the stuffs it is made of, I would never have even dreamt about ridding it of each and every one of all the thanks I have put therein. I thank Peter Fawcett for having had decided to supervise my projects on languages though in the end I had to do them all by myself and on my own. I thank Graham Davies again for the full two years of financial support, the inspiration, compassion and encouragement received. I thank Mona Baker for having given me an office and access to a unix machine at the translation group. Though I had to return them all to her for my works for Graham, it had been a wonderful experience for me to have done some work for her translational corpus.

I thank Louis Papworth, Lesley McDonagh, Pauline Howarth, Stewart Grace, Helen Cooke for getting office-procedural things done. For chemical engineering helpdesk I thank Jacqie Kitson, Simon Perry and Fran Stanley. I thank Aṃnạc Gịtȧbarrṇạ, Dr's Michael Anthony Allen, Stuart Holmes, Bronislav Librovich, Leo Lue, Gerard Markx, Andrew Nowakowski, Jinȧvudh and Kriangsakd́xi Śirisambanḋha for the encouragement. For the same I thank Zoë Speakman.

After an unofficial correction to this thesis in September 2003 I thank Ted Roberts again and also Ian Bradley, both for having read the previous versions of this thesis and given some comments all of which are valuable. It is remarkable not only that their names are of exactly the same length but also that both are very kind. As a result of this, the thesis owes much of its good points to them whereas all of its shortcomings are because of Me who is itself imperfect being merely a tiny subset of the Whole.

It has taken me ten years to do this PhD. I would have quitted before now had I not realised that to do a Doctor of Philosophy is to solve a percolation problem. This many number of years means that the list of people whom I need to thank has long time ago percolated. It also means that I can not thank everybody that I want to here. I can easily thank everyone for everything in five words, that is 'I thank God for this,' but I decided to make it more, though by doing more I have meant less. This I shall venture now to remedy. Going back in time deeply I thank Shakespeare for his *Sonnets*, which all writers should read and come to understand. Similarly I thank Rumi for his *Mathnawi*. I thank Muhammad the Prophet, Jesus the Messiah and Buddha the Enlightened. I thank God the Superset. He is to me not only the Creator but also the Supervisor. I may owe you and him and her, this and this and that, but we all owe God our being.

Kit Tyabandha
Woodthorpe, Manchester
31^{st} August 2004 .

Notation

$0(n)$. a zero vector of n dimensions.

$0(n,n)$. an $n \times n$ zero-matrix.

$1(n)$. a one vector of n dimensions.

$1(n,n)$. an $n \times n$ one-matrix.

a. surface area of a cell.

a^{ij}. cofactor of a_{ij}, $a^{ij} = (-1)^{i+j} \det(a_i, b_i)$.

$\aleph$. neighbour.

α. surface area per unit volume.

β. area of face.

c. cell; concentration.

d. length.

D. diffusion coefficient.

δ. thicness of the boundary layer.

ϵ. porosity.

ϵ_0. permittivity of free space, $\epsilon_0 = 8.854 \times 10^{-12}$ Fm^{-1}.

η. viscosity of liquid.

e. edge; the charge of an electron, $e = -q$. See q.

$E(x)$. expected number of x

f. form factor, f_c, of a cell, f_f, of a face; face, nf, n-face.

h. Planck's constant, 6.626×10^{-34} J $\cdot$ s.

he. he; he or she.

$\mathrm{I}(n)$. an $n \times n$ identity matrix.

I_r. integers in the interval r *eg*, $I_{[0,\infty)}$.

J. flux.

κ. Kurtosis.

$\mathcal{K}$. Kurtosis.

λ. wavelength. m.

μ. weight of particles in the cake per unit of filter surface; fluid viscosity; magnetic moment.

$\mu(\cdot)$. mean; also μ_g the geometrical mean, μ_h the harmonic mean.

$m^n(\cdot)$. the n^{th}-moment of.

$\mathrm{mad}(\cdot)$. mean absolute deviation.

$\mathrm{med}(\cdot)$. median.

$\mathcal{M}^n(\cdot)$. the n^{th}-moment of.

N. original total number, for example N_c is the total number of cells, and N_v of vertices, created which may include those that are out of bound.

n_x. number of x. n_c, number of cells. n_e, number of edges. n_v, number of vertices.

n^x_y. number of x of a y. n^e_c, number of edges of a cell. n^f_c, number of faces of a cell. n^v_c, number of vertices of a cell.

$n^x_{y,z}$. number of x per y of z. $n^e_{f,c}$, number of edges per face of a cell.

ω. angular velocity.

p. probability; (with subscript) percolation probability. p_c, p_b, p_v, p_e, percolation probability of cells, bonds, vertices, and edges respectively. alternatively, p^c_c, p^b_c, p^v_c, p^e_c,.

$p(\cdot)$. probability density function, probability distribution.

P. pressure.

q. the charge of a proton, 1.6×10^{-19} C; feed rate.

Q. flow rate, $Q = \mathrm{d}V/\mathrm{d}t$.

r. radius. r_v, r_e, r_f, radius to a vertex, edge, or face, respectively.

ρ. density; number density.

r. specific resistance.

§. (s) section or chapter.

s. perimeter, for example s_c the perimeter of cell; cell perimeter.

σ. spin variable.

θ. time.

v. velocity.

vertice. any vertex.

V. volume.

W. fluid velocity.

x. coordinates; coordination number.

$\bar{x}$. mean value of x, *cf* $E(x)$.

z. face perimeter.

§ 1. Preface and introductions

§ 1.1 Preface

The present thesis contains two parts, that is the body and the appendix. In the former are the works written in Mathematics and English while in the latter are programmes written in a variety of languages, for example Matlab, TeX, and so on. Also in the appendix are put those works which have been published since they may not be in the body. The body of the thesis is divided into chapters. Similar to background processes which trigger a percolation, Chapters 1 to 3 contain those things that ultimately lead to original results which start from Chapter 4 onwards.

The following are some of the contributions of this thesis. Chapter 3 (Voronoi tessellation) which begins on page 65 explains why for a Voronoi cell in two dimensions the number of vertices is even (page 75) and the number of edges is divisible by three (page 75.) A new operator $\mathcal{V}(\cdot)$ is introduced (page 97) which finds a Voronoi tessellation of a Voronoi tesselation. Another operator in page 67 find an n^{th}-order covering lattice of a Voronoi tessellation. There are also in this chapter original analysis and statistics of Voronoi tessellations. Chapter 4 (Percolation) finds Voronoi percolation for two dimensions (§ 4.2, page 119), three dimensions (§ 4.3, page 121) and 2-d section of 3-d Voronoi tessellation (§ 4.4, page 123.) The problem of percolation of n-gons in continuum is solved (Theomem 4.3, page 130) in § 4.7 which starts from page 127. Percolation of 2-homohedral tilings begins on page 136. Chapter 5 (page 157) looks at the possibility of introducing arbitrarily shaped particles in filtration simulation (page 163.) Beginning on page 167 Chapter 6 studies continuum percolation within network percolation (§ 6.5, page 183 to § 6.11, page 195.) While the covering operator $C_n(\mathcal{V})$ (page 67) changes the surface of $\mathcal{V}$ according to its surrounding, the centre of gravity operator $\mathcal{G}(\mathcal{V})$ introduced in § 6.3, page 177, adjusts the position of its nucleus in relation to the same, $\mathcal{V}$ being a Voronoi cell. Application of percolation to the study of road traffic can be found from page 201, that is Chapter 7. Appendix A.2, A.3 and A.4, respectively in page 222, 225 and 230, give network percolation programmes based on a new algorithm. The programme in Appendix A.5 (page 233) finds continuum percolation of n-gons. The programme in Appendix A.6 (page 235) can create any tiling in two dimensions which repeats itself, and also finds the corresponding percolation threshold. TeX macros in Appendix B.5 (page 321) is used for writing a PhD thesis. Nothing in the references mentions how to do this. What you are reading now is the example produced by TeX using these macros. It does cross-reference, citation, and index. To make the reading easier less complicated macros are put in Appendix A.12 (page 256.) All programmes in Appendix A, that is from Appendix A.1 in page 221 to Appendix A.33 in page 308, are original. The translations of Dirichlet's and Voronoi's works (namely § D.1, G. L. Dirichlet, 1848, p. 334; § D.2, G. F. Voronoi, 1908 (I), p. 341; § D.3, G. F. Voronoi, 1908 (II), p. 381; and § D.4, G. F. Voronoi, 1909, p. 421) are original.

§ 1.2 A general introduction

This work began in 1995 under the co-supervision of now Emiritus Professor David John Bell and Emiritus Professor Graham Arthur Davies. The main theme of it is the percolation on porous media and Voronoi tessellation. The objectives are to study the characteristics and nature of Voronoi tessellation and percolation, and then look at their application on porous media, which includes such things as membranes.

As a brief background to this project, the dissertation for my master degree in Control and Information Technology, dated September 1995, was on the percolation on a two-dimensional Voronoi tessellation. From 1996 to 1998 I had been doing a PhD in Japan on topics related to Control System. Parts of my work there are recorded in the few *technical reports* that I wrote during that time and presented to my supervisor, now Emeritus Professor Katsuhisa Furuta. In September 2000 I came back to UMIST to do a PhD in Translation but a few months later changed my mind to resume this research on Tessellation with Graham Davies.

Throughout my research I kept hand written notes which I called *work books.* All results exist in two places, either in the thesis or in the form of the TeX documents which I call *work notes.* Materials sometimes move from the work notes to the thesis. Apart from simulation results, these work notes also contains ideas for future works. All these worknotes were written during 2001. Their importance is that they represent the essential momentum or the underlying potential at work behind every percolative process. But most of the breakthroughs in this project came during

2002. I solved numerically a problem on the two-dimensional continuum percolation of squares on the plane. My algorithm can also give solutions for the continuum percolations of n-polygons, for $n \geq 3$ and $n \in I^+$.

The idea of writing these worknotes came from a book I had read about the work notes written by Michael Ventris while he was deciphering the Mycenean Linear B script. I wrote no worknotes during 2002. During the first half of 2003, however, I resumed my writing again and this time produced not worknotes but books which I published under the trademark name of ⋟Kittix.

I started to use Latex in 1994 to write a design exercise report for Dr. Martin Zarrop of the Control System Centre at UMIST. Back then, most people I know liked the programme better than its progenitor TEX, which most thought was difficult. I used Latex for writing my master degree dissertation in 1995, and continued using while in Japan and until early 2001 when I first turned to TEX then held fast to it. In the past when I still used LaTEX I could hardly adjust, but has to accept the existing formats. One awkward example is seen in my technical report number 6 where one reference (Fitzgerald *et al*, 1971) appears everywhere as "[FCKK71]", which looks very awkward.

Topics are divided into Voronoi structure and geometry, percolation, and membrane science. Voronoi tessellation has been used to model the array of the somata and cone cells in retina of mammals. (Ammermuller et al 1993; Ahnelt et al, 2000; Zhan et al, 2000) Curcio et al (1992) found anisotropy in cone spacing. For readers who are interested in the theoretical works on stochastic geometry may look at Møller (1989).

None of the papers and articles in § C has appeared in print elsewhere. It may be true that nothing worthwhile is ever without trouble. This project has had its share of problems. But 'What happened is history!', as Graham once said to me, or 'It is over!' as I to students in a lab I demonstrated with whom at times I had sympathised regarding problems beyond their power which affected them, and to whom I had passed down all the essentials of a programmer, there is no point keep on complaining about the past. Being a subset of the Superset I shall not judge. Being a me then the Me in Christ in God, how could I judge other subsets myself being merely another subset. It is not for a part to judge a fellow part, but let the Whole be the only One who judges, and let that alone suffice! Having done my tasks I am now only a writer who believes in the one Creator from whom all things come and to whom the same return. As Percolation may compass percolations my works is in Science in Him.

I completed a two-dimensional percolation programme and used it to compute the critical probability of a Voronoi graph in time for my M.Sc. dissertation. I used the C programme developed by Nicholas Jackson and maintained by Riaz Jafferali to generate the Voronoi network, and wrote my own programme on Matlab to find the percolation probability. Before I went to Tokyo in September of the same year, we agreed that I would carry on doing some more work along the same line while I was in Japan. Graham suggested that I looked at the viscous fingering, a phenomenon which occurs when a low viscosity fluid penetrates a high viscosity one in the form of thin branches, which is of great importance in the replacement of oil in a reservoir by water.

I telneted across half the globe from Tokyo to log on to the workstations at the Chemical Engineering Department in Manchester, but instead of writing another programme to do viscous fingering I only managed to find the percolation probability of the road network in Bangkok using the programme I from my M.Sc. I also carried further the work I had previously done with Martin Zarrop on object location using self-tuning control; and had an idea that if we could construct a network of people with an internal flow of money, we would be able to explain economic crises such as hyperinflation by using the percolation theory. On the topic of percolation and Voronoi tessellation I presented one paper at a conference in St. Louis (1996); on object location at two conferences, one in Atagawa (1996) and another one in Singapore (1997); and on economic modelling one at a conference in Tokyo (1997).

The original objectives of the present project are the following. Firstly, to study the statistics of the three-dimensional Voronoi tessellation, namely the number of vertices, coordination probabilities of nodes, number of faces, volumes, area of each face, number of sides and edges of faces, and perimeter. Next, to 'formulate a simulation of Voronoi tessellation,' that is to say, polygons in two dimensions and polyhedra in three dimensions; then analyse the lattices for the number of cells, sides, and edges. In two dimensions find the fractional expected area per cell, and expected number of vertices and edges. In three dimensions find the expected number of vertices, edges, faces, the fractional expected volume per cell, area of faces and fractional area per cell. Then section the Voronoi tessellation in two dimensions using a straight line and repeat the same thing in three dimensions using a random plane. In two-dimensional section of three-dimensional lattices using orthogonal planes, find the number of cells, edges, and the area. And then compress the lattice in one plane, with compression ratios $0 \leq c \leq 0.8$, and find the volume and surface area of cells for $c = 0$, 0.4, and 0.6; plot a section structure for $(\text{x–y})_{c_z=0}$, $0 < z < 1.0$, $c_y = 0$, 0.4, 0.6; $(\text{x–z})_{c_y=0}$, $0 < y < 1.0$, $c_z = 0$, 0.4, 0.6; compare the number of vertices, edges, and the area with those obtained from the case when $c_z = 0$, $c_y = 0$, and $z = 1$. Study transformations of Voronoi structures, for example the effect on the statistics when Voronoi graphs have a thickness. For each of the expected values mentioned above, find the first, second, and third moments. And finally, from the observation I have made that a three-dimensional tessellation have a bias towards an even number of vertices, find an explanation. To conclude the study of Voronoi tessellation, find affine transformations on two- and three-dimensional lattices and use the AVS programme to print out sections of 3-d lattices †. Then for affine transformations, for example stretching, write algorithms to reposition the structure. For 2-d lattices, find the critical percolation probability (CPP), compare these with previous data and carry on to find the CPP on higher dimension lattices. To look at the application to the technical problems of membranes, foams, plant cells, *etc*, and if possible to compare my results with models to describe other physical and natural phenomena. Relevant to the application in foams is perhaps the population balance equations. Care should be taken to ensure that any application mentioned is described in details how it can fit in with real problems. Of no less importance is to study methods of adjusting structures to fit the applications. These methods include modifications to Poisson point processes and, again, affine transformations of the cellular structure.

Filters used in aluminium smeltering must withstand a temperature higher than 700°Csince aluminium alloy melts at 650–680°C. These filters are used in order to separate the oxide froth which otherwise would form pits when the aluminium is cast. The process of manufacturing filters often introduces asymmetry elements to their structure, for instance there may be elongation in one dimension because of gravity, or in cases where material is drawn out to make the filter the structure may become distorted from the shear force of drawing. The texture of bread is anisotropic because the internal pressure from gas produced by the yeast press es the dough against its own surface tension. The elongated cells inside bamboos appear in various sizes, which explain why bamboos are at the same time strong and flexible. Certainly one sees the Voronoi tessellation wherever one looks. I feel that I have learnt so much from Graham.

The following summarises our original outline of the project. The objective is to study a *foam like* porous medium. Describe the statistics of structures in three-dimensional space, state the reason why we need the statistics and the roles they play in application. For the Voronoi tessellation, created from the Delaunay triangulation of Poisson points, give algorithms which generate the structure from one- up to n-space ‡, and give examples of these [structures found in nature]. Give the analyses of the static geometrical data, namely the length scale $l < L$ in one dimension; the area $a < A$, n_1, n_2, L in two dimensions; and the volume $v < V$, n_1, n_2, n_3 in three dimensions, where n_{d+1} is the number of the d-dimensional entities, $L = V^{\frac{1}{3}}$, $A = V^{\frac{2}{3}}$, and $V = \sum_i v_i$. Include the translation that I have made of the seminal papers by Voronoi and Dirichlet, a section on the transformations made on the structures, possible future developments of the project, and the applications.

Over the course of doing the project its title has changed from the original *Voronoi tessellation and porous media*, to *Division of space by Voronoi graphs, application to the models of porous membranes*, then to *A twentieth-century definition of the ancient theme, Voronoi network and percolation in porous media*, and to *On suspension blockages of filtering membranes as continuum percolation under van der Waals influences in centroidal Voronoi networks*, and finally into the present name.

I have observed that a Voronoi cell in three dimensions always have vertices and edges in an integral multiple of two and three respectively. I found the reason for this and have given one proof for the first observation, and two for the second one (§ 3.7).

Related to the present study is the stress-strain analysis in the bonds of a Voronoi structure in three dimensions as well as the study of randomly placed rods. Also interesting is the problem of percolation of randomly placed squares, which he thought would be useful in the growth process of synthetic zeolites. The zeolite problem mentioned above I have found to be identical to the so-called *continuum percolation* of squares. I have developed an algorithm to solve this not only for squares but also for the general case of n-gons. The programme uses some of the ideas behind the basic movements in the Ayudhȧyạ sword fighting. The algorithm has been intuitively arrived at from my own experience of several years holding a square, hind shield in one hand while in another brandishing a Thai sword for a troupe when our sword school demonstrated for that matter. It was only afterwards that I found the rigorous proofs for all its parts. However, I have not carried the work further to three dimensions because, for one thing, I personally do not believe the crystal growing to be random. Crystals grown within the same solution should have some kind of quantum coordination which allows them to align themselves with one another when they meet. The random appearance seems to be only their various habits.

---

† This has been done using Matlab instead.

‡ In one variation, $n \leq 5$.

The nature of discoveries and progresses in science is according to Bacon (Francis Bacon, 1620) *a birth of Time rather than a birth of Wit.* This is the same idea of percolation and the description he gave is the very picture of the theory. According to him major scientific progresses come in revolutions which are sparsely distributed in both time and regions. There have only been three periods of major progress out of the *five and twenty centuries over which the memory and learning of men extends*, namely the Greeks, the Romans and the nations of Western Europe. These are narrow limits of time, the periods in between of which are unfavourable to development. A discovery or an invention, then, comes as a chance accident in the scale of an individual, and as a certainty when looking from a distance.

When the time is right and all the hidden momentum built up, theories will come on by itself as a rule. This does not negate the excellence of an individual, but in a society where there are enough multitude of individuals the show will always go on, with or without a particular genius. This idea can very well explain cases of multiple discoveries. According to Kekulé in his *Benzolfest* speech in 1890, when he ascribed his conception of the cyclic nature of Benzene in dreams, *certain ideas at certain times are in the air* and *if one man does not enunciate them, other will do so soon afterwards.*

To see the relationship of this with percolation it is possible to look at two different things in turn, first at the discoveries and then at the discoverers. With a unit being that of a *discovery* the connection to percolation is that big discoveries come as connections of other smaller and less obvious ones. A theory often has more than one perspective, and which one of them comes to the fore first depends much on which combination happens to percolate through first. The discoveries of Schrödinger and Heisenberg in Quantum Physics can bear witness to this both in the *combination* and the *multiple discoveries* parts of this argument.

Let us turn our attention now to the scientist and look at the one who does the discovering instead. The theory of percolation tells us that at the point of discovery he is by no mean the sole integral ingredient. If he does not do it, then someone else will certainly do. In order to see this, I did four simulations for the cell, bond, vertice, and edge percolations on a two-dimensional Voronoi network and then another four with the same respective blockage of each case but considering the inverse phase instead. The number of units considered for the four cases are $n_c = 200$, $n_b = 416$, $n_v = 298$, and $n_e = 426$. With the order of simulations as described above, at just one step before percolation occurs there are respectively 10.6, 13.4, 11.5, 1.1, 11.24, 10.0, 19.4, and 7.9 percents among the remaining units which will readily trigger the onset of percolation. In other words, these are atoms which are able to link up existing clusters and form a percolating cluster.

The formation of mobs is an interesting phenomenon comparable with phase change in physics. What happens is that an agglomerate of individuals becomes one and a single creature, the underlying mechanism of which still baffles any effort towards understanding it. Likely enough it has got something to do with psychology and the mind. But to me at least, the phenomenon is percolative. Having gained some acceptance from my previous writings (Tiyapan, 1995–1996, KNT3(iv), KNT4(i) and [kbukt]; respectively §'s E.5, E.6 and E.7 of Tiyapan, 2003, KNT8(iii)) I gave my new work which briefly discusses the mechanism underlying the formation of mobs to an editor of the Sakkayaphab journal whom I know. At that time a political turmoil unequal in its degree and extent has been going on for five years. Whether by fate or by design there has been a successful but tragic use of mob in Bangkok. The word *mob* has joined the list of those synonym to *distrust*, namely *communist* or, in western community now for that matter, *Islam* and *evil.* Whether because of this or something else, the article (Tiyapan, 1996, KNT4(vii); or § E.15 of Tiyapan, KNT8(iii)) simply and mysteriously got lost; no one would admit having seen it, and the translation of another subsequent article of mine (1996, KNT4(viii); or § E.16 of Tiyapan, KNT8(iii)) has not been without a noticeable negligence. Thus to me distrust is also percolative. The list of things one finds over-distrusted without reasonable explanation goes on indefinitely, homosexuality, communism, *etc.* The same seems to be the case with bad habits. My father used to teach me using the following poem,

Bad habit gathers by unseen degree
like brook makes river, river runs to sea.

Looking back, it could have been the title of that article, *on pragmatists and idealists*, which has somehow convinced the editors into believing that it was political which to me is nonsensical. I only meant literary, even if at times philosophical. I include it here because it contains a curve showing a critical emotional transition.

The formation of the United States, the European Union, or the Commonwealth comes from the trust which acts to join countries together like glue boxes in TeX. Like all binding forces, trust is mutual and spreads in the same way as a growing cluster does. The cluster grows bigger as one or more members are added, and it becomes stronger as the level of the mutual trust increases. In a similar way, distrust is also mutual and also spreads . If I distrust you and you distrust me, I will make sure that I remain as far away from you as possible while you will certainly avoid me by all means in return.

Only these two are possible, so there are only two phases to consider, that of trust and distrust. The relationship where one trusts while the other distrusts would not be stable, since the former will soon learn to join the latter. Trust forms clusters of one phase, distrust another. The size of these clusters vary in a way similar to those in percolation of geometrical networks. The strength of the glue is analogous to the probability either of becoming or remaining a member of a cluster.

The rise of dictators, the proliferation of weapons of mass destruction, *etc*, these things I believe are the products of changes of something hidden within the underlying structure. Unless we find out what is happing in the background, these things will unavoidably occur. I believe that this unseen thing behind the scene is governed by some phenomena similar to that of percolation. I think that the key towards understanding many unexplainable phenomena is to investigate, in the light of the percolation theory, the working of agglomeration of countries or states like those of the United States and the European Union.

The belief that scientific discoveries are a birth of time, rather than of wit (*cf* Larsen, 1993), is the same as the idea of percolation. We know because we remember. And all the various discoveries of our time together with the knowledge we possess of the past bring us closer to another discovery. Scientific discoveries, then, is the collective product of humanity rather than property of a single person (*cf* Merton, 1965).

And because all species are also the product of percolation in time, our knowledge and consciousness, too, are the product of the universe. One may say that it is a personification when we refer to a collective noun, for instance a mob, as an individual. But the truth is that, under the percolation theory, it is in fact a separate individual without any need for the use of a simile. The renormalisation group theory tells us that there exists a structure in a bigger scale that behaves like the individual components that comprise it. It seems, therefore, that for humans these collective beings of ours are still primitive compared with each of us as an individual. This is the reason why, whenever we come together, we alway make wars. In our case, then, we seem to be conscientiously percolated only individually not collectively. In the case of bees, on the other hand, it is the other way round. This is why a colony of bees does things which make far better sense than a bee does. But one can not say that even a *colony* of bees has consciousness, because there seems to be no morals in what it does. I do not know whether there are other beings in the universe both the collective and individual beings of whom have percolated conscientiously. But I believe they exist, in which case they should be more intelligent than us, though this is by no means necessarily the case.

§ 1.3 Mathematics

A map, denoted by $f : \mathrm{X} \mapsto \mathrm{Y}$, is an assignment of an image y in the target Y to a preimage x in the source X, where f is called a function and $f(x) = y$. A composition of $f : \mathrm{X} \mapsto \mathrm{Y}$ and $g : \mathrm{Y} \mapsto \mathrm{Z}$ is the map $h : \mathrm{X} \mapsto \mathrm{Z}$ such that $h(x) = g(f(x))$ for every x in X. There are three types of function, each one having at least three names. Table 1.1 lists these three types together with the sources of their various names. The prefix *epi-* in Greek means *on* and *isos-* means *equal.*

Anglo-Saxon		*Greek*		*Latin*	
an to an	one to one	monomorphos	monomorphism	injectus	injective
on to	onto	epimorphos	epimorphism	superjectus	surjective
an to an and on to	1–1 and onto	isomorphos	isomorphism	bijectus	bijective

Table 1.1 *Three types of function and the source of their names.*

A concrete category is a set of function that is closed under composition and contains the identity map for every source and target. The maps in C are the morphisms of a category $\mathcal{C}$, and both its sources and targets are objects. The topological category $\mathcal{T}$ is a category of topological spaces and continuous maps.

Furthermore, a functor is a map of categories which preserves composition and takes identities to identities. A covariant functor has $F(f \circ g) = F(f) \circ F(g)$, whereas a contravariant functor has $F(f \circ g) = F(g) \circ F(f)$. Homology is a covariant functor from $\mathcal{T}$ to the category of abelian groups and homomorphisms, while cohomology is a contravariant functor from $\mathcal{T}$ to the category of rings and ring homomorphisms.

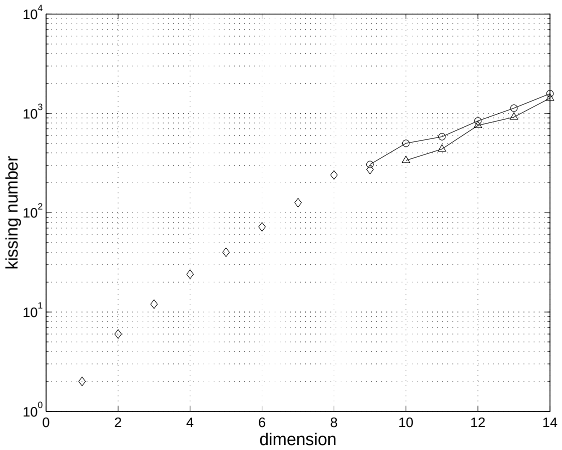
A relation is an equivalent relation, denoted by $\sim$, if and only if three axioms, namely that of reflexivity, symmetry and transitivity are satisfied. A set of all x in A such that $x \sim a \in A$ is the equivalence class of a. A partition of A is formed by the set of equivalence classes of equivalence relations, and is a family of disjoint subsets of A covering A. Then A is the union of the equivalence classes, and each equivalence class has an empty intersection with any other. That is, every a in A is also in an equivalence class $E(a)$, and $E(a) \cap E(b) \neq \emptyset$ implies that $E(a) = E(b)$. Conversely, every partition of A gives an equivalence relation on A, and a is equivalent to b if and only if a and b are in the same subset of the partition.

With the discovery of the equation $f_0 - f_1 + f_2$ by Euler in 1752 a new era of mathematic began, and that is the era of topology. The Euler's equation in its equivalent forms are written as $\sum_{i=0}^{d}(-1)^i f_i(P) = 1$ or $\sum_{i=-1}^{d}(-1)^i f_i(P) = 0$ (*cf* Grünbaum, 1967).

Partitions of numbers in the theory of numbers (*cf* Hardy and Wright, 1979) has a similar idea to what we shall do in § 6.11 for finding the Voronoi sections and their grids. Moreover, the random nature of these two problems, that is percolation and the number theory, as well as their omnipotence seem to point out that there could be some closer relationship between the two than mathematicians and physicists believe at present.

Also there is another similarity between these two fields, that is both are easy to state but very difficult to solve.

Sphere packing studies the density of packings of hard spheres. Here the kissing number problem studies the number of spheres that can be arranged such that all of them touch one central sphere of the same size. In other words it is the maximum number of neighbours that a sphere can have. It is also called the ligancy or Newton-, contact or coordination numbers.



For n dimensions the lower bound for the kissing number is $\eta = \zeta(n)/2^{n-1}$, where

$$\zeta(n) = \frac{1}{\Gamma(n)} \int_0^\infty \frac{x^{n-1}}{e^x - 1} \mathrm{d}x$$

is the Riemann zeta function. In Figure 1.1 the a diamond symbol represents an exact value for a lattice, a triangle the lower bound for a lattice, while a circle the lower bound for a nonlattice. The lines linking these symbols are for the ease of look only.

Figure 1.1 Kissing number versus dimension.

On the other hand, the covering problem is a problem that tries to find the least dense way to cover a space in n dimensions with overlapping spheres of the same size while the quantising problem studies how to place points in such a way that minimise the average second moment of their Voronoi cell, which finds application in the conversion from analogue to digital (Conway, 1988). In two dimensions the hexagonal lattice solves the packing, kissing, covering and quantising problems. But the packing problem of hard spheres is still unsolved. Everybody knows that the solution is approximately 0.7405 or exactly $\pi/3\sqrt{2}$ but no one has been able to prove it.

For any lattice the fundamental polytope, *i.e.* a polytope consisting of points $\sum_i^n \theta_i \nu_i$ where $0 \leq \theta_i < 1$, is its fundamental region. The proportion of space occupied by the spheres is the volume of one sphere divided by the volume of fundamental region, while the latter quantity is equal to $(\det \Lambda)^{1/2} = \det M$. Here $M = [\nu_{i,j}]$, $1 \leq \nu_{ij} \leq m$, is the generator matrix for lattice Λ and each ν_i

is a basis vector having ν_{ij} coordinates.

§ 1.4 Geometry

If the equation of a plane be $ax + by + cz + d = 0$, then $(a, b, c)^T$ is a vector normal to it and the parametric equations of this line is $x = at + p$, $y = bt + q$ and $z = ct + r$, $t \in R$, and (p, q, r) is a point on the line.

The simplest geometrical figure is a circle while the simplest of all polygons is a triangle. The degree of freedom of triangles increases from the equilateral to the isosceles and the right triangles to the scalene triangles. While spending the summer of 1990 in a traineeship through AIESEC the author was introduced to a geometrical puzzle which, as he later knew, is called the *flexatube.* It could have come from ancient China. This puzzle is made up of sixteen right isosceles triangles tiled into four squares, each comprising of four triangles, which are in turn joined together to form a loop. It can be easily made up using some hard papers, a pair of scissors and cello tape. There are in total twenty hinges, four of which are as long as the hypothenuse while the other sixteen have their length equal to the shorter side of the triangle. By turning these rigid triangles upon their hinges the inside surface of the strip can become the outside and vice versa. Tiyapan found one solution and later on the same year learnt about another. It has not yet been proved whether these two are the only possible solutions. Geometry plays not a small part in our everyday's life, for instance the symbol for the men's toilets in Budapest is an equilateral triangle, while that for women's toilets is a circle. Thus the geometrical information, having passed through the eyes, goes directly to the brain and we understand with no words being needed.

The median of a triangle is the line joining its vertex with the mid point of the opposite side. All the medians of a triangle intersect one another at its centroid. The lines connecting all mid points of a triangle divide it into four equal triangles. Let the length of the median from the corner A to its opposite side a of a triangle be m_a, then $m_a = (2b^2 + 2c^2 - a^2)^{1/2}/2$, and similarly for m_b and m_c. We have the relations $\mathrm{A}(\triangle m_a m_b m_c) = 3\mathrm{A}(\triangle abc)/4$, where $\mathrm{A}(\triangle\cdot)$ is the area of a triangle, and $m_a^2 + m_b^2 + m_c^2 = 3(a^2 + b^2 + c^2)/4$.

One end of a median is at a vertex, the other one at the mid point of a side. The lines which bound the first ones is the triangle itself, those which join the second ones create the medial triangle. A bisector of a medial triangle divides the perimeter of the original triangle into two equal parts. The incentre of the medial triangle, the Spieke point, is the c.g. of the wire-framed triangle $\triangle ABC$. The incentre, the geocentre and the Spieke point, all lie on a single straight line. The geocentre on the Euler line is one third away from the circumcentre to the orthocentre, that point where the altitudes of the triangle intersect.

In general dimension we talk about spheres. A sphere in d dimensions has its volume, V, proportional to r^d and its surface, A, to r^{d-1}, so that $V \propto A^{d/(d-1)}$.

The area of sphere in three dimensions is $A = 4\pi r^2$ and the volume $V = \frac{4}{3}\pi r^3$. When $V = 1$, $r = -\left(-3/\pi\right)^{1/3}/2^{2/3}$, $\left(3/\pi\right)^{1/3}/2^{2/3}$, or $(-1)^{2/3}\left(3/\pi\right)^{1/3}/2^{2/3}$ which are numerically $-0.310175 - 0.537239\imath$, 0.62035, $-0.310175 + 0.537239\imath$ in that order. Therefore $\alpha = A|_{V=1} = 6^{2/3}\sqrt[3]{\pi} = 4.83598$

When $V = 8$, $r = \left(6/\pi\right)^{1/3}$ and therefore $A = 46^{2/3}\pi^{1/3} = 19.3439$. In order to find α, the surface area per unit volume, one divide the area by $V^{2/3}$, in other words $\alpha = V^{1/3} \cdot A/V = A/V^{2/3}$.

The same is true for other polyhedra. For example in a tetrahedron where x is the length of the side, the vertices can be $(0,0,0), (x,0,0), \left(\frac{x}{2}, \frac{\sqrt{3}}{2}x, 0\right), \left(\frac{x}{2}, \frac{\sqrt{3}}{6}x, \frac{1}{2}\sqrt{\frac{93}{35}x}\right)$. When $V = 1$, one can obtain x by solving the equation

$$1 = \frac{1}{6}\,\mathrm{abs}\left(\begin{vmatrix} 0 & 0 & 0 & 1 \\ x & 0 & 0 & 1 \\ \frac{x}{2} & \frac{\sqrt{3}}{2}x & 0 & 1 \\ \frac{x}{2} & \frac{\sqrt{3}}{6}x & \frac{1}{2}\sqrt{\frac{93}{35}x} & 1 \end{vmatrix}\right).$$

This gives $x = 2\left(\frac{35}{31}\right)^{\frac{1}{6}} = 2.0409$ as the only real positive answer. When x is doubled, V increases from 1 to 8, which means that one would be dividing A by $V^{\frac{2}{3}}$ to obtain α.

The perimeter of a triangle is $s = 3d_e$ and the area $A = (\sqrt{3}/4)l^2$. When $A = 1$, $d_e = \pm 2/3^{1/4} = \pm 1.51967$. Therefore $s = 4.55901$.

Vertices shared by two cells make up a common face between them. Two way have been tried for finding the edges. The first one was by looking at all neighbouring cells of every cell in turn three at a time. The edges are then made up of those vertices that are common among these three cells. Only those edges which have exactly two vertices are considered. They are called *good edges* as contrasted with edges on the boundary. This is a much longer way than the second one, which is to consider vertices common to any two faces of a cell. Similar to the first case, such vertices forms a good edge if and only if there are only two of them. The two methods above give exactly the same list of edges, so they confirm each other. It has been tested that all edges having more than two vertices are boundary ones, that is they have at least one vertex outside the boundary of the unit cube considered.

By drawing some of the cells as a solid using *fill* command it has been tested that the result from `convhull` covers the entire cell surface. This confirms the step where areas are calculated.

The hexagon or honeycomb is perhaps the pattern which is most frequently found in nature. Even though the world we live in is three-dimensional, cells normally divide and spread in two dimensions in the form of layers. Moreover, they are packed in these layers in patterns which most often resemble the honeycomb (*cf* Williams and Bjerknes, 1972).

An octagon is an eight-sided polygon. It is the shape of the cross section of every chimney in the mills built in Manchester during its industrial era of the nineteenth century, as well as that of the terrets in the Main Building of UMIST. Perhaps one of the reasons for its popularity is that it looks strong while having the style of a good taste. May be the reason why it looks strong is that it possesses eight axes of symmetry, on top of another symmetry around the origin.

There are nine regular polyhedra. Among these are five regular convex solids known to the ancient Greek called Platonic polyhedra. They are tetrahedron, cube, dodecahedron, octahedron, and icosahedron. They have regular congruent faces and regular polyhedral angle vertices. Their face angles and their dihedral angles at every vertex are equal. The other four regular polyhedra have only been discovered much later and are not convex. They are called the Kepler-Poinsot polyhedra and are nonconvex. The small stellated dodecahedron and the great stellated dodecahedron were found by Kepler (1571–1630). The great icosahedron and the great dodecahedron were found by Poinsot (1777–1859). The small stellated dodecahedron and the great dodecahedron do not satisfy Euler's equation. The process of creating it by extending nonadjacent faces until they meet is called *stellating.* There are also polyhedra called quasi-regular.

The semi-regular polyhedra are called the Archimedean polyhedra. Here all faces are regular polygons but not all are of the same kind. Every vertex is congruent to all others. They comprise of an infinite group of prisms, an infinite group of antiprisms or prismoid, and another thirteen polyhedra. Each prism or prismoid is made up of two regular polygons on parallel planes where the vertices are aligned in the former case or shifted half way to the next neighbouring vertices in the latter case. Each vertex in prisms is joined to a corresponding vertex of the opposite polygon, while in prismoid it is joined to two corresponding vertices. All faces of an Archimedean solid are regular and all its polyhedral angle vertices congruent.

On the other hand the Archimedean duals have the property that all their faces are congruent to one another and all their polyhedral angles regular. These solids are important in crystallography. They are vertically regular and include an infinite group of dipyramids, an infinite group of trapezohedra, and additionally thirteen other polyhedra.

The surface area per unit volume α of a solid can be computed from the actual volume V and the actual surface area A as $\alpha = V^{1/3}A/V = AV^{-2/3}$. Another way of finding the perimeter per unit area of an n-gon follows the steps listed in Algorithm 1.1. Here θ is the angle made by the lines from the centre of gravity of an n-gon to its two consecutive vertices, α half the angle between two edges, h the distance from the c.g. to each edge, that is the height of one of the n identical triangles all of which have a vertex at the c.g., a the area of each of such triangles, $A(d)$ the area of the n-gon in terms of the edge length, s the perimeter and d the edge length d_e. Algorithm 1.1, however, may be reduced to two steps, namely solving for $(nd^2/4)\tan[(n-2)\pi/(2n)]$ and then $s = nd$. Of course, if n increases towards infinity then s approaches $2\sqrt{\pi} \approx 3.5449$.

Algorithm 1.1 *Perimeter per unit area of regular polygons.*

```
$\theta \leftarrow 2\pi/n$;
$\alpha \leftarrow (\pi - 2\pi/n)/2$;
$h \leftarrow (d/2)\tan\alpha$;
$a \leftarrow (1/2)dh$;
$A(d) \leftarrow na$;
solve $A(d) = 1$ for $d$
$s \leftarrow nd$;
```

□

Polygon	n_e	A	s *(numerical)*
Triangle	3	$(\sqrt{3}/4)d^2$	4.55901
Square	4	d^2	4
Pentagon	5	$5(1+\sqrt{5})d^2/\left[4(10-2\sqrt{5})^{1/2}\right]$	3.8119
Hexagon	6	$3\sqrt{3}d^2/2$	3.7224
Heptagon	7	$(7/4)d^2\tan(5\pi/14)$	3.6721
Octagon	8	$2d^2\tan(3\pi/8)$	3.6407
Nonagon	9	$(9/4)d^2\tan(7\pi/18)$	3.6198
Decagon	10	$5\left[(5+\sqrt{5})/2\right]^{1/2}d^2/(-1+\sqrt{5})$	0.3605
Undecagon	11	$(11/4)d^2\tan(9\pi/22)$	3.5944
Dodecagon	12	$3(2+\sqrt{3})d^2$	3.5863

Table 1.2 *Perimeter per unit area of n-gons.*

In a similar fashion the surface area and volume of regular solids can be found, but first we need to know more about these solids. Table 1.3 lists some of the important properties of regular solids. Here cyc$(\cdot)$ is the cyclical permutations and $\tau = (1+\sqrt{5})/2$, *i.e.* the golden ratio. Regular polyhedra can also be represented by Schlafi's symbol as $\{p, q\}$ where p and q are respectively the face- and vertex figures.

solid	*aka*	n_v	n_e	n_f	x_v	d_e	r_v	r_e	r_f	*dual*
tetrahedron	3|2 3	4	6	4	$(\pm1,\pm1,\pm1)$ even or odd -1's	$2\sqrt{2}$	$\sqrt{3}$	1	$1/\sqrt{3}$	itself
cube	3|2 4	8	12	6	$(\pm1,\pm1,\pm1)$	2	$\sqrt{3}$	$\sqrt{2}$	1	octahedron
octahedron	4|2 3	6	12	8	cyc$(\pm1,0,0)$	$\sqrt{2}$	1	$1/\sqrt{2}$	$1/\sqrt{3}$	cube
dodecahedron	3|2 5	20	30	12	cyc$(0,\pm\tau,\pm1/\tau)$, $(\pm1,\pm1,\pm1)$	$2/\tau$	$\sqrt{3}$	τ	$\tau[(\tau+2)/5]^{1/2}$	icosahedron
icosahedron	5|2 3	12	30	20	cyc$(\pm1,0,\pm\tau)$	2τ	$\sqrt{2+\tau}$	1	$(6+5\tau)^{1/2}/3$	dodecahedron

Table 1.3 *Some important properties of regular solids.*

The ratio between the edge length and distance to face, the surface area, the volume and the ratio between surface area and volume of some solids are shown in Table 1.4.

solid	d_e/r_f	A	V	α
Tetrahedron (3^3)	$2\sqrt{6}$	$24\sqrt{3}r^2$	$8\sqrt{3}r^3$	7.2056
Cube, (4^3)	2	$24r^2$	$8r^3$	6
Octahedron, (3^4)	$\sqrt{6}$	$12\sqrt{3}r^2$	$4\sqrt{3}r^3$	5.7191
Dodecahedron,(5^3)	$\frac{4\sqrt{10}}{(3+\sqrt{5})(5+\sqrt{5})}$	$300\frac{2}{5-\sqrt{5}}^{\frac{1}{2}}\frac{(1+\sqrt{5})r^2}{25+11\sqrt{5}}$	$100\left(\frac{2}{5-\sqrt{5}}\right)^{\frac{1}{2}}\frac{(1+\sqrt{5})r^3}{25+11\sqrt{5}}$	5.3116
Icosahedron, (3^5)	$3(1+\sqrt{5})\left[6+\frac{5}{2}(1+\sqrt{5})\right]^{\frac{1}{2}}$	$180\sqrt{3}\frac{(3+\sqrt{5})r^2}{17+5\sqrt{5}}$	$60\sqrt{3}\frac{(3+\sqrt{5})r^3}{17+5\sqrt{5}}$	8.0484
Sphere	∞	$4\pi r^2$	$(4/3)\pi r^3$	4.8360

Table 1.4 *Surface area per volume of regular solids.*

Algorithm 1.2 finds the values given in Table 1.4. Here p is the face figure, $n = n_f$ the number of faces, $r = r_f$ the distance from the centre of the polyhedron to each face, $d = d_e$ the edge length, $s = A/V$ and $\tau = \left[1+\sqrt{5}\right]^{1/2}/2$ the golden mean. Furthermore, a and v are respectively the area of each face and the volume of the pyramid whose base is the face and the apex of which is the centre of the polyhedron.

Algorithm 1.2 *Area per volume of regular polyhedra.*

```
$\theta \leftarrow 2\pi$;
$\alpha \leftarrow (\pi - 2\pi/p)/2$;
$h \leftarrow d\tan\alpha/2$;
$a \leftarrow pdh/2$;
$v \leftarrow ar/3$;
$A(r) \leftarrow na$;
$V(r) \leftarrow nv$;
solve $V = 1$ for $r$;
$s(r) \leftarrow A(r)/V(r)$;                                                □
```

Since Algorithm 1.2 will be used to proof Theorem 1.1, we shall give it a formal proof.

Proof. Each face of a regular polyhedron is a regular polygon by definition. The lines connecting the centre and c.g. of a face with all its vertices divide 2π radian into p equal portions, where p is the face figure. Each portion then represents the angle in radian made by two consecutive vertices of the face as seen from its centre. Let this angle be θ, then it follows that $\theta = 2\pi/p$. From the symmetry around the c.g., all these angles together with the sides of the face form p identical isosceles triangles whose two equal triangles are at the vertices of the face. Moreover, this angle, α, is half the internal angle of a face vertex. Then, because we know that all the internal angles of a regular polygon sum up to $2\pi(n-1)/n$ radian, it follows that $\alpha = (\pi - 2\pi/p)/2$. The area of each face is therefore $a = p(d_e h/2)$, where $h = (d_e/2)\tan\alpha$ is the distance from the centre of the face to its edges. The volume of the pyramid which have the face as its base and the c.g. of the polyhedron as its apex is thus $v = ar_f/3$, and the polyhedral surface area and volume are respectively $A = n_f a$ and $n_f v$. Comparing the above with Algorithm 1.2 completes the proof. □

From the results in Table 1.4 we can see that the sphere has the ratio α less than every regular polyhedron; in fact one could conjecture that it has the smallest s of all solids. Icosahedron, on the other hand, is a regular polyhedron which has maximised α. So now we know, for instance, that a virus wants to maximise its surface area.

The ratio α here is not simply obtained by dividing the surface area by the volume of a solid, as Algorithm 1.2 also tells. We take the volume of a solid to be the unity first, then proceed to find its correspondent surface area. As an example to show that these two values are not the same, consider an icosahedron whose $d_e = 2\sqrt{2}$ and $r_f = 1/\sqrt{3}$. From the values of d_e and r_f the surface area and volume are respectively 362.765 and 37.8252, which result in A/V being 9.5906 which is not the same as our α.

The numerical ratios given in both Tables 1.2 and 1.4 are rounded approximates, to make them easier to read. The exact values can easily be obtained by following the steps of calculation in Algorithms 1.1 and 1.2.

After having worked with Algorithm 1.2 it turns out that α depends on r_f. A little investigation confirms this, and Theorem 1.1 arises as a result. A platonic solid is a regular solid and vice versa.

Theorem 1.1. *Let α be the surface area of a Platonic solid and r_f the distance from its centre to each face. Then $\alpha r_f = 3$.*

Proof. There are five and only five such solids, therefore we find α of every one of them. Assuming that Algorithm 1.2 together with the resulting A's and V's shown in Table 1.4 are correct. Then simply divide each $A(r)$ by the corresponding $V(r)$ for the regular solids from the table to get $\alpha = A(r)/V(r) = 3/r$. □

Collorary 1.1[1] follows immediately from Theorem 1.1 and the definition of α, but such interesting consequence as it is deserves being called a collorary.

Corollary 1.1[1]. *In a regular polyhedron, $Ar_f = 3V$.*

The duality among regular polyhedra is shown in Table 1.3. In particular, the tetrahedron is dual to itself, the octahedron and the cube are dual to each other and likewise the dodecahedron and the icosahedron. The icosahedron is a popular shape among viruses. As regarding duality, it is worth noting that all pyramid are self-dual.

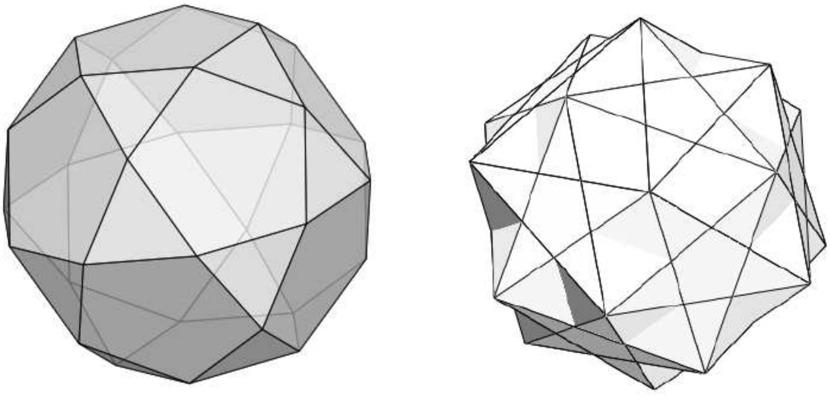
Deltahedra are polyhedra which have all faces equilateral triangles. They have $2n$ faces, $3n$ edges and $n+2$ vertices. There are eight deltahedra, namely regular tetrahedron (4 faces), triangular dipyramid (6), regular octahedron (8), pentagonal dipyramid (10), snub disphenoid (12), triaugmented triangular prism (14), gyroelongated square dipyramid (16) and icosahedron (20).

A cube is sometimes called a hexahedron because it has six faces. But there are other polyhedra which also have six faces, for instance the triangular dipyramid (5 vertices, 9 edges), the pentagonal pyramid and the tetragonal antiwedge (6, 10), the hemiobelisk and hemicube (7, 11) and the pentagonal wedge (8, 12).

polyhedron	n_v	n_e	n_f
antiprism	$2n$	$4n$	$2n+2$
antiwedge	$2n-2$	$4n-6$	$2n-2$
cupola	$3n$	$5n$	$2n+2$
cupola pyramid	$3n+1$	$7n$	$4n+1$
cupolarotunda (ortho-, gyro-)	$5n$	$10n$	$5n+2$
deltohedron	$2n+2$	$4n$	$2n$
dipyramid	$n+2$	$3n$	$2n$
hemiprism	$2n-1$	$3n-1$	$n+2$
ortho(,gyro-)bicupola	$4n$	$8n$	$4n+2$
ortho(,gyro-)birotunda	$6n$	$12n$	$6n+2$
prism	$2n$	$3n$	$n+2$
pyramid	$n+1$	$2n$	$n+1$
rotunda	$4n$	$7n$	$3n+2$
rotundapyramid	$4n+1$	$9n$	$5n+1$
wedge	$2n-2$	$3n-3$	$n+1$

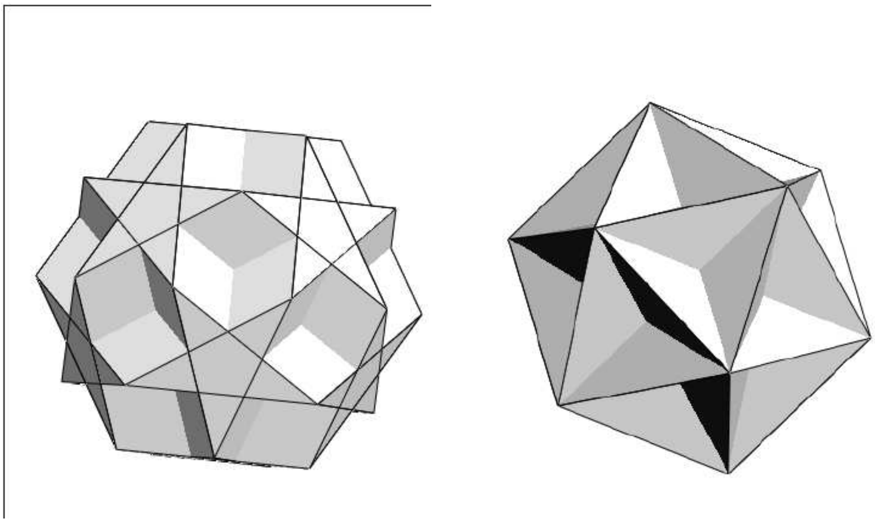
Table 1.5 *The number of vertices, edges and faces of some polyhedra*

The nearest neighbour and minimum spanning tree have been applied to the problem of taxonomy in botany. Clayton (1972), working on the characters of plants to manually classify them (*eg* Clayton, 1970) with the use of only the binary dendrogram and trial and error, adopted a numerical method which finds the minimum spanning tree in a multi-dimensional character space. Since taxonomy can be considered as a kind of dictionary, it is possible to apply a similar approach to machine translation and the compilation of dictionaries.



(a) (b)

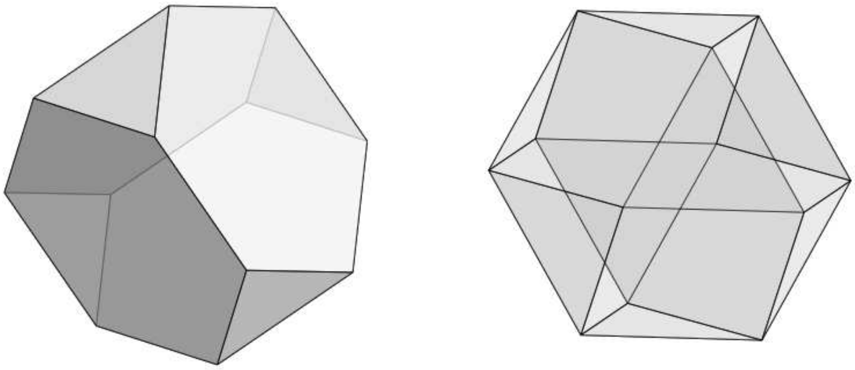
Figure 1.2 (a) *Icosidodecahedron, rhombic triacontahedron,* 2|3 5. (b) *Small ditriagonal icosidodecahedron, small triambic icosahedron,* 3|5/2 3



(a) (b)

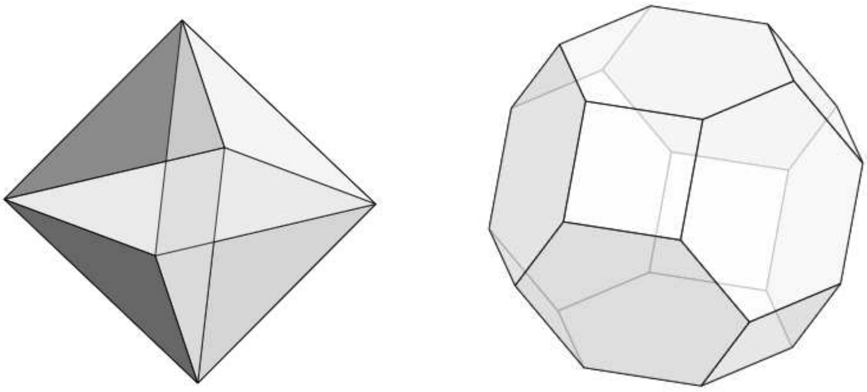
Figure 1.3 (a) *Great dodecadodecahedron, medial rhombic triacontrahedron,* 2|5/2 5. (b) *Great dodecahedron, small stellated dodecahedron,* 5/2|2 5

The polyhedra from Figure 1.4 to 1.6 are semi-regular.



(a) (b)

Figure 1.4 (a) *Truncated tetrahedron, triakistetrahedron,* 2 3|3. (b) *Octahemioctahedron, octahemioctacron,* 3/2 3|3



(a) (b)

Figure 1.5 (a) *Tetrahemihexahedron, tetrahemihexacron,* 3/2 3|2. (b) *Truncated octahedron, tetrakishexahedron,* 2 4|3

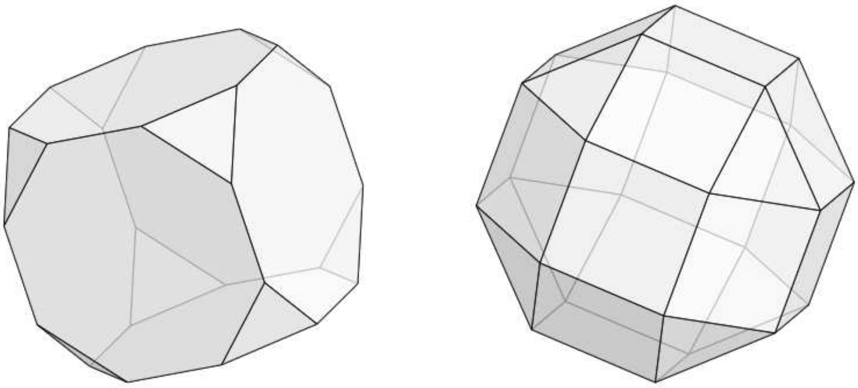

(a) (b)

Figure 1.6 (a) *Truncated cube, triakisoctrahedron,* 2 3|4. (b) *Rhombicuboctahedron, deltoidal icositetrahedron,* 3 4|2

Polyhedra in Figure 1.7 are snub polyhedra.

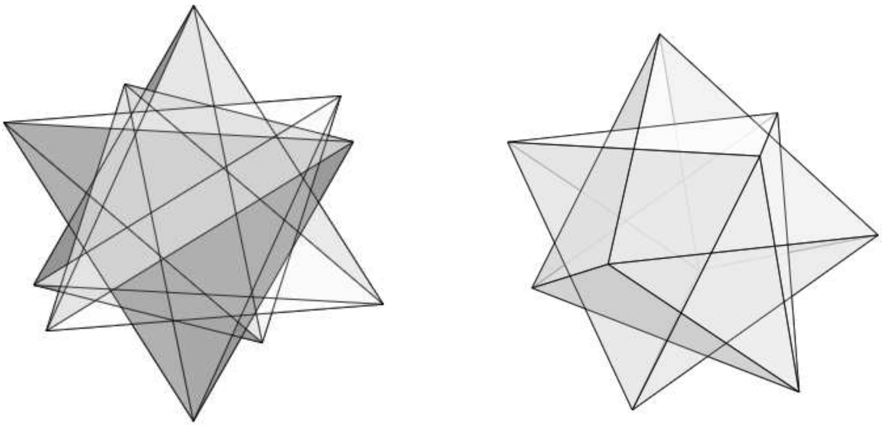

(a) (b)

Figure 1.7 (a) *Pentagrammic crossed antiprism, pentagrammic concave deltohedron,* |2 2 5/3. (b) *Pentagrammic antiprism, pentagrammic deltohedron,* |2 2 5/2

The surface of Fullerene is made up of pentagons six-sided figures. Its shape represents that of the geodesic domes developed by Buckminster Fuller, and hence the name *Fullerene.* The latter may either be hexagons or figures all the six sides in each one of which form two sets of three sides having an equal length. The simplest Fullerene, the carbon-60 molecule, has the same shape as that of a football and a handball. With some thought the reason for this is not difficult to see. With its thirty-two faces it closely resemble the sphere. Also the two different shapes of all its components are symmetrically distributed and therefore enable colouring with only two different colours, namely one for each of the two shapes. To see how this helps, suppose one made a football in the shape of a bloated dodecahedron. Then it would be impossible to colour it using more than one colour at the same time of giving it a symmetrical appearance when viewed from more than a few directions. With the fullerene shape and the colouring scheme mentioned, however, the football looks symmetrical when viewed from 54 different directions symmetrically distributed around it. These directions corresponds to those when one looks at it in the direction perpendicular to the centre of each of its faces and when in the direction through the middle of each of the 22 edges lying between two hexagonal faces.

Making polyhedron models is an educating experience. Contrary to the general believe that you need to make an accurate drawing for the required parts (Wenninger, 1971), this needs not be so. Examples of this are the origami models of polyhedra where complex polyhedron structures are made from interlocking pieces each of which is made by folding a piece of paper of a rectangular or square shape.

A set of elements with the sum and the product of any two elements defined is a commutative ring if under these two operations it satisfies the following postulates: closure, uniqueness, commutative, associative, and distributive laws, identity (zero and unity), and additive inverse. An integral domain is an ordered domain if its positive elements satisfy the laws of addition, multiplication, and trichotomy. A subset of an ordered domain is well-ordered if every nonempty subset of it contains a smallest member. $a|b$ means that $.b$ is divisible by a. The Euclidean algorithm or division algorithm states that $a = bq + r, 0 \leq r < b$. Two integers are relatively prime if their only common divisors are ± 1. $a \equiv b (\bmod m)$ if and only if $m|(a - b)$. The commutative ring Z_2 is the properties of multiplication and addition of even (0) and odd (1) numbers.

+	*0*	*1*
0	0	1
1	1	0

·	*0*	*1*
0	0	0
1	0	1

The following is Z_5.

+	*0*	*1*	*2*	*3*	*4*
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

·	*0*	*1*	*2*	*3*	*4*
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

There is a close link between geometry and algebra. Geometrical surfaces can be described as algebraical equations. For example, for circles and polygons the equations are binary quadratic, while for spheres and polyhedra they are ternary quadratic. Even one-sided surfaces can be described algebraically. The equation of Klein bottle, when deformed into a sphere with two circles removed and replaced by two cross-caps, is a quartic equation

$$a^2(x^2 + y^2)(b^2 - x^2 - y^2) = z^2(a^2x^2 + b^2y^2),$$

while the Steiner surface is also a quartic one

$$y^2z^2 + z^2x^2 + x^2y^2 + xyz = 0.$$

Two surfaces is homomorphic to each other if it is possible to continuously transform one into the other. All convex polyhedra are homomorphic to a sphere. The Steiner surface is homomorphic to the heptahedron, which is an Archimedean polyhedron with diametral plane.

In the plane, a second-degree equation gives either two straight lines, a circle, an ellipse, a parabola, or a hyperbola. In space, it can give two planes, cylinders and cones (circular, elleptic, parabolic, or hyperbolic), aphere, spheroid, ellipsoid, two hyperboloids, and (elliptic or hyperbolic) paraboloid.

Partition, *tessellation* and *division of space* are the same thing. In the context of set theory, a partition of set X is a family of sets A_1, A_2, ..., A_k which are subsets of X, such that $A_i \neq \emptyset$; $A_i \cap A_j = \emptyset$; $\bigcup_i A_i = X$, where i, j = 1, 2, ..., k and $i \neq j$. (*cf* Berge, 1958) A further condition that makes any tessellation a Voronoi one is that, for all i there exists a unique point a_i within A_i such that every point in A_i is closer to a_i than to any other a_j, $j \neq i$.

Voronoi tessellation in three dimensions can be constructed by imagining each region as a spherical cell growing outwards to meet neighbouring cells and continue growing to fill the gaps. The centre of each sphere is a unique nucleus point of the region such that it is closest to any point belonging to that region than any nuclei points. If the rate of growth is the same from every cell, the resulting partitions will be planes which can be described by ternary quadratic equations. However, if this rate differs from one cell to another, the partitions will be curved surfaces and the result is a non-Voronoi tessellation. It is possible to impose a constraint of minimum distance between neighbouring nuclei. Such cases can be looked at as spheres of an equal nonzero radius expanding away from nucleus centre points. If the radii differ from one sphere to another, or if some nonspherical solids are used instead of spheres, the tessellation obtained will be non-Voronoi.

Consider the case where all spheres are of equal size. If these spheres already touch their neighbours before the expanding starts, the case is that of packed spheres expanded to form a Voronoi tessellation. There are two types of close-packing: cubic (face-centred) and hexagonal. In both cases each sphere has twelve neighbours. Both cases have the same density, which is $\frac{\pi}{3\sqrt{2}}$. The Voronoi regions produced from the cubic case are rhombic dodecahedra and the faces are rhombuses. In the case of hexagonal close-packing, the corresponding regions are trapezo-rhombic dodecahedra and the faces are either rhombics or trapezia. Where the spheres meet with their three neighbours in the layer above and their three neighbours in the layer below, the faces are rhombics. Where they meet with the six neighbours on the same layer they are trapezia.

For geometrical calculation, an example of a definitive book is that written by Salmon (1912). The gamma function,

$$\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}\mathrm{d}t, \quad \mathrm{Re}(z) > 0, \tag{$\mathbf{1}$)i}$$

got its name from Legendre and is known as the Euler gamma function or simply the second Euler function. The formula $\Gamma(z+1) = z\Gamma(z) = z!$ recursively calculates the gamma function from, for instance, $\Gamma(1/5) \approx 4.5908$, $\Gamma(1/4) \approx 3.6256$, $\Gamma(1/3) \approx 2.6789$, $\Gamma(2/5) \approx 2.2182$, $\Gamma(1/2) = \sqrt{\pi} \approx 1.7725$, $\Gamma(3/5) \approx 1.4892$, $\Gamma(2/3) \approx 1.3541$, $\Gamma(3/4) \approx 1.2254$, and $\Gamma(4/5) \approx 1.1642$. The Stirling's formula was found by de Moivre which approximates the gamma function. The gamma function expansions is

$$\Gamma(x+1) = \lim_{k\to\infty} \frac{k^x 1\cdot 2\cdot 3\cdots k}{(x+1)(x+2)\cdots(x+k)}, \tag{$\mathbf{2}$)i}$$

and the gamma function of negative numbers can be obtained from

$$\Gamma(-z) = \frac{-\pi}{z\Gamma(z)\sin\pi z}. \tag{$\mathbf{3}$)i}$$

The incomplete gamma function is

$$\Gamma(z,x) = \int_0^x e^{-t}t^{z-1}\mathrm{d}t = \int_x^\infty e^{-t}t^{z-1}\mathrm{d}t, \tag{$\mathbf{4}$)i}$$

and the normalised or regularised incomplete gamma function is $\Gamma(z,x)/\Gamma(z)$

Archimedean solids are dual to Catalan solids. The fullerene C_{60} is the truncated icosahedron, an Archimedean solid. Table 1.6 shows some statistics of the Archimedean solids.

Archimedean solid	n_v	n_e	n_f	*dual, Catalan solid*
truncated tetrahedron	12	18	8	triakis tetrahedron
cuboctahedron	12	24	14	rhombic dodecahedron
truncated cube	24	36	14	triakis octahedron
truncated octahedron	24	36	14	tetrakis cube
rhombicuboctahedron	24	48	26	deltoidal icositetrahedron
snub cube	24	60	38	pentagonal icositetrahedron
icosidodecahedron	30	60	32	rhombic triacontahedron
great rhombicuboctahedron	48	72	26	disdyakis dodecahedron
truncated icosahedron	60	90	32	pentakis dodecahedron
truncated dodecahedron	60	90	32	triakis icosahedron
rhombicosidodecahedron	60	120	62	deltoidal hexacontahedron
snub dodecahedron	60	150	92	pentagonal hexacontahedron
great rhombicosidodecahedron	120	180	62	disdyakis triacontahedron

Table 1.6 *Archimedean solids*

The analogue of polyhedra in four dimensions is sometimes called polychora, with the 4-d equivalent of the Euler-Descartes formula being $n_v - n_e + n_f - n_c = 0$ where n_c is the number of its 3-d facets called cells. For dimensions higher than four the analogies are polytopes. An n-dimensional polytope is bound by hyperfaces of polytopes of $(n-1)$ dimension which join at hyperedges of $(n-2)$ dimensions.

The Euler characteristic χ is 1 for a point, invariant in a topological homeomorphism and additive for disjoint sets. For Euclidean space in n dimensions, $\chi = (-1)^n$ as that of ordinary open polytopes. The χ of all ordinary closed polytopes is the χ of a closed n-dimensional ball, and $\chi = 1$ for any n dimensions. If the hypercell itself, *i.e.* the interior of the polytope, is not counted then the right hand side of the formula becomes 2 when n is odd and 0 otherwise, for example $n_v - n_e + n_f - n_c + n_t - n_p + n_h = 2$ in 7 dimensions, where t, p and h are respectively the tetrafaces (sometimes called hyperpoints), pentafaces (hyperedges) and hexafaces (hyperfaces).

As a revision of the sixth form mathematics, selections are *combination* if the order is irrelevant, and are *permutation* otherwise. The formula is for the former ${}^n\mathrm{C}_k = n!/[n!(n-k)!]$, and ${}^n\mathrm{P}_k = n!/(n-k)!$ for the latter.

Definitions which are useful when describing the time- and storage complexities of an algorithm are $O(f(n)) = \{g(n) : \exists c, n_0 \in R^+, g(n) \le cf(n)\,\forall n \ge n_0\}$, $\Omega(f(n)) = \{g(n) : \exists c, n_0 \in R^+, cf(n) \le g(n)\,\forall n \ge n_0\}$, $\theta(f(n)) = \{g(n) : \exists c_1, c_2, n_0 \in R^+, c_1 f(n) \le g(n) \le c_2 f(n)\,\forall n \ge n_0\}$ and $o(f(n)) = \{g(n) : \forall c \in R^+ \exists n_0 \in R^+, g(n) \le cf(n)\,\forall n \ge n_0\}$, the most commonly used for the purpose being the $O(\cdot)$.

A geometry, in Klein's view, is a set S and a subgroup G of the group $B_{ij}(S)$ of all bijections from S to itself. The elements of S are points, and G acts on S by mapping points to points. Two subsets of S are equivalent if there is an element of G which takes one set into the other. In general, S has an extra structure which $B_{ij}(S)$ preserves. A map $f : \mathrm{R}^m \mapsto \mathrm{R}^n$ is linear if it maps a linear combination of vectors to the same linear combination of the images. A matrix (a_{ij}), $1 \leq i \leq m$ and $1 \leq j \leq n$, transforms each basis element b_i to a combination of the basis elements b_j. The map is a bijection if f^{-1} exists or equivalently if the determinant of its matrix is non-zero. The general linear group, $GL(n, \mathrm{R})$, is the set of all invertible linear transformations from the vector space R^n to itself. The special linear group $SL(n, \mathrm{R})$, a subgroup of $GL(n, \mathrm{R})$, is the set of all invertible transformations with determinant 1. The orthogonal group, $O(n)$, the set of all orthogonal transformations T. The special orthogonal group $SO(n)$, a subgroup of $O(n)$, is the set of all orthogonal transformations whose matrix has determinant 1.

The group $l(\mathrm{R}^n)$ of all isometries of R^n consists of composites $T_a \circ L$, where $T_a : x \mapsto x + a$ is a translation and L an orthogonal map. The set of all translations forms a normal subgroup of $l(\mathrm{R}^n)$ which is isomorphic to the group R^n under addition.

A summetry group of the subset X of $S = \mathrm{R}^2$ is the subgroup of $l(\mathrm{R}^2)$ whose all elements map to themselves. Rotation by π/n generates a subgroup which is isomorphic to C_n, the cyclic group of order n. The dihedral group D_n, whose order is $2n$, is the group of symmetries of a regular n-gon. For $n \geq 3$, D_n is a non-abelian group.

The affine group, $A(\mathrm{R}^n)$, is the group of all affine transformations or affinities of R^n, that is to say, the transformation of the form $T_a \circ L$ where T_a is a translation and $L \in GL(n, \mathrm{R})$. Affine transformations preserve no distance, angle, area or volume. But they preserve collinearity, parallelism and ratios. A similarity transformation or similitude, $T_a \circ \lambda L$ with $L \in O(n)$ and $\lambda > 0$, is an affine map which preserves angles.

Affine theorems are theorems which can be proved by only those concepts which are preserved by affine transformations. In other words, they are theorems which can be proved by vector methods without using norms, dot- or vector products. Examples of such theorems are the coincidence of the medians of a triangle, Ceva's theorem and Menelaus's theorem.

The homogeneous coordinates of a point x on the affine line are (α, β), where $x = \alpha/bta$. Here a line OA through the origin and a point A on the line $y = 1$ is described by a vector (α, β) in it. The point at infinity has homogeneous coordinates $(1, 0)$. The projective line is denoted by RP^1.

The projective group, $PGL(n, F) = GL(n, F)/\{\lambda I | \lambda \in F - \{0\}\}$ where F is any field, is the set of all projective transformations or projectivities. The standard reference points on RP^1 are ∞, 0 and 1. There is a unique projective transformation which takes any three distinct points to any other three distinct points. Let a, b, c and d be points in RP^1, and θ the map which takes a, b and c respectively to ∞, 0 and 1. Then the cross-ratio is $\theta(d) = (a, b; c, d) = (d-b)/(d-a) \cdot (c-a)/(c-b)$. This cross-ratio is preserved by projective transformations.

The general form of an algebraic surface is $f(x, y, z) = 0$l, where $f(x, y, z)$ is a polynomial in x, y and z. A surface of order one is a plane, of order two a quadratic surface, for example ellipsoids and hyperboloids, and of order three a cubic surface.

§ 1.5 Physics

Percolation has been introduced and developed in the 1950s. (Hammersley et. al., 1954; Broadbent et. al., 1957) A typical physical problem which gives rise to problems in percolation is that of finding the probability that the centre of a sufficiently large porous stone gets wet when immersed in water. The internal structure of such a stone can be viewed as comprised of pores and solids. Water can seep through clusters of pores but is blocked when solids are encountered. Therefore the water can only reach the centre of the stone if there exists an open path, in other words a single cluster of pores, which leads it there. When the size of the rock is sufficiently large, this cluster is called an infinite cluster.

If on the other hand we consider the solids instead of the voids, we can reason that since the stone is rigid there have to exist at least one infinite cluster of solids. Then, because there must be rocks of various degrees of porosity which allow water to pass through, there must be a range of porosity which at one end is a nonpermeable stone where there exists an infinite cluster of solids but not an infinite cluster of pores, while at the other end it is the other way round, that is there exists an infinite cluster of pores but not one of solids. In other words, the latter case is the case where the rock has disintegrated into pieces. If one considers the pores and the solids as being two phases opposite to each other, then one can reason that the total volume of the two stays the same, whereas the ratio between them could vary.

The study of blockages in porous media can be translated into the study of percolation on the media when pores randomly turn into solids. In other words, it is the study of the formation of an infinite cluster of one phase within the infinite mass of the other. More often the latter infinite mass is taken for granted as always exists and remains the same. The domain of consideration is thus reduced to only the original pore spaces. Then, the study becomes that of inversion between two phases, namely the infinite cluster of free pores and the finite clusters of free pores. While the existence of an infinite cluster or finite clusters of blockages is of no consequences when one is only interested in the percolation point, it is considerably important when one wants to contemplate on the behaviours on either side of that critical point. It is important also when one studies changes in the rate of flow through the media.

One example is the study of traffic networks. Traffic congestion can be described as three degrees of flowability: free-flowing, congested, and stand-still. In a free-flowing traffic there are infinite clusters of roads and finite clusters of blocked roads. In a congested traffic there are both infinite clusters of roads and infinite clusters of blocked roads. In a stand-still traffic there are no infinite clusters of roads, while there are infinite clusters of blocked roads.

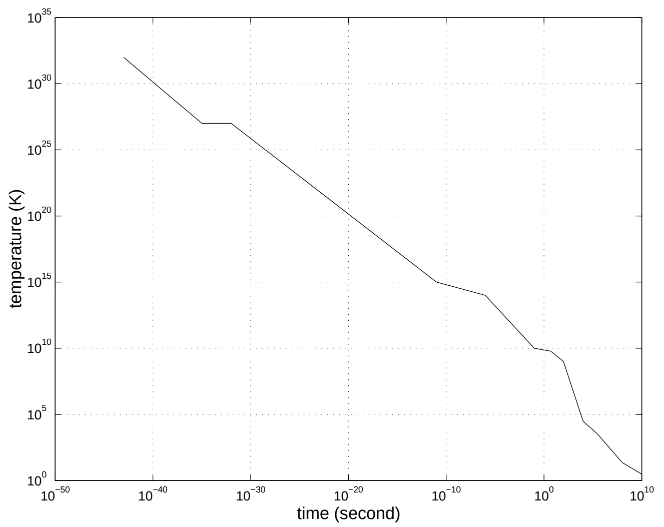
Applications of the percolation theory includes the study of forest fires and epidemics, the study of defects in semiconductors, the study of the effective resistance of a disordered mixture of two materials, the study of the Curie point of spontaneous magnetisation in ferromagnets and the Ising model, and the study of the rigidity of networks.

Phenomena which contain phase transitions are normally related to percolation theory. Examples of physical phase transition includes the boiling point and the point of evaporation, the transition of materials into superconductivity state, the triggering of a chain reaction in nuclear fission, and the triggering of nuclear fusion of hydrogen burning in a new-born star. Apart from these, there are philosophical phase transitions which include the nature of understanding, the nature of scientific discoveries and technological progress, the making of mobs, revolutions and wars, enlightenment, and, of course, love. Nature is essentially made up of numerous local linear or gradual relationships bound together in a larger scale by abrupt transitions.

The word *percolation* is derived from the word *percolator*, a coffee making machine. The connotation is therefore that of fluid flowing through a porous medium. In dynamic and everyday applications the emphasis is usually on the fluid part, and one studies the point of transition when the flow becomes blocked. There is another side of the same system where one studies the structure of underlying network. In that case the emphasis is on the solid part, and one is interested to find the point where particles percolate to form a solid structure.

Thus the study of percolation falls broadly into two branches, the continuum percolation and the network percolation. The former finds applications in crystallisation where one would like to know the number density of crystals that will make, for example, a synthetic zeolite membrane; in etching of metallic surface where one would like to find the estimates of the shapes and sizes of the etch pits which overlap and form clusters according to etching time and concentration of the acid; or in telecommunication networks where one would like to determine the number of mobile phone cells which percolates a city. The latter finds applications in filtration where one would like to discard filters just before they become blocked or in traffic networks where one could compare the robustness to standstill between two traffic network configurations.

Phase transition occurs in various areas. Wherever it is seen, it has the same characteristic of an intensely exponential change in a short period within a long stretch of a much more gradual one. The rate of change at the point of transition is generally astronomical. The sudden switch from one phase to another found in the study of percolation is a phase transition. The assumption of the existence and occurrence of infinite clusters as an explanation for these transitions can be justified by considering the promptness and the scale of theoretical transitions in a cosmological setting, namely the turning on of the general relativity at Planck time 10^{-43} seconds after the bigbang, the Grand Unified phase transition at 10^{-35} seconds and the start of nucleosynthesis at 1 second (cf. Croswell, 1995).



Temperature of the universe according to the Big Bang theory. The temperature of the universe decreases exponentially and its trend against time has to be plotted on a log-log scale to be comprehensible for otherwise it would lie almost on top of the coordinate axes. Such a plot as shown in Figure 1.8, produced from the data taken from Vaas (2002), roughly approximates a straight line with a negative slope. The percolation at the big bang could possibly be a phase change between matter and antimatter.

Figure 1.8 Temperature of the universe.

There is an idea in cosmology which could easily be as old as the oldest known religion, that describes the universe as alternating in cycles of destruction and creation. One current theory holding that idea tries to explain the big bang as the collision between two universes in five dimensional space-time. I believe that the universe undergoes phase changes periodically and percolates at the point the same one of which is differently called the big bang, the big crunch, or the big bounce depending how one looks at it. What we see now is only one side of the coin. We can not calculate pass the singularity at the big bang using our present theories because it is another phase complementary to us. That is why that creative singularity baffles us. I believe a theory must exist which can explain both or, if there are more than two, all phases as well as the transition point. And when we finally discover it we will understand the cosmic history and, hopefully, percolation.

In an asymmetric membrane the top dense layer determines the transport rate while the porous sublayer acts as a support. Because the permeation rate is inversely proportional to the thickness of the transport barrier, asymmetric membranes show a much higher permeation rate than homogeneous symmetric membranes of the same thickness.

Soap is solution of a sodium salt of a fatty acid, for example sodium stearate ($C_{17}H_{35}COO^{-}Na^{+}$). The films of soap bubbles are made up of a monomolecular layer of amphipathic ions, which have a hydrophilic and a hydrophobic parts. The former forms a hydrophilic polar carboxyl head, which in the case of sodium stearate is the COO^{-}. The latter forms a tail of hydrocarbon chain, in this case $C_{17}H_{35}$.

	Laboratorical scale		**Industrial scale**	
Process	*Origin*	*Application*	*Origin*	*Application*
microfiltration	Germany, 1920	bacteria filter		
ultrafiltration	Germany, 1930	laboratory use	America, 1960	concentration of macromolecules
hemodialysis	Netherlands, 1950	artificial kidney		
electrodialysis			America, 1955	desalination
hyperfiltration			America, 1960	sea water desalination
gas separation	1860	gas treatment	America, 1979	hydrogen recovery
membrane distillation			Germany, 1981	concentration of aqueous solutions
pervaporation			Germany and Netherlands, 1982	dehydration of organic solvents

Table 1.7 *Origin of membrane processes.*

J K Mitchell observed gas separation in 1831. In 1860 Thomas Graham made a systematic study of it (*cf* Ismail *et al*, 2002).

There are three categories of synthetic membranes: porous, nonporous, and carrier. Micro- and ultrafiltration use porous membranes while nonporous membranes are used for dialysis, vapour permeation, gas separation and pervaporation. All membranes are required to be mechanically, thermally, and chemically stable. The means of separation, however, varies. For porous membranes it is the pore dimension, for nonporous membranes the difference in diffusivity or solubility, and for carrier membranes the properties of carrier molecules.

Methods	*pore size (μm)*	*porosity*
coating		
phase inversion		
sintering	0.1–10	10–20%
stretching	0.1–3	90% max
leaching	0.005 min	
etching	0.02–10	10% max

Table 1.8 *Methods of preparation of membranes.*

In cosmology, the structure of the Abell Clusters shows a nonrandom distribution which suggests the existence of a second-order grouping, the clusters of clusters of galaxies (Abell, 1958). Apart from the Milky Way and the Andromeda galaxies (M31 or NGC224) there are at least 44 other galaxies in our Local Cluster. The Local Cluster is on the edge of the Local Supercluster which is flattened in shape and the approximate centre of which is at the Virgo Cluster. The Local Supercluster is separated by a nonspherical void of low galaxy density from the Pisces-Perseus supercluster. The latter lies nearly perpendicular to our line of sight, has the shape of a linear filament, and spans over 90° across the sky. The study of superclusters of galaxies often requires the use of superclusters of computers.

Abell (1958) plotted the distribution of the clusters in what he called, the 'Aitoff equal area projection'. By this he probably means the Hammer, aka Hammer-Aitoff, projection because that is what is normally used for astronomical maps in galactic coordinates as well as for maps of the whole world. Inspired by the Aitoff projection, it was created by H. H. Ernst von Hammer as a modification of the Lambert Azimuthal Equal Area projection. The Hammer projection is equal area while the Aitoff projection is not. Both are similar to each other in that both are modified azimuthal projections where the central meridian is a straight line half the length of the equator, the only point free from distortion is the centre point, and there is a moderate distortion throughout. From investigations I conclude that it is a normal practice in Astronomy and Cosmology to call the Hammer projection as the Aitoff projection.

Powles and Quirke (1984) analyse the numerical trajectory of a molecule in liquid circles by using an empirical fractal parameter called the Richardson coefficient, α. In their simulation they used the Lennard-Jones (12,6) intermolecular potential together with a reduced density and pressure. For the length of molecular trajectories in an argonlike liquid they found $\alpha = 0.65$, in comparison with a similar analysis done on a randomised Koch curve of order 6 which gives $\alpha = 0.25$ and the exact $\alpha = 0.2618\ldots$ for K_∞, the Koch curve of an infinite order. The length of a fractal curve is $L(\epsilon) \propto \epsilon^{-\alpha}$, where ϵ is the step distance or scale. The graph between $\log(L(\epsilon)/\sigma)$ and $-\log(\epsilon/2)$ is a straight line with a positive slope represented by the equation $\log(L(\epsilon)/\sigma) = -\alpha\log(\epsilon/\sigma) + K$ where $K = \log k - (1+\alpha)\log 2$, k being a constant. The Leonard-Jones (n,6) potential is the Van der Waals potential between two atoms which is described in the form $V(r) = C_n/r^n - C^6/r^6$ where C_n and C_6 are constants, and r is the distance between the centres of the two atoms. Similarly, the Leonard-Jones (12,6) potential is $V(r) = 4\epsilon\left((\sigma/r)^{12} - (\sigma/r)^6\right)$. It is sometimes written as $V(r) = \epsilon\left((R/r)^{12} - 2(R/r)^6\right)$, where $R = R_i + R_j$, $\epsilon = \sqrt{\epsilon_i\epsilon_j}$, R_i and ϵ_i being the radius and respactively the interaction energy of each atom. In other words, in the last formula ϵ is the geometrical mean of the interaction energy of each atom while R is the atomic cross-section which is a property of a pair of atoms.

§ 1.6 Cosmological structure

The Lorentz transformation was first written down by Voigt in 1887 as $x' = x - vt$, $y' = y/g$, $z' = z/g$, and $t' = t - vx/c^2$, then by Larmor in 1898, and finally by Lorentz in 1899. Poincaré stated in 1898 on the measure of time that there is no absolute equality of two time intervals. He named the Lorentz transformation after Lorentz and showed that together with the rotations they form a group. Working on the transformations, Einstein announced the special relativity in 1905 as a theory merited by its simplicity and beauty rather than being an explanation of experimental results. For the special relativity and the Lorentz transformations Minkowski founded a four dimensional non-Euclidean space to represent space-time which Einstein adopted used as a basis for his general relativity which appeared in 1915.

Whether the universe is homogeneous or isotropic depends on the scale in which one consider. If the scale is large enough then they can be considered both so but not otherwise. Similarly, an infinite Voronoi network originated from Poisson generators may be considered as being both isotropic and homogeneous because all the irregularities averages out.

Cosmology is related to the study of membranes and filters. Molecules of liquids in a membrane experience forces due to induced dipoles and fixed dipoles. The induced dipoles results in the Lennard–Jones potential whereas the fixed dipoles give rise to a permanent dipole moment between fluid molecules or between molecules of very fine particles.

Inter molecular interactions are approximated by considering the charged parts of the molecule as point charges. The force between charged parts of each molecule and those of its neighbours can be estimated by the Coulomb potential, $V = q_1q_2/(4\pi\varepsilon_0 r)$, and the Coulomb force between each pair of molecules is $F = q_1q_2/(4\pi\varepsilon_0 r^2)$.

Analogous to the $1/r$, Coulomb potential in electrostatics is the $1/r$ potential in Swarzshield's expansion for the ten metric or gravitational potential of Einstein for the effect of an elementary concentration of mass in a space-time continuum that is asymptotically flat, that is

$$\mathrm{d}s^2 = \sum_{\alpha\beta} g_{\alpha\beta}\mathrm{d}x^\alpha \mathrm{d}x^\beta = \left(1 - \frac{2Gm}{(c^2r)}\right)\mathrm{d}r^2 + r^2[\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\psi^2] - \left(1 - \frac{2Gm}{(c^2r)}\right)\mathrm{d}T^2. \qquad (\mathbf{5})_{\mathrm{i}}$$

However, this is not as applicable as the Coulomb potential since the equations in the gravitation theory are nonlinear and therefore the superposition principle does not apply.

The Big Bang can be nothing but a change of phase of the universe. What the other phase may be we can not know, because there is a singularity which divides us from the Yonder Side. But one thing is indisputable, that is if we want to understand the universe, or in plainer words to make any sense out of it, we need to understand the singularity. And since we have to include the percolation theory in our final calculations, we might as well make it the beginning of our quest for a grand unified theory. Whether it is the percolation theory that we already know or another one not yet found is of no consequence. A percolation theory will still be a percolation theory no matter what form it may take, or indeed whatever name you may call it by. Our percolation theory is no percolation theory in a sense that it leaves out half of the picture, that is to say, the singularity that it still does not know how to explain. A true percolation theory is a theory which can includes singularities in its calculation while leaving out nothing that we know already. Give it some other name if you like, but *that* is the real percolation theory.

We now know many things in details, for example how stellar equations must account for mechanical, energy and thermal equilibria and that nuclear reactions imply conservation of charge, nuclear number and lepton number (*cf* Cooper *et al*, 1985). But at the Big Bang every one of these is supposed to break down. Because of this, our picture of the universe will always be incomplete until we can come to terms with that singularity which is our theoretical creator, the Big Bang.

§ 1.7 Filtration

Filtration is the operation of separating a heterogeneous mixture of a fluid and particles of solids by means of a filtering medium which lets the fluid pass through but not the particles. The name filtration comes from the art of wine making. There are two processes involved, namely the flow of the fluid through the cake and the medium, and the filtration where particles are deposited on or in the medium. The objective is to understand how the rate of flow depends on the properties of both the suspension and the medium, and on the operational conditions.

Both the cake and the medium are porous. As is the case with the cosmological structure (*cf* § 1.6), a filtering medium can be considered as being homogeneous on the large scale, whereas on the small scale it nearly never is (*cf* Heertjes, 1964). In other words, on the scale of the particles and the pores everything concerned is inhomogeneous to a high degree, that is to say, the slurry, the flow, and the cake. The micro-inhomogeneity in the cake can lead to a macro-inhomogeneity. Furthermore, the interaction among the particles, cake, medium and fluid makes the study of filtration ideal ground for numerical studies.

Heertjes (1964) describes the flow through a filter by the Fanning equation, $v = \mathrm{dV}/\mathrm{d}t = (1/\eta\gamma)\mathrm{d}(\Delta P)^\delta/\mathrm{d}R$, where η is viscosity and R the resistance. For viscous flow $\gamma = \delta = 1$, whereas for turbulent flow $\gamma = 0.11$ and $\delta = 0.55$. Both the cake (c) and the medium (m) have the say, so $R_c + R_m = R$, $\Delta P_c + \Delta P_m = P$, and $\Delta P_c/R_c = \Delta P_m/R_m$. The specific resistance of the cake is $r = \mathrm{d}R_c/\mathrm{d}w$, and therefore $R_c = \int_0^w r\mathrm{d}w = w(r)w$, where $w(\cdot)$ represents the mean value in contrast with w which is the weight of particles in the cake per unit surface of filter. From slurry (s) the filtrate passes through the slurry-cake (*sc*-) and the cake-medium (*cm*) interfaces. A volume $V_{sc} = (1 + c/\rho_s)$ of slurry is needed to produce V_{sc}. Here c is the concentration- and ρ_s the density of particles. Assuming the amount of particles contained in the filter to be negligible, then $w = cV_{sc}$. Then from $V_c = V_m$, $V_{sc} = V_m + V_R$, $V_R = w \cdot \bar{\epsilon}/[\rho_s(1 - \bar{\epsilon})]$ and $\epsilon_s = \rho_s/(\rho_s + c)$ we have

$$w = \frac{cV_m}{1 - \frac{(1-\epsilon_s)}{\epsilon_s}\frac{\bar{\epsilon}}{(1-\bar{\epsilon})}} \qquad (\mathbf{6})_{\mathrm{i}}$$

The filtration coefficient λ in $-\mathrm{d}c/\mathrm{d}l = \lambda c$ is not a constant but change with time because of particles adsorbed by the bed. If one assumes that particles are bound to the wall by London-van der Waals force only, then $\lambda = K(\epsilon_0 - \sigma)$ where σ is the specific deposit of solids in filter bed described as volume of solid per unit filter volume, ϵ_0 the initial porosity of the bed and K a function of London-van der Waals constant, d, ϵ, η and W. The equation of continuity is $W_0(\mathrm{d}c/\mathrm{d}l) = \mathrm{d}(\sigma)/\mathrm{d}\theta$. Cake is stabilised by the flow force and consolidates when it has reached a critical thickness and the velocity dropped below a critical value. The cake pressure is highest at the interface with the filter, so it is here that it starts to consolidate. Vibration is normally used to loosen it.

In the Diffusion Limited Aggregation model (*cf* Houi and Lenormand, 1986) particles a_i are placed on the lattice while particles b_i move towards them from a distance. These travelling particles stick to the first thing they meet, thus forming clusters, but they disappear whenever they stray too far away from the clusters. The density $\rho \sim nd^2/(hl) = na/(\tilde{x}l)$, $\tilde{x} = x/a$, approaches a fractal power law $\rho \sim \tilde{x}^{D-2}$ with the fractal dimension D such that the deposit is homogeneous when $D = 2$ and heterogeneous when $D < 2$; l is the length of the filter, x the thickness of the deposit and d the particle diameter. They consider two models, one to study the effects of random motion while the other that of ballistic trajectory. In the first model particles move in a square network, jumping from one site to one of its nearest neighbours with a probability q towards the filter and p in the other three directions. The diffusive- and the ballistic probabilities are respectively $4p$ and $1 - 4p$. The ratio between convective and diffusive displacement, a Péclet number, is defined as $P_e = (1 - 4p)/4p$. For the ballistic motion $P_e \to \infty$ while for the Brownian motion $P_e = 0$. In their second model particles move through space and $P_e = |u|/|r|$, where u is a constant displacement vector in the flow direction while r is in random direction. Particle A sticks to B when $\alpha < \beta$ or rolls on it if $\beta < \alpha < \gamma$. In this latter situation, A will stick to B if it is prevented from reaching the angle γ comparative to A, but if at last $\alpha > \gamma$ A and B will separate.

The hydro dynamic forces act to transport particles through the medium. When the solid parts come close together, there is a viscous resistance which increases with the inverse of the separation between them. The van der Waals force acts at a close range, is always attractive and is theoretically infinite when particles touch a solid. Brownian motion affects particles smaller than 1μm and results in a heterogeneous deposit, whereas the ballistic trajectory occurred in sedimentation or filtration of big particles yield a deposit that is homogeneous.

Prefilters used to protect fibre bed coalescers from the damage caused by suspended solid can become a bottle neck from being ladened with the solids itself. Chan (1990) was interested in such problem as the processing of hydrocarbon liquids on offshore platforms. In this process the four phase system, that of hydrocarbon gas, condensate, glycol and solids, is treated. The gas is separated and distillated to fractionate off butane and propane for uses in petrochemical manufacture. The other fluids contain hydrocarbon condensate and glycol which is added to prevent gas hydrates, a solid phase, from forming. But ethylene glycol has to be separated before it enters the purification system, for otherwise it would foul heat exchangers and the trays of distillation column. This can be conveniently done using a fibre bed coalescer. But solids suspended in one of the constituents, gas condensate glycol, will deposit and block these beds unless removed first by using a prefilter, which would then in turn become blocked and causes the bottle neck to the whole process.

Dead end filtration is simpler to simulate on the computer, but crossflow filtration is used more often in industry. Hydrophobic polymers in general have the advantage of good chemical and thermal stability, but hydrophilic polymers are becoming more and more attractive as membrane materials because they tend to have less adsorption. Adsorbed layer means more resistance to flow and a decline in flux. Moreover, these layers are difficult to remove by normal cleaning methods. Cellulose and its derivatives are among the best known hydrophilic polymers used as membrane materials. Examples of these are cellulose acetate, cellulose triacetate, cellulose tripropionate, cellulose nitrate, cellulose acetate-butyrate and ethyl cellulose. They are used in micro-, ultra- and hyperfiltration as well as in dyalysis and gas separation. Cellulose is hydrophilic but not soluble in water. It has a regular chain structure and is quite crystalline. Cellulose nitrate and cellulose acetate are used in micro- and ultrafiltration. Cellulose esters have excellent membrane properties except with regard to their sensitivity to biological-, chemical- and thermal degradations. They are made by air-casting or dry phase inversion. In phase inversion, a polymer is transformed from a liquid- to a solid state. Solidification is often started by liquid demixing, the transition of one liquid into two liquids. Phase inversion techniques include evaporation and precipitation the widely used for membrane of which is immersion precipitation (*cf* Schumacher, 1996). The top layer of an asymmetric membrane is dense and therefore responsible for most of the filtration. The sublayer is porous and provides the support to the top layer. Macrovoids are often found in the sublayer which lead to weak spots in the membrane and must be avoided, especially in high pressure applications which use dead end filtration. Tetrahydrofunate and acetone give membranes with a dense top layer as the result of delayed demixing. Dimethylsulfoxid and Dimethylformiade give membranes with a more porous structure from instantaneous demixing. Analogous to this is how pumice is very porous from the instantaneous solidification when it forms.

During the separation the flux through the membranes declines or decreases with time from adsorption, concentration polarisation, fouling, gel layer formation and pore pluggings. This is especially severe in micro- and ultrafiltration, with the decline in the flux often exceeding ninety per cent.

Darcy's law gives the volumetric flux of a pure liquid through a membrane, $J = \Delta P/(\eta R)$ where R is the overall resistance of the membrane which includes the resistances from adsorbed particles, cake, concentration polarisation, gel, pore blocking and membrane. Adsorption decreases the pore radius according to the Hagen-Poiseuille equation, $\Delta r/r = 1 - (J/J_m)^{1/4}$. At steady state the convective transport balances the permeate flow past the membrane and the diffusive back flow which results from the accumulation of solute at the membrane surface, $Jc + D\partial c/\partial x = Jc_p$ where D is the diffusion coefficient. Then from the boundary conditions $c = c_m$ at $x = 0$ and $c = c_b$ at $x = 1$, $(c_m - c_p)/(c_b - c_p) = \exp(J\delta/D)$. In other words, $c_m/c_b = \exp(J/k)/[R_n + (1 - R_n)\exp(J/k)]$ where the mass transfer coefficient $k = D/\delta$ and the intrinsic retention $R_n = 1 - c_p/c_m$. When the solute is completely retained by the membrane, $R_n = 1$ and $c_p = 0$, and therefore $c_m/c_b = \exp(J/k)$.

Across the filtering medium there is a driving force, in other words the pressure drop. There are four driving forces, centrifugal, gravity, pressure and vacuum. Filters in use in practice are either surface- or depth filters. In the former the solids are deposited on the surface in the form of a cake, thus the name *cake filtration*, while in the latter they are deposited inside the medium, thus *deep bed filtration*.

In filtration, Darcy's law is often written $Q = A\Delta p/(\mu R)$, where $R = L/K$ is the medium resistance, L the thickness- and K the permeability of the bed. If there is a cake, $R = R + R_c$ where R_c, the cake resistance, is $R_c = \alpha w$, α being the specific cake resistance in mkg^{-1} and w is the mass of cake deposited per unit area. Cakes are normally compressible, so α changes with Δp_c and is approximated as α_a, where $1/\alpha_a = (1/\Delta p_c)\int_0^{\Delta p_c} \mathrm{d}(\Delta p_c)/\alpha$. There is an experimental empirical relation $\alpha = \alpha_0/(\Delta p_c)^n$, where n is the compressibility index, and $\alpha_a = (1-n)\alpha_0(\Delta p_c)^n$.

The mass of cake deposited is $wA = cV$, where c is the concentration of solids in the suspension. For incompressible cake, $Q = \Delta pA/\left[\alpha\mu c(V/A) + \mu R\right]$ or equivalently $\mathrm{d}t/\mathrm{d}V = \alpha\mu cV/(A^2\Delta p) + \mu R/(A\Delta p)$. If Δp is constant, $t = a_1V^2/(2A^2\Delta p) + b_1V/(A\Delta p)$, where $a_1 = \alpha\mu c$ and $b_1 = \mu R$. The experimental determination of α and R works with this equation in the form $t/V = aV + b$, where $a = a_1/(2A^2\Delta p)$ and $b = b_1/(A\Delta p)$; or rather in a more detailed form $(t - t_s)/(V - Vs) = \alpha\mu c(V + V_s)/(2A^2\Delta p) + \mu R/(A\Delta p)$, where t_s is the starting time at the beginning of the truly constant pressure period.

In constant rate filtration, Q is kept constant,

$$Q = \frac{\Delta p(t)A}{\frac{\alpha\mu cV(t)}{A+\mu R}}.$$

In other words, $\Delta p = \alpha\mu cQ^2t/A^2 + \mu RQ/A$, or $\Delta p = a_1v^2t + b_1v$, where $v = Q/A$ is the approach velocity of the filtrate.

In many cases, operation demands constant rate followed by constant pressure, in which case $\Delta p = a_1v^2t + b_1v$ for $t < t_s$ and $\Delta p = \Delta p_s$ a constant for $t \geq t_s$. This amounts to $V = Q_1t$ for $V \leq V_s$ and $(t - t_s)/(V - V_s) = a(V + V_s) + b$ for $V > V_s$.

If a centrifugal pump is used, it is the case of variable pressure and variable rate. Here the equation is $V = A(\Delta pA/Q - \mu R)/(\alpha\mu c)$, and the filtration time necessary is $t = \int_0^V \mathrm{d}V/Q$.

For compressible cakes, $\Delta p = \Delta p_c + \Delta p_m$, where $\Delta p_m = \mu RQ/A$ and $\Delta p_c = \alpha_a\mu cVQ/A^2$.

And $\Delta p_c = (1-n)\alpha_0 \Delta p_c^n \mu c V Q / A^2$, that is to say,

$$\frac{\mu c V Q}{A^2} = \frac{(\Delta p_c)^{1-n}}{(1-n)\alpha_0}.$$

Then, for the constant rate filtration, $(\Delta p_c)^{1-n} = \alpha_0 (1-n) \mu c Q^2 t / A^2$. And for the variable pressure and variable rate operation,

$$V = \frac{A^2}{((1-n)\alpha_0 \mu c)} \cdot \frac{(\Delta p - \Delta p_m)^{1-n}}{Q}, \qquad \mathbf{(7)}_{\mathrm{i}}$$

where Δp_c, Δp_m, V, Q and t are all variable.

The relationship between the specific cake resistance, porosity and specific surface is the Kozeny-Carman equation, $\alpha = K_0 S_0^2 (1-\varepsilon)/(\rho_s \varepsilon^3)$, where K_0 is the Kozeny constant which is approximately 5 for the lower porosity ranges, S_0 is the specific surface of the particles making up the bed, that is the ratio between the surface area and the volume of solids, ρ_s the solid density and ε the porosity, that is the ratio between the volume of voids and the volume of cake.

Fluid within porous media is essentially stagnant and the flow is laminar. Perhaps second only to the Hubble's constant in Cosmology in the matter of elusiveness is the k-factor which accounts for the tortuosity in porous filters, which, according to Kozeny and Carman, is $k = k_0 (L_e/L)^2$, where k_0 is the shape factor, L_e/L the tortuosity factor, L_e being the interstitial length followed by the streamline and L is the thickness of the bed. There are many other formulae (*cf* Piekaar and Clarenburg, 1967), for instance $k = k_0/\rho$ suggested by Sullivan where $\rho = \overline{(\sin^2 \phi)}$ is the orientation factor.

Transport mechanism in filters is due to diffusion, gravity and hydrodynamic force. The efficiency is minimum at about the particle size of 1 μm. Filter normally runs about 24 hours between washes, with a rate between 5 and 15 $\mathrm{m \cdot h^{-1}}$. Wash rates are approximately 0.5 $\mathrm{m \cdot min^{-1}}$. (*cf* Ives, 1977).

§ 1.8 Statistics

Poisson distribution, defined by $p(x,\lambda) = \left[\lambda^x e^{-\lambda}/x!\right] \mathrm{I}_{[0,\infty)}$, is the binomial distribution, $p(x,n,p) = {}^n\mathrm{C}_x \theta^x (1-\theta)^{n-x} \mathrm{I}_{[0,n]}$, when n goes to infinity, θ goes to zero, while $n\theta = \lambda$. Here θ is the probability of success of each trial. It is used when counting the number of occurences of a random event. Analogously Poisson point process, which has $p(x = n(v)) = \left[\lambda |v| e^{-\lambda |v| x}/x!\right] \mathrm{I}_{[0,\infty)}$, is the binomial point process, $p(x = n(v)) = {}^n\mathrm{C}_x \theta^x (1-\theta)^{n-x} \mathrm{I}_{[0,n]}$, when the volume V goes to infinity, while $n/|V| = \lambda$. Here $\theta = |v|/|V|$ is the probability of points within V being placed in $v \subset V \subset R^d$, and λ the density or intensity of points. Therefore the density of point of a Poisson point process is constant by definition. A point process is a procedure which generates points on a domain within a space of d dimensions.

The Poisson point process thus derived has the properties that $0 < p_{n(v)=0} < 1$ for $0 < |v| < \infty$, $\lim_{|v| \to 0} p(n(v) \geq 1) = 0$, $n(v_i)$ mutually independent and $n(\bigcup_n v_i) = \sum_n n(v_i)$ when A_i are disjoint, and $\lim_{|v| \to 0} [p(n(v) \geq 1)/p(n(v) = 1)] = 1$.

The weighted mean of a group of data is $\mathbf{x} = \sum_i f_i x_i / n$ and the weighted variance is $\sigma^2 = \sum_i f_i (x_i - \mathbf{x})^2/n$, where f_i is the occurrence frequency of x_i and $\sum_i f_i = n$. Likewise the r^{th}-moment around the average is $m_r = \sum_i f_i (x - \mathbf{x})^r / n$, while the r^{th}-moment around the origin is $m'_r = \sum_i f_i x^r / n$ (*cf* Spiegel, 1975). Some relations among these various moments are $m_1 = 0$, $m_2 = m'_2 - {m'_1}^2$, $m_3 = m'_3 - 3m'_1 m'_2 + 2{m'_1}^2$, and $m_4 = m'_4 - 4m'_1 m'_3 + 6{m'_1}^2 m'_2 - 3{m'_1}^4$.

The variance when normalised by $n-1$ gives the best unbiased estimated variance if the sample has a normal distribution. On the other hand the variance which is normalised by n is identical with the second moment of the sample about its mean.

The log-normal distribution is closely related to the normal distribution. If $\ln x$ has a normal distribution with μ and σ^2, then x has a log-normal distribution with μ and σ^2. In other words, a log-normal distribution curve will appear as a normal curve when plotted with a log scale in the x axis. Its probability distribution function is

$$f(x|\mu,\sigma) = \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{(2\sigma^2)}}. \qquad \mathbf{(8)}_{\mathrm{i}}$$

It is positive definite, and therefore attractive in some areas of application where this is required, for example the amount of rain fall or particle size distribution. Both the log-normal and the normal distributions require only two parameters to describe, that is its mean and variance.

The most common drop size distribution in agitated heterogeneous liquid-liquid systems are the normal and the log-normal distributions (Giles *et al*, 1971). A straight line is obtained when the diameter of drops, d,with normal size distribution is plotted against cumulative percentage frequency, or in the case where they have log-normal size distribution, when $\log d$ is plotted against cumulative percentage frequency.

Monte Carlo methods (*cf* Hammersley and Morton, 1954) use random numbers. They have found applications in a wide variety of fields, ranging from numerical analysis to recreation. Random numbers sometime come from nature, for instance the generator which creates random numbers from a resistance noise suggested by A. M. Turing, in which case they are truly random in nature. In 1927 L. H. C. Tippett compiled a table of random numbers, and in 1955 RAND Corporation (*cf* Knuth, 1998). For the premium bonds lottery ERNIE is used in the UK since 2^{nd} June 1957 to generate a 9-digit sequence of random numbers by exploiting the random frequency instability in a free-running oscillator. ERNIE was upgraded in 1973 and the present version, introduced in 1988, is Mark 3.

Normal distribution is a family of curves which have two parameters, namely the mean μ and the standard deviation σ. The standard normal distribution, $\Phi(x)$, has $\mu = 0$ and $\sigma = 1$. It is related to the error function by the relation $\mathrm{erf}(x) = 2\Phi(x\sqrt{2}) - 1$. The central limit theorem states that, as the sample size increases to infinity, the sum of independent samples from some distribution of finite mean and variance converges to the normal distribution.

§ 1.9 Poisson process

The Poisson process is the probability model with one parameter, which represents all processes in which points occur randomly in time. The gamma experiment is to run the process to find the time t_k of the k^{th} arrival, whereas the Poisson experiment is to run it to find the number of arrivals n_t in the interval $(0, t]$, $t \geq 0$. These two experiments give rise to two sets of random variables dual to one another, and $n_t \geq k$ if and only if $t_k \leq t$. The regeneration property says that the process after any time t is independent of the process before t and is probabilistically the same as the original process. The interarrival time is $x_1 = t_1$, $x_k = t_k - t_{k-1}$ for $k = 2, 3, \cdots$.

In a Poisson process the number of changes in each of the non-overlapping intervals is independent from that in the others. Let ν be the number of one change, $h = 1/n$ a sufficiently small interval and n the number of trials. Then the probability of exactly one change in h is $p = \nu h = \nu/n$. The probability of two or more changes in h is zero. The number of k changes occurring in n trials is the continuous limit of the discrete binomial distribution

$$p(k) = \frac{n!}{k!(n-k)!}\left(\frac{\nu}{n}\right)^k\left(1 - \frac{\nu}{n}\right)^{n-k} \qquad (\mathbf{9})_{\mathrm{i}}$$

§ 1.10 Phase transition

In the Ising model each spin has two possible states, that is up and down, and the hamiltonian is $H = J_0 \sum_{<i,j>} \sigma_i \sigma_j$ where the summation is over the nearest neighbours. Since it has been exactly solved, the Ising model provides a good model for the understanding of phase transition. This model can represent the transition from ferro- to paramagnetic at the critical temperature where the correlation length becomes infinite. Characteristic to the Ising model is the peak in the specific heat at the critical temperature.

The two-dimensional xy model is a model of spins confined to a plane, the hamiltonian of which is $H = J_0 \sum_{<i,j>} \cos(\theta_i - \theta_j)$. This model can represent the superconducting and the superfluid films. For this model there is no phase transition showing long-range ordering. One example is the two-dimensional Coulomb gas model where the vortex-antivortex pairs, which are bound to each other at low temperature, increases in number as the temperature increases and become separated at the KT temperature that marks the phase transition.

It had been generally believed that no phase transition can exist for the xy model when Kosterlitz *et al* (1973) showed that there is another kind of phase transition, arisen from the topological excitation of vortex-antivortex pairs instead of from the long-range ordering found in a spontaneous magnetisation. They consider the two-dimensional model of gas with charges $\pm q$ where the interaction potential is

$$U(|\mathbf{r}_i - \mathbf{r}_j|) = -2q_i q_j \ln\left|\frac{(\mathbf{r}_i - \mathbf{r}_j)}{\mathbf{r}_0}\right| + 2\mu \qquad (\mathbf{10})_{\mathrm{i}}$$

when $r > r_0$, and 0 when $r < r_0$. The problem is reduced to that of solving an equation of the form $(\mathrm{d}y/\mathrm{d}x) = -e^{-xy}$. The application mentioned there is in the xy model of magnetism, the solid-liquid transition, and the neutral superfluid, but not in a superconductor and a Heisenberg ferromagnet.

The frustrated xy model, the hamiltonian of which is $H = J_0 \sum_{<i,j>} \cos(\theta_i - \theta_j - A_{ij})$, occurs when a magnetic field is applied perpendicular to the two-dimensional plane of the xy model. The frustration parameter, $f = \Phi/Phi_0$, is a measure of the average external magnetic flux. When $f = 1/2$ the model is called the fully frustrated xy model. The local chirallity, $m(r_i) = \frac{1}{2\pi}\sum(\theta_i - \theta_j - A_{ij})$, which describes the property of the ground state, where it can either be $+\frac{1}{2}$ or $-\frac{1}{2}$. The network configuration at $\mathrm{T} < \mathrm{T}_c$ is that of a draught board, and has Z_2 symmetry. This regularity is broken by the formation of domain walls in an Ising phase transition at T_c.

Renormalisation group method has shown that there exist larger structures that behave like a smaller one. This means that the same structure can recur infinitely many times in infinitely many different scales, and that is the same idea that makes fractal geometry. This is why the study of cluster structure and the use of fractal dimension to characterise clusters becomes important (*cf* Stauffer and Aharony, 1985).

Percolation in Physics means Statistical Mechanics. Much of the contemporary vocabularies in the former has come from the lexicons of the latter from Ising Models to Renormalisation group and then to finite size scaling (*cf* Pathria, 1996).

§ 1.11 Random processes

A synonym to *random* is *stochastic* (*cf* Miles, 1972). Any algorithm which employs a random element is called Monte Carlo. Random processes can have various types of distribution. The beta distribution has a probability density function $f_{X_{\alpha,\beta}}(x) = x^{\alpha-1}(1-x)^{\beta-1}/B(\alpha,\beta)$, $0 \leq x \leq 1$, where $\alpha > 0$ and $\beta > 0$ are shape parameters, and $B(\alpha,\beta)$ is the beta function. There are three types of shape; the bridge shape has $\alpha > 1$ and $\beta > 1$, the J-shape $\alpha \leq 1$ and $\beta \geq 1$, or $\alpha \geq 1$ and $\beta \leq 1$, and the U-shape $\alpha < 1$ and $\beta < 1$.

The 200 generators used in Figure 1.9 are randomly chosen with beta distribution with the shape parameters $\alpha = 2.7$ and $\beta = 3$, that is bridge shape. Both x and y in Figure 1.10 have J-shaped distribution with the shape parameters $\alpha = 2$ and $\beta = 0.8$. The density is unbounded at $x = 1$ and at $y = 1$ because $\beta < 0$ for both. The shape parameters in Figure 1.11 are $\alpha = 0.5$ and $\beta = 0.3$, that is U shape. The density is unbounded at $x = 0$, 1 and at $y = 0$, 1 because α is also less than zero.

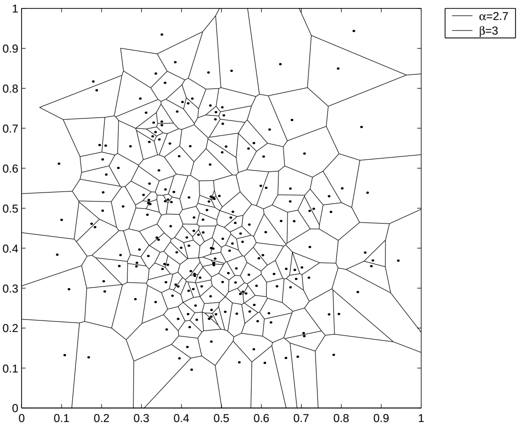


Figure 1.9 Voronoi graph with bridge-shaped beta distribution..

The random variable of the F-distribution is $F_{n_1,n_2} = n_2X/(n_1(1-X))$, where X is a beta variate with $\alpha = n_1/2$ and $\beta = n_2/2$.

The probability density function of some of the distributions of random numbers found in practice are shown in Table 1.9. For the negative binomial distribution, p is a probability of success and is constant and r the number of successes required before stopping. With rq/p as the mean, $q = 1 - p$, and the variance rq/p^2,it is used to model consecutive trials. For the noncentral$f$ distribution, S_{n_1,μ_1} and S_{n_2,μ_2} are chi-squared random variables which are independent and noncentral, $S_{n,\mu} = \chi^2_{n-1} + (Z_n - \mu^{1/2})^2$.

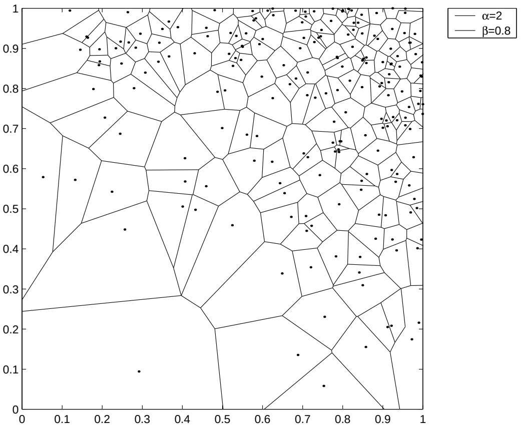


Figure 1.10 Voronoi graph with J-shaped beta distribution..

It has the mean $\nu_2(\delta+\nu_1)/\nu(\nu_2-2)$, where $\nu > 2$, and the variance

$$2\left(\frac{\nu_2}{\nu_1}\right)^2\left[\frac{(\delta+\nu_1)^2+(2\delta+\nu_1)(\nu_2-2)}{(\nu_2-2)^2(\nu_2-4)}\right],$$

where $\nu_2 > 4$. This means that it is the general case of the f-distribution, which is the case where $\delta = 0$. The parameters ν_1 and ν_2 are degrees of freedom.

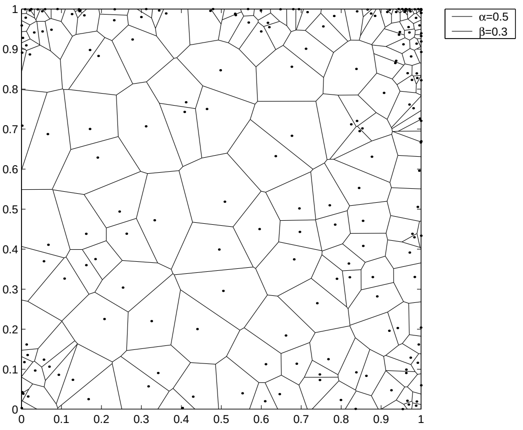


Figure 1.11 Voronoi graph with U-shaped Beta distribution..

The noncentral t distribution has $F(x|a,b)$ as the incomplete beta function with parameters a and b, and ν the degrees of freedom. It is the generalisation of the student's t-distribution. It's mean is

$$\delta\left(\frac{\nu}{2}\right)^{\frac{1}{2}}\frac{\Gamma\left(\frac{(\nu-1)}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)}, \qquad \mathbf{(11)}_{\mathrm{i}}$$

where $\nu > 1$, and it's variance $\nu/(\nu-2)(1+\delta^2)-\nu/2\delta^2\left[\Gamma((\nu-1)/2)/\Gamma(\nu/2)\right]^2$. The noncentrality parameter for all noncentral distributions, *viz.* the noncentral f-, t- and chi-square distributions is represented by δ.

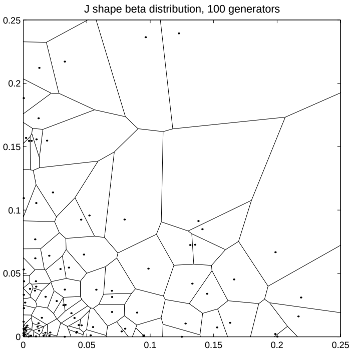
The mean of the normal distribution is μ and the variance σ^2. The standard normal distribution has $\mu = 0$ and $\sigma = 1$. The probability mass function of the Poisson distribution is sometimes written $y = f(x|\lambda) = \lambda^x/x!e^{-\lambda}\mathrm{I}_{0,1,\ldots}(x)$. It has a value when x is a nonnegative integer. Otherwise the density function is zero.

The Rayleigh distribution has a parameter b. Its mean is $b\sqrt{x/2}$ and its variance $(4-x)b^2/2$. The uniform distribution has the mean $(a+b)/2$ and the variance $(b-a)^2/12$. For the standard uniform distribution $a = 0$ and $b = 1$. The discrete uniform has the mean $(N+1)/2$ and the variance $(N^2-1)/12$.

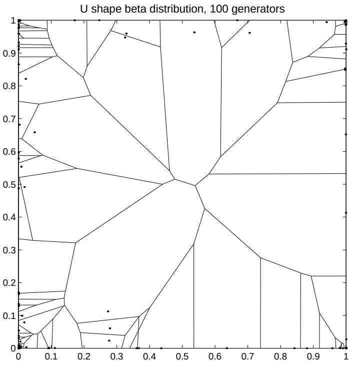
The Weibull distribution is sometimes written $y = f(x|a,b) = abx^{b-1}e^{-ax^b}\mathrm{I}_{0,\infty}(x)$. The Weibull distribution with a single parameter has $a = 1$. The three-parameter Weibull distribution has a p.d.f. $f_X(x) = c(x-a)^{c-1}e^{-((x-a)/b)^c}b^{-c}$ when $x \geq a$, otherwise $f_X(x) = 0$.. The mean is $a^{-(1/b)\Gamma(1+b^{-1})}$ and the variance is $a^{-2/b}\left[\Gamma(1+2b^{-1})-\Gamma^2(1+b^{-1})\right]$.

Beta	$f_{X_{\alpha,\beta}}(x) = x^{\alpha-1}(1-x)^{\beta-1}/B(\alpha,\beta)$, $0 \leq x \leq 1$
Binomial	$f_{X_{n,p}}(x) = {}^n\mathrm{C}_x p^x q^{n-x}$, $x = 0,1,\ldots,n$, $0 \leq p \leq 1$ and $q = 1-p$
Chi-square	$f_{\chi^2_n}(x) = e^{-x/2}x^{(n/2)-1}/2^{n/2}\Gamma(n/2)$, $0 \leq x \leq 1$
Exponential	$y = f_{x\|\mu} = e^{-x/\mu}/\mu$
F	$f_{F_{n_1,n_2}}(x) = n_1^{n_1/2}n_2^{n_2/2}x^{(n_1/2)-1}/B(n_1/2,n_2/2)(n_2+n_1x)^{(n_1+n_2)/2}$
Lognormal	$y = f_{x\|\mu,\sigma} = e^{-(\ln(x)-\mu)^2/2\sigma^2}/x\sigma\sqrt{2x}$
Negative Binomial	$y = f_{x\|r,p} = {}^{r+x-1}\mathrm{C}_x p^r q^x \mathrm{I}_{0,1,\ldots}(x)$,
Noncentral f	$F_{n_1,n_2} = \sqrt{S_{n_1,\mu_1}/n_1}/\sqrt{S_{n_2,\mu_2}/n_2}$
Noncentral t	$\mathrm{P}(-t < x < t\|(\nu,\delta)) = \sum_{j=0}^{\infty}\left[(\delta^2/2)^j e^{\delta^2/2}/j!\right]\mathrm{I}\left(x^2/(\nu+x^2)\|1/2+j,\nu/2\right)$
Noncentral Chi-square	$F(x\|\nu,\delta) = \sum_{j=0}^{\infty}\left[(\delta/2)^j e^{\delta/2}/j!\right]\mathrm{P}(\chi^2_{\nu+2j} \leq x)$
Normal	$y = f(x\|\mu,\sigma) = e^{-(x-\mu)^2/2\sigma^2}/(\sigma\sqrt{2\pi})$, $\mu > 0$
Poisson	$f_X(x) = \mu^x e^{-\mu}/x!$, $x \geq 0$
Rayleigh	$y = f(x\|b) = (x/b^2)e^{(-x^2/2b^2)}$
T	$y = f(x\|\nu) = \Gamma((\nu+1)/2)/\Gamma(\nu/2)\cdot 1/\sqrt{\nu x}\cdot 1/(1+x^2/\nu)^{\nu+1/2}$
Uniform	$y = f(x\|a,b) = (1/(b-a))\mathrm{I}_{[a,b]}(x)$, $b > a$
Discrete Uniform	$y = f(x\|N) = (1/N)\mathrm{I}(1,\ldots,N)(x)$
Weibull	$f_X(x) = ax^{a-1}e^{-(x/b)^a}/b^a$, $x \geq 0$ and $a,b > 0$

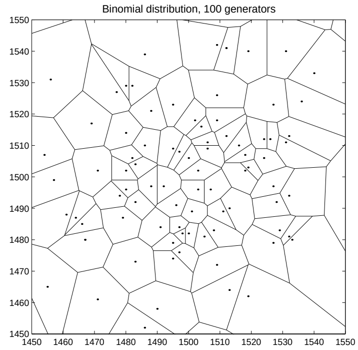
Table 1.9 *Probability density functions.*



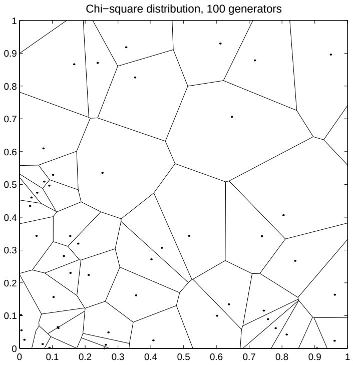
(a)



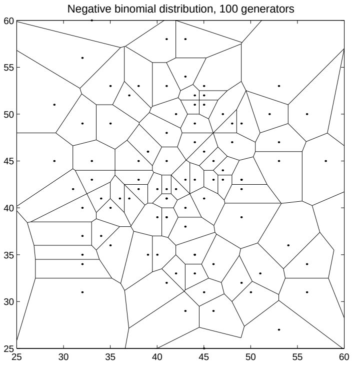
(b)



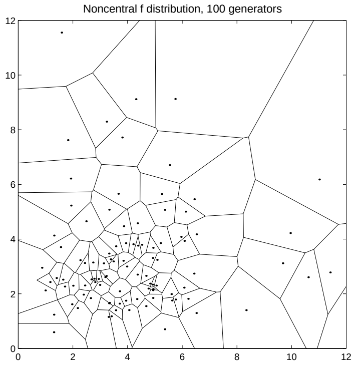
(c)



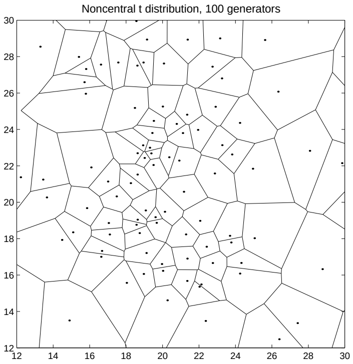
(d)



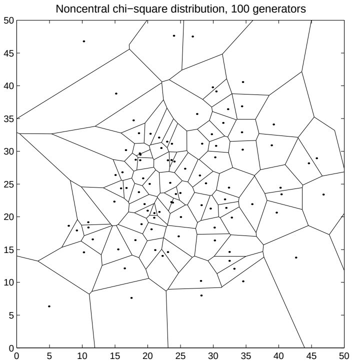
(e)



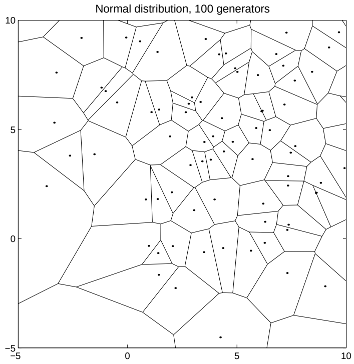
(f)



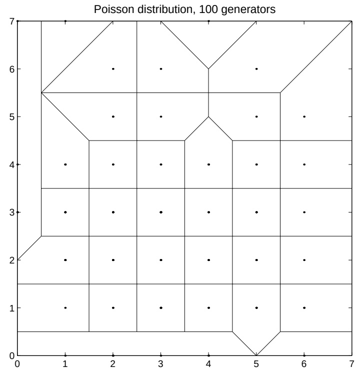
(g)



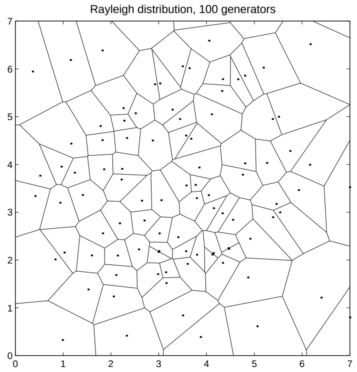
(h)



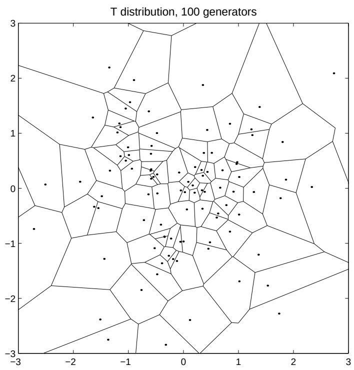
(i)



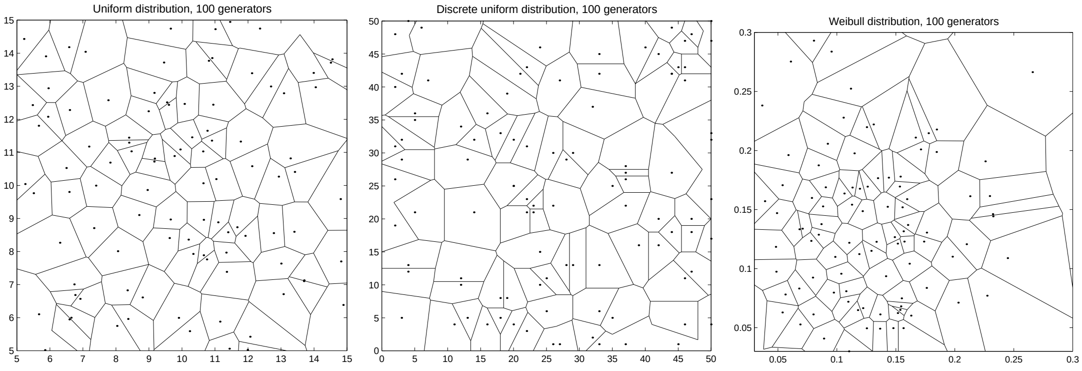
(j)



(k)



(l)


Uniform distribution, 100 generators
Discrete uniform distribution, 100 generators
Weibull distribution, 100 generators

(m) (n) (o)

Figure 1.12 *Voronoi graphs of various random p.d.f.'s, whose distributions are* (a) *J-shaped Beta,* $\alpha = 0.5$ *and* $\beta = 10$*, in both x and y,* (b) *U-shaped Beta,* $\alpha = 0.1$ *and* $\beta = 0.2$, (c) *Binomial,* $n = 3{,}000$ *and* $p = 0.5$, (d) *Chi-squared,* $n = 1$, (e) *negative binomial,* $p = 0.7$ *and* $r = 100$, (f) *noncentral f,* $\nu_1 = 7$ *and* $\nu_1 = 12$, $\delta = 20$, (g) *noncentral t,* $\nu = 10$ *and* $\delta = 20$, (h) *noncentral chi-square,* $\nu = 7$ *and* $\delta = 20$, (i) *normal,* $\mu = 5$ *and* $\sigma = 4$, (j) *Poisson, 100 generators* (x, y), $\lambda = 3$, (k) *Rayleigh,* $b = 3$, (l) $t, \nu = 7$, (m) *uniform,* $a = 15$ *and* $b = 5$, (n) *discrete uniform, from 100 points,* $N = 50$ *and* (o) *Weibull,* $a = 50$ *and* $b = 2$.

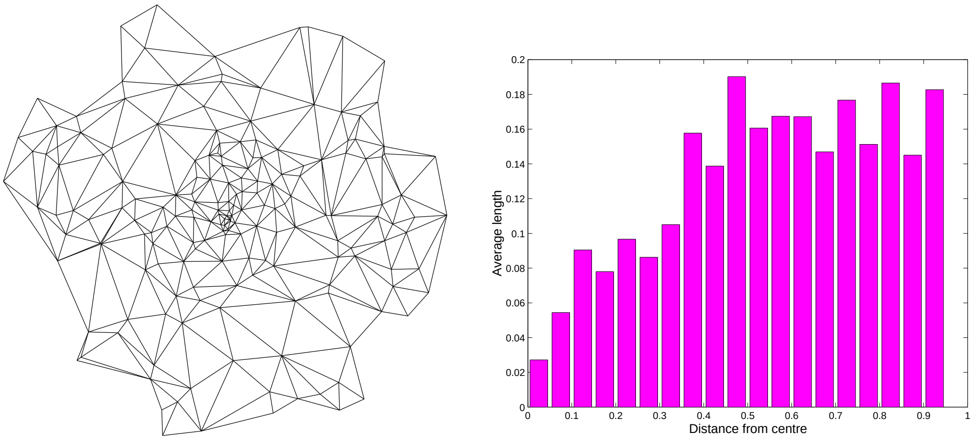
There are five methods for generation of Poisson processes, two of which are the time-scale transformation and the thinning algorithm. The first one changes a heterogeneous process $T_{(i)}$ in $[0, t_0]$ into a homogeneous process $\tau_{(i)}$ by a new time scale $\tau = \in t_0^T \lambda(u)du$, where λ is the rate function. We have $\tau_{(i)} = \Delta(T_{(i)})$ and $T_{(i)} = \Delta^{-1}(\tau_{(i)})$, for $\tau_{(i)} \leq \Delta(t_0)$. If $\lambda(t) = e^{\alpha+\beta t}$, then $\Delta(T) = e^{\alpha}(e^{\beta T} - 1)/\beta$ and $T_{(i)} = (1/\beta)\ln(\beta\tau/e^{\alpha} + 1)$. The second one follows Algorithm 1.3 with the input $\lambda^*, \lambda(\cdot), t_0$.

Algorithm 1.3 *Thinning algorithm.*

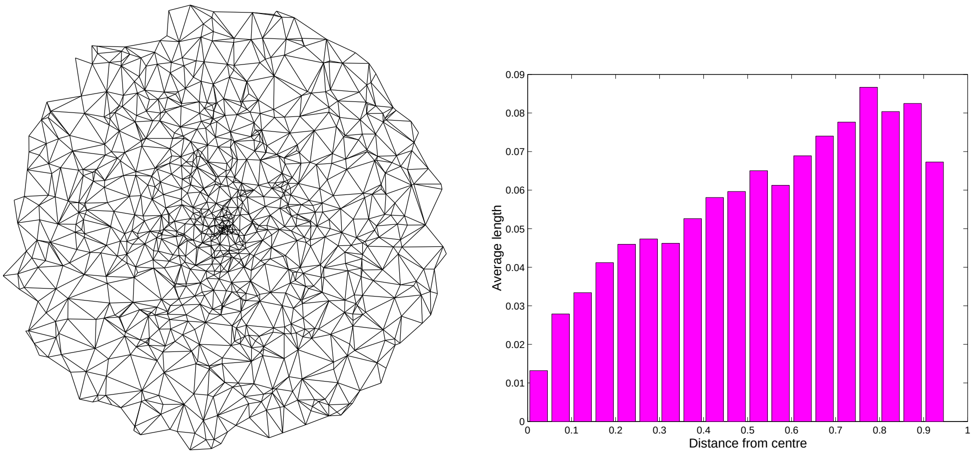
```
$T \leftarrow 0$;
while $T \geq t_0$ do
   generate $U$, $V \sim U(0,1)$;
   while $V > \lambda(T)/\lambda^*$ do
      $T \leftarrow T - \ln U/\lambda^*$;
   endwhile
endwhile
```

□

Figure 1.13 shows that the result from the point process in two dimensions involving two random variables, *i.e.* $r(\theta) = r\angle\theta$, is not homogeneous even though it may be on average isotropic with respect to the centre for a very large network. In Figure 1.14 where the domain is also circular but each point is simply a 2-d Poisson point process, the average distance is constant for a large system. Figure 1.15 is also 2-d Poisson point process, but the space here is square. In all of the figures, *i.e.* Figure 1.13–1.15, (a) and (b) start from 200-, whereas (c) and (d) from 1,000 point generators. Boundary effects are reduced by excluding those vertices and edges along the border.

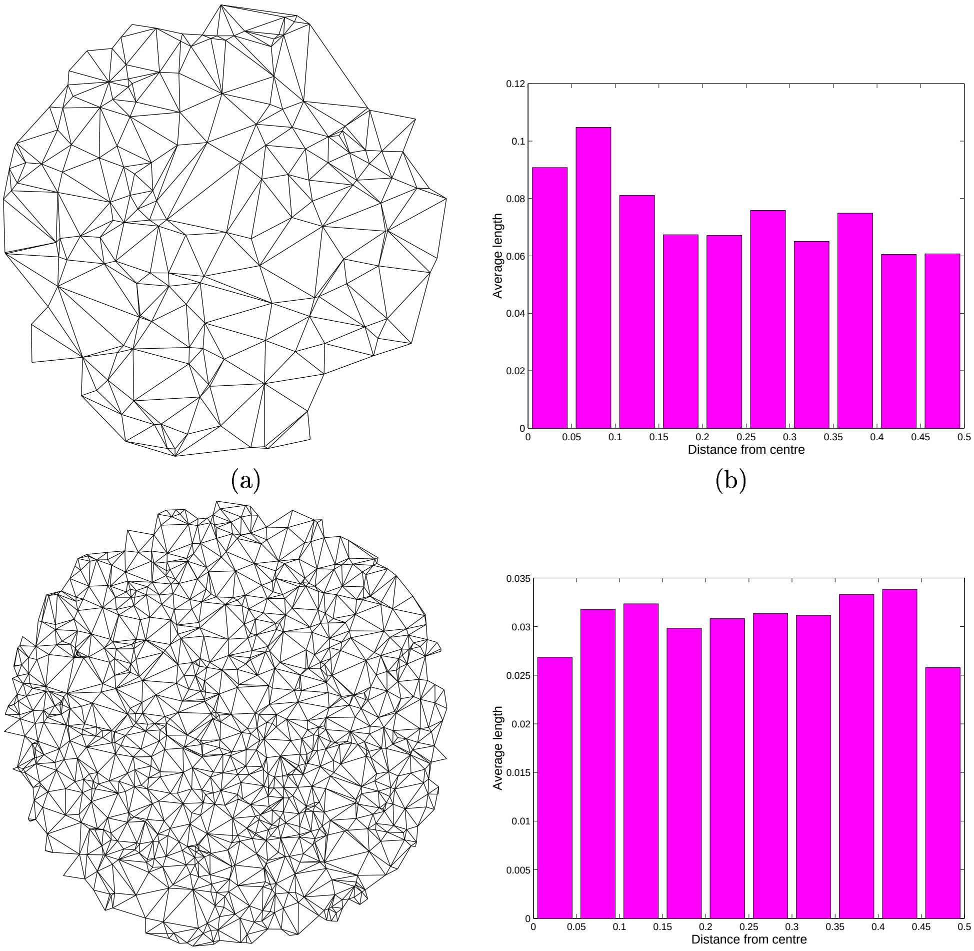


(a) (b)



(c) (d)

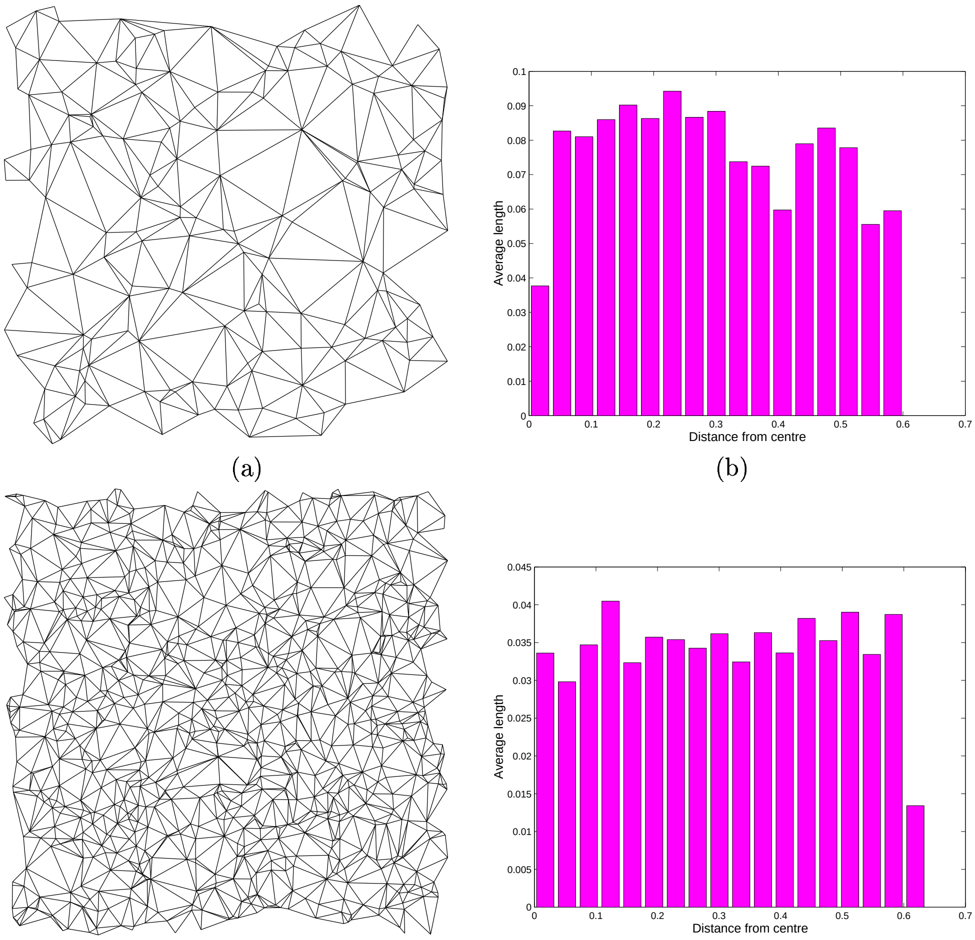
Figure 1.13 *Point process $r\angle\theta$, where both r and θ are independent random generators; (a) the Delaunay triangulation, 182 vertices shown, (b) average distance between neighbours vs distance from centre of the network, (c) another similar network, 947 vertices shown in total, (d) average edge length of the Delaunay triangulation.*



(a) (b)

(c) (d)

Figure 1.13 *Point process x, y, where x and y are independent random generators in a circular domain; (a) the Delaunay triangulation with 184 vertices, (b) average edge length vs distance from centre of the network, (c) another similar network, 909 vertices shown in total, (d) average edge length of the Delaunay triangulation.*



(a) (b)

(c) (d)

Figure 1.13 *Point process x, y, where x and y are independent random generators within a square domain; (a) the Delaunay triangulation with 168 vertices, (b) average edge length vs distance from centre of the network, (c) another similar network, 816 vertices shown in total, (d) average edge length of the Delaunay triangulation vs the distance from centre.*

The codes used in carrying out these investigations are listed in § A.25. Figure 1.13 effectively tells us that the random point process $r\angle\theta$ gives particle distribution shaped like a bell with density decreasing away from the centre. One might almost say that this is similar to what a spiral galaxy looks like. There is still much room to explore what particle distribution the different topologies of random process give rise to, for example, 'What would the distribution in 2-d be of points that are generated from random points on a line swept in circle at a constant speed?' The algorithm is as simple as Algorithm 1.4, but the room for imagination and the scope of exploration are unlimited. There are only two variables, that is the nuclei positions x and the triangulation edges e.

Algorithm 1.4 *Triangulation edge length distribution.*

```
$x \leftarrow$ random process;
$e \leftarrow$ **find** triangulation of $x$;
$(x, e) \leftarrow (x - \partial x, e - \partial e)$;
**compare** $|e|$;
```

□

§ 1.12 Structures in nature

Prusinkiewicz and Lindenmayer (1990) model the structures in plants after having briefly discussed about the difference between the Chomsky grammars and the L-system, both of which being a mean for doing string rewriting, but the latter is based on the Turtle geometry which makes it convenient for the geometric rewriting of fractals. The states of a turtle consists of its position coordinates and the direction in which it is facing. Meinhardt (1995) models the patterns on sea shells by using mathematical based on the partial differential equations governing the system of activator, inhibitor, and substrate. Starting from a homogeneous initial condition, small deviations therein undergo a positive feedback and therefore increase. Activator catalyses both the production of itself and that of its inhibitor. The latter acts as a negative feedback which limits and makes the reaction local.

Random tissue in three dimensions has four edges, six faces and four cells meeting at each vertex. It is thus surrounded by four cell nuclei, as well as by six bonds forming a tetrahedral cage. The four edges meeting at a vertex resemble a caltrop, and similarly the lines from it to the four nuclei. If a vertex had more than four edges it would have been structurally unstable, because then it can be split into two normal vertices by an infinitesimal deformation. Continuous random networks, for example models of covalent glasses like vitreous silica, are excluded from this restricted class since theirs may be more than three non-planar faces to an edge even if they still have four edges to a vertex (Revier, 1982).

Unlike crystallography, the ideal random structure is by no means unique because it is the solution of a statistical problem. There are, however, certain geometrical and topological invariances, the most famous of which is possibly the Euler's theorem. In two dimensions this theorem states that $f - e + v = \chi$, where χ is an Euler-Poincaré characteristic and integer of order one, χ being for instance 1 and 2 respectively for plane and sphere; in three dimensions it is $f - e + v = 2$. The valence relations, $\sum n f_n = 2e = 3v$, hold for 2-d and 3-d alike.

Stumbling upon some observations, Theorem's 3.3 and 3.4 resulted. These two theorems help explain together with Algorithm 3.2 on page 86, the valence relations. Theorem 1.2 is also another product obtainable from applying both the Euler's theorem and the valence relations (*cf* Prause, 2000).

Theorem 1.2. *The average number of edges per polygon in a large pattern is six.:*

Proof. From Euler's theorem, $f - e + v = 1$, and the valence relations, $\sum_n f_n = 2e = 3v$, it follows by applying the latter to the former that $f - \sum nf/2 + \sum nf/3 = 1$. Since $\bar{n}$ is the average number of edges per face, it follows that $f - \bar{n}f/2 + \bar{n}f/3 = 1$. Then $f(1 - n/6) = 1$, and consequently $1 - 1/f = n/6$. As the network becomes large, f becomes infinite and as a result $\bar{n} = 6$. □

It is interesting to note that Theorem 1.2 posts no restrictions on whether the network is random or regular. The only assumption made is that three and only three edges meet at each vertex. One is almost tempted to say that, as the size of a network becomes infinite, nature somehow follows this theorem and make sure that each of the polygons has six edges on average.

When I translated the three papers by Voronoi (Tiyapan, 2001) I used the term *vertex* to mean a vertex of a specific polygon or polyhedron. For any vertex, I used the term *vertice*, and for more than one vertex vertices. It turns out that I am not the only one who concerns himself with the word. Moore and Angell (1993), for instance, use *apex* for a single corner, *vertex* to mean any point in a tessellation in two or three dimensions where its apices meet.

Rivulets flowing inside a pack bed have been studied by several authors (*cf* Porter, 1968). They are said to flow independently of each other, with no mixing among one another. Diffusion theory has been used to treat a random walk process.

The van der Waals force in combination with double layer repulsion play an important part in the study of filter and particle movements in porous media. Both are electrical forces and can be used to explain the particle capturing mechanism.

Percolation is related to chemical engineering contexts (*cf* Mohanty *et al*, 1982). But the necessary background in stochastic processes for the purpose of simulation has already been described earlier, for example the modelling of colmatage, the retaining of particles suspended in a fluid flowing through a porous medium (Litwiniszyn, 1963 and 1967). In general, however, few authors in all engineering fields relate their works directly to the percolation theory. The majority of studies in this area are based on dynamics and fluid dynamics theories (*cf* Mulder and Gimbel, 1990; Kock and Judd, 1965).

The reason for the lack of percolation material in engineering literature may be because the percolation process generally works behind the scene and only shows itself as critical phenomena. Most engineering studies are concerned with things under some operational condition, within the range of which percolation seems to be absent. By contrast, in Physics where extreme conditions are considered, there is an enormous and increasing amount of publications which are directly under the topic of percolation. But this by no means means that within more pacific ranges no places exist for percolation. In fact it is precisely this lack of mentioning in the literature that induces one to this kind of research. Because our idea of criticality is by tradition closely linked to the idea of time, percolation seems to be present only when there are instantaneous changes. But if in a steep s-curve we only rotate the axis clockwise by $\pi/2$ radian, such that to make the time axis vertical instead of horizontal, then we will see that in place of one critical point in the middle of the graph connecting two different levels, there are now two critical phenomena on both sides, one on each side, and in the middle a flat region where time hardly changes. Looking at it this way percolation seems to be a symmetry between time and space. In physical systems spaces percolates, but in the dual world where criticality is continuity it is the time instead which percolates.

Having said that, without rotating the time-space axes backwards and forwards too often one should still find ways to investigate what a pacified percolation does. In this regard, the study of economics seems to be an ideal place to start, if simply because one knows there exists such thing as hyperinflation but one never wants to study that when it happens. This automatically forces the researcher to find ways of doing researches which would not ruin his pocket or put his life in jeopardy. Another ground with a good prospect is in traffic congestion, even though its worst effect is not yet devastating, apart from what it sometimes does to the economy.

With these digressions in mind, if we now turn our thought at this point to our chemical engineering studies, we will not fail to see how ideal filtration fits the no-ruins requirement. For here we have a process which percolates routinely, often needs to be backflushed, but where the effect it produce is probably that of giving engineers a headache and, at its worst, putting a decent company out of business. But even with this advantage, I still believe that the study of filtration should not concentrate only on the fouling of filters, but should try to understand both the percolated and nonpercolated situations, preferably the latter for the lack of it in literature, and to connect what happens in a working filter to what happens, or does not happen, in a fouled one.

The revolutionising discovery made by F. August Kekulé (Kekulé, 1865, cited in Wotiz, 1993) that benzene has cyclic nature gave rise to the structural theory of organic chemistry.

§ 1.13 Computational geometry

The altitude lines of a triangle are concurrent. The bisectors of the angles of a triangle are concurrent. Ceva's theorem says that, all the three lines in a triangle which contain a vertex and a point on its opposite side are concurrent if and only if no two among them are parallel and the product of the three ratios of division of the sides made in one direction around the circumference of the triangle is one. In Gergonne's theorem, the three lines of a triangle which are made by the vertices and the points of tangency of the incircle on the side opposite to them are concurrent. The intersection between the three lines tangent to the circumcircle of a triangle and the sidelines opposite to them are collinear.

A point is an extreme point of a plane convex set s unless it lies in a triangle which has vertices in s but is no vertex of the triangle. A ray from inside a bounded convex figure intersects the boundary of the latter at exactly one point. Consecutive vertices of a convex polygon exist in sorted angular order about any interior point. A subfacet of a simple polytope is shared by two and only two facets. Two facets share a subfacet if and only if the latter is determined by $d-1$ vertices in their set; these two facets and the subfacet are called adjacent. A line segment defined by two points is an edge of the convex hull if and only if all other points of the set lie on, or to one side of it.

The diameter of a convex figure is the largest distance between parallel lines of support. The diameter of its convex hull determines the diameter of a set. Every vertex of the Voronoi graph is the intersection of three of its edges. Every nearest neighbour of a Voronoi polygon defines an edge.

VT and the triangulation of its nuclei are dual to each other. A Voronoi graph on n points has at most $2n-5$ vertices and $3n-6$ edges. The convex hull of a Voronoi graph on n can be found in linear time.

Modern programming philosophy puts much emphasise on modularity of a programme and on information hiding of modules. Though undoubtedly information hiding can be good for the finished products, during the course of development it sometimes works against yourself when you try to pinpoint an error in order to debug. Some modularisations are more about hierarchies than simplicity. Whenever this is the case, it is necessary to unconventionally seek a simpler path.

The explicit equation in 2-d is $y = mx + c$ or $ax + by + c = 0$. Imposing the constraint $a^2 + b^2 = 1$, that is multiplying all the terms by $(a^2 + b^2)^{-1/2}$, puts the equation into the canonical or normalised form. This makes $a = \cos\alpha$, $b = \cos\beta$ and $c = -r$, where a and b are directional cosines, *i.e.* the cosines of the angles that the normal line makes with the x and y axes respectively. Examples of possible conventions are to have a normal line point towards outside of the region, to have the line direction always to the right of the normal vector, or to keep c positive always.

A parametric form of line equation in 2-d is by introducing a third variable t and write the equations as $x = x_0 + ft$ and $y = y_0 + gt$, where (x_0, y_0) is the point on the line corresponding to $t = 0$. A line through a point p which makes angles α and β with the x and the y axes respectively has the parametric equations $x = x_p + t\cos\alpha$ and $y = y_p t\cos\beta$. One convention is to vary t from 0 to 1 over a line segment, another is to normalise it by multiplying its coefficient by $(f^2 + g^2)^{-1/2}$.

An implicit line equation $ax + by + c = 0$ can be turned into a parametric form as $x = -ac/(a^2+b^2)^{1/2} + bt$ and $y = -bc/(a^2+b^2)^{1/2} - at$. And a parametric line described by $x = x_0 + ft$ and $y = y_0 + gt$ is converted into the implicit form as $-gx + fy + (x_0g - y_0f) = 0$

The implicit plane equation in three dimensions is $ax + by + cz + d = 0$. The parameters can be found by using Cramer's rule, $a = \det(1, y_i, z_i)$, $b = \det(x_i, 1, z_i)$, $c = \det(x_i, y_i, 1)$ and $d = \det(x_i, y_i, z_i)$, which gives $a = y_1z_{32}+y_2z_{13}+y_3z_{21}$, $b = z_1x_{32}+z_2x_{13}+z_3x_{21}$, $c = x_1y_{32}+x_2y_{13}+x_3y_{21}$ and $d = x_1(y_2z_3 - y_3z_2) + x_2(y_3z_1 - y_1z_3) + x_3(y_1z_2 - y_2z_1)$.

A normalised form has a constraint $a^2 + b^2 + c^2 = 1$. This amounts to multiplying its implicit equation by $(a^2+b^2+c^2)^{-1/2}$ to get $\alpha x + \beta y + \gamma z + \delta = 0$. Here α, β and γ are cosines of the angles which the normal to the plane makes with the coordinate axes. The distance between two parallel normalised planes is $\delta_2 - \delta_1$. A normalised implicit plane equation can be used to represent a planar half-space by multiplying every tems by -1 and then assign a convention that the vector formed by the direction cosines always points towards the outside or the inside of the region.

The distance from a point to a plane, if the plane is $ax + by + cz + d = 0$ and the point is (x_p, y_p, z_p), is $r = \left[(ax_p + by_p + cz_p + d)^2/(a^2 + b^2 + c^2)\right]^{1/2}$. The intersection of two planes, from the planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, is $x = x_0 + ft$, $y = y_0 + gt$ and $z = z_0 + ht$ where $f = \det(b_i, c_i)$, $g = \det(c_i, a_i)$ and $h = \det(a_i, b_i)$, $i = 1$ and 2.

The intersection of three planes is found by Algorithm 1.5, Here the minor matrices a^{ij} is δ_{23}^{ab}, δ_{23}^{ac}, or δ_{23}^{bc} as the case may be.

Algorithm 1.5 *Intersection among three planes.*

```
$\Delta \leftarrow \sum_{(a,b,c)} (-1)^{i+j} a_{ij} a^{ij}$;
if $|\Delta| < \epsilon$ then
    at least two of the planes are parallel;
else
    $x \leftarrow (b_1\delta_{23}^{dc} - d_1\delta_{23}^{bc} - c_1\delta_{23}^{db})/\Delta$;
    $y \leftarrow (d_1\delta_{23}^{ac} - a_1\delta_{23}^{dc} - c_1\delta_{23}^{ad})/\Delta$;
    $z \leftarrow (b_1\delta_{23}^{ad} + a_1\delta_{23}^{db} - d_1\delta_{23}^{ab})/\Delta$;
endif                                                          □
```

The intersection between a line and the plane $ax + by + cz + d = 0$ is $(x_1 + x_{12}a, y_1 + y_{12}a, z_1 + z_{12}a)$, where $a = -(ax_1 + by_1 + cz_1 + d)/(ax_{12} + by_{12} + cz_{12})$, $x_{12} = x_2 - x_1$ and similarly for y_{12} and z_{12}.

The area of a circle is πr^2 and that of its segment is $\theta r^2/2$. A segment is its pie cut reaching its centre while a sector is a plane slice through the sphere. The area of a sector is this area subtracted by that of a triangle, or $r^2(\theta - \sin\theta)/2$. The centre of gravity or the centroid lies on the bisector of the central angle with the distance of $4r\sin(\theta/2)/3\theta$ for a sector and $4r\sin^3(\theta/2)/3(\theta - \sin\theta)$ for a segment.

The volume of a pyramid is $Ah/3$, where A is the area of base and h is the height of the pyramid. The volume of a sphere is $4\pi r^3/3$, and the distance from its centroid to the sphere centre is $(3r/4)\left[\sin^4(\theta/2)/(2 - 3\cos(\theta/2) + \cos^3(\theta/2))\right]$ That of a sector of a sphere is $(\pi r^3/3)(2-3\cos(\theta/2)+\cos^3(\theta/2))$ The volume of a tetrahedron is $V = (1/6)\det(x_{12}, x_{13}, x_{14}; y_{12}, y_{13}, y_{14}; z_{12}, z_{13}, z_{14})$, where $x_{ij} = x_j - x_i$,or $V = (1/6)\det(x_i, y_i, z_i, 1)$. The former is limited to the case of three dimensions, and is in fact $V = (1/6)(a \times b) \cdot c$, where a, b and c are respectively the lines from O to A, B and C in a tetrahedron OABC.

Generalising the latter to higher dimensions, we have the volume of a d-dimensional simplex $V = (1/d!)\det(x_{ij}, 1)$, where x_{ij} is now $(x_j)_i$, $1 \le i \le (d+1)$ and $1 \le j \le d$. Here the information found in existing literature seems to be wrong, some lists the multiplying factor as $1/d$, some simply uses 1/6 throughout all $(d \ge 3)$! § 3.13 mentions in more detail how I arrived at the value used here.

Some of of the algorithms found in literature are the following. The algorithm to find whether a point is inside a polygon, Algorithm 1.6. The arbitrary line l here, which is taken for simplicity to be horizontal, passes through z.

Algorithm 1.6 *Point inside a polygon.*

```
$r \leftarrow 0$;
for $i = 1$ to $n$ do
   if edge $i$ and $l$ not parallel then
      if $i$ intersects $l$ to the left of $z$ at any point except its lower extreme then
         $r \leftarrow r + 1$;
      endif
   endif
   if $r$ odd then
      $z$ is internal to $p$;
   else
      $z$ is external;
   endif
endfor                                                                          □
```

To find the inclusion in a convex polygon, $q \in p$ being a known fixed point within the polygon, find the wedge in which z lies by doing a binary search and test whether $\angle(zqp_{i+1})$ is a right turn-while $\angle(zqp_i)$ a left turn angle. If $\angle(p_ip_{i+1}z)$ is a left turn angle, then z is inside p.

The Euclidean minimum spanning tree may be obtained by Algorithm 1.7.

Algorithm 1.7 *Euclidean minimum spanning tree.*

```
$f \leftarrow 0$;
for $i$ from 1 to $n$ do
   $s(p_i) \leftarrow 0$;
   $f \leftarrow p_i$;
endfor
while $f$ contains more than one number do
   $t \leftarrow f$;
   if (s(t)=j) then
      clean up;
      $j \leftarrow j + 1$;
   endif
$(u, v) \leftarrow$ shortest unselected edge incident on $t$, $u \in t$;
$t' \leftarrow$ tree in $f$ containing $v$;
$t'' \leftarrow$ merge $(t, t')]$;
delete $(t')$ from $f$;
$s(t'') \leftarrow \min(s(t), s(t')) + 1$;
$f \leftarrow t''$
endwhile                                                                        □
```

To rotate $v = (v_1, v_2)^{\mathrm{T}}$ to $v' = (v'_1, v'_2)^{\mathrm{T}}$, use $v^{\mathrm{T}} = Av$ where $A = [\cos\theta, -\sin\theta; \sin\theta, \cos\theta]$ is the transformation matrix and θ is the anti-clockwise angle of the rotation.

To rotate a general line in three dimensions by θ around an arbitrary axis, the transformation matrix becomes $A = T^{-1}RT$, where

$$R(\theta) = R_x^{-1}(\alpha)R_y^{-1}(\beta)R_z(\theta)R_y(\beta)R_x(\alpha)T = R_x(-\alpha)R_y(-\beta)R_z(\theta)R_y(\beta)R_x(\alpha).$$

The translation T translates one point of the line to the origin and R_x rotates around the x-axis by α which puts u on to the xz axis, R_y around y-axis by β and puts u' on to the z axis, and R_z around z-axis. Since the vector v is (x_{12}, y_{12}, z_{12}), we have the a, b, c and the unit vector $u = (a, b, c)/|v| = (x_{12}, y_{12}, z_{12})/|v|$. Then it follows that $\cos\alpha = c/d$, $\sin\alpha = b/d$, $\cos\beta = d$ and $\sin\beta = -a$. All of these can perhaps be summarised as a linear procedure in Algorithm 1.8.

Algorithm 1.8 *Rotation in three dimensions*

```
$v \leftarrow (x_{12}, y_{12}, z_{12})$;
$m \leftarrow |v|$; $a \leftarrow x_{12}/m$;
$b \leftarrow y_{12}/m$;
$c \leftarrow z_{12}/m$;
$d \leftarrow (b^2 + c^2)^{1/2}$;
$r_{xx} \leftarrow c/d$;
$r_{xy} \leftarrow b/d$;
$r_{yx} \leftarrow d$;
```

```
$r_{yy} \leftarrow -a$;
$T \leftarrow [\mathrm{I}(3), -(x_1, y_1, z_1)^{\mathrm{T}}; 0(3)^{\mathrm{T}}, 1]$;
$R_x \leftarrow [1, 0(3)^{\mathrm{T}}; 0, r_{xx}, -r_{xy}, 0; 0, r_{xy}, r_{xx}, 0; 0(3)^{\mathrm{T}}, 1]$;
$R_y \leftarrow [r_{yx}, 0, -r_{yy}, 0; 0, 1, 0, 0; -r_{yy}, 0, r_{yx}, 0; 0(3)^{\mathrm{T}}, 1]$;
$R_z \leftarrow [[\cos\theta, -\sin\theta; \sin\theta, \cos\theta], 0(2,2); 0(2,2), \mathrm{I}(2)]$;
$R \leftarrow T^{-1} R_y^{-1} R_z R_y R_x$; $v' = Rv$.                                   □
```

Also in three dimensions, the rotation around the x-axis is

$$R_x = [\{r_{11} = 1, m_{23} = [\cos\theta, -\sin\theta; \sin\theta, \cos\theta]\}],$$

around y-axis is

$$R_y = [\{r_{22} = 1, m_{13} = [\cos\theta, \sin\theta; -\sin\theta, \cos\theta]\}]$$

and around z-axis is

$$R_z = [\{r_{33} = 1, m_{12} = [\cos\theta, -\sin\theta; \sin\theta, \cos\theta]\}].$$

The minor containing parts of the i^{th} and j^{th} rows and columns is m_{ij}.

The right hand coordinate system is where a 90° rotation around the x-, y- and z-axis bring respectively the y- to z-, z- to x- and x- to y-axis. Scaling and translating a vector v in three dimensions amounts to calculating $[v'; w] = A[v; w]$, where A is respectively $[\mathrm{I}(3)s, 0(3); 0(3)^{\mathrm{T}}, 1]$ and $[\mathrm{I}(3), \Delta v; 0(3)^{\mathrm{T}}, 1]$, $\Delta v = (\Delta x, \Delta y, \Delta z)^{\mathrm{T}}$.

A quaternion can be described as a pair (s, v) of a scalar s and a vector $v = (a, b, c)$. The rotation by θ around an axis in the direction of a unit vector u is then the quaternion $(\cos\theta/2, u\sin\theta/2)$. Let $q = (s, v)$. Then $q^{-1} = (s, -v)$. The multiplication of quaternions is $q_1 q_2 = (s_1, v_1) \cdot (s_2, v_2) = (s_1 s_2 - v_1 \cdot v_2, s_1 v_2 + s_2 v_1 + v_1 \times v_2)$, where the cross product is described in minors as $v_1 \times v_2 = [\delta^{yz}; -\delta^{xz}; \delta^{xy}]$.

Let q represent a rotation. Then a vector p is rotated to p' by $P' = qPq^{-1}$, where P and P' are respectively $(0, p)$ and $(0, p')$. In simplified words, this means $p' = s^2 p + (p \cdot v)v + 2s(v \times p) + v \times (v \times p)$. Then we have the transformation matrix for the general rotation around u in three dimensions,

$$R_u(\theta) = \begin{bmatrix} (1 - 2b^2 - 2c^2) & (2ab - 2sc) & (2ac + 2sb) \\ (2ab + 2sc) & (1 - 2a^2 - 2c^2) & (2bc - 2sa) \\ (2ac - 2sb) & (2bc + 2sa) & (1 - 2a^2 - 2b^2) \end{bmatrix}$$

where $s = \cos\theta/2$ and $v = (a, b, c) = u\sin\theta/2$. Furthermore, if q_1 is a rotation by θ_1 around v_1 and likewise q_2 by θ_2 around v_2, then $q_3 = q_2 q_1$ is a rotation by $\theta_3 = 2\cos^{-1} s_3$ around v_3 such that $\sin\theta_3 \geq 0$.

Quaternion is an extension of complex number to higher dimensions where there are three imaginary parts instead of one. It is defined as $q = s + ia + jb + kc$, where a, b, c and s are real numbers, $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$.

Let a plane be described by $(v - p) \cdot n = 0$, where p is a point on-, and n a perpendicular to the plane. This means that, for all points v lying in the plane (p, n), $(v - p) \cdot n = 0$. If the plane (p, n) is transformed into (Ap, m), then Av lies in (Ap, m) and consequently $A^{\mathrm{T}} m = n$ or $m = (A^{\mathrm{T}})^{-1} n$.

A vector normal to the surface remains normal if it is transformed by $(A^{\mathrm{T}})^{-1}$. When a transformation matrix A has the property $A = (A^{\mathrm{T}})^{-1}$, for example rotation, all surface normals remain normal. But in general this equality does not hold, so we have for instance the non-uniform scaling where these normals cease to be normal.

To test for the intersection between a ray and a triangle using Plücker's coordinates is described in Algorithm 1.9.

Algorithm 1.9 *Ray and triangle intersection.*

```
find Plücker's coordinates for vertices and the ray;
test ray against each of the edges;
if ray hits an edge, passes all of them clockwisely or all of them counter-clockwisely then
    ray intersects the triangle;
else
    ray and triangle intersect not;
endif                                                                □
```

If these vertices are v_1, v_2 and v_3, and the ray is $r = r_{12} = r_2 - r_1$, where r_1 and r_2 are any two points on the ray, then Plücker's coordinates for the vector v_{12} are $(u, v) = (v_2 - v_1, v_2 \times v_1)$, and similarly for v_{23}, v_{31} and r_{12}. For the test between the ray and each edge, find $c = u_r \cdot v_i + v_r \cdot u_i$.

Then the ray r counter-clockwisely passes the edge i, hits it or clockwisely passes it respectively as $c < 0$, $c = 0$ or $c > 0$.

A 3-d line can be represented by the six numbers that come with coordinates of two distinct points, or by the eight numbers that come with the coordinates of two distinct planes. Plücker's coordinates, however, provides a mean which suits geometrical computation better than both of these. It redefines the coordinates as $u = p - q$ and $v = p \times q$, where p and q are two points on a line, neither quantity of which depends on p or q.

Let a tetrahedron has its vertices at a_i, $i = 1$ to 4. Then the centre of its circumsphere is at $a_1 + \delta$ and its corresponding radius $r = |\delta|$, where

$$\delta = \left[|a_{12}|^2(a_{13} \times a_{14}) + |a_{13}|^2(a_{14} \times a_{12}) + |a_{14}|^2(a_{12} \times a_{13})\right] / 2|a_{12}^{\mathrm{T}}; a_{13}^{\mathrm{T}}; a_{14}^{\mathrm{T}}|.$$

A triangle $\Delta p_1 p_2 p_3$, where $p_i = (x_i, y_i)$ and $i = 1$ to 3, has an area $A = |x, y, 1(3)|$, which is positive if and only if $\Delta p_1 p_2 p_3$ forms a counter clockwise cycle or $\angle p_1 p_2 p_3$ is left-turned. Or equivalently the area is $A = \det(x_{12}, x_{13}; y_{12}, y_{13})$, which is positive if the points are in anti-clockwise order of the indices and negative otherwise. Or the area is $A = [s(s - d_1)(s - d_2)(s - d_3)]^{1/2}$, where the the semi perimetre s is $s = \sum_i d_i/2$, where d_i, $i = 1$ to 3, are the lengths of the three sides of the triangle.

For a convex polygon, the area can be found by adding together the n triangles formed by any two adjacent vertices and one fixed point within the polygon. Here n is the number of vertices it contains. Or it can be found by adding the $(n - 2)$ triangles formed by one fixed vertex and any two adjacent vertices of those remaining. Another way of finding the area is $A = (1/2)\sum_{i=0}^{n-1}(x_i y_{i+1} - y_i x_{i+1})$, where (x_i, y_i) are vertices. Rearranging to make it faster and more accurate, $A = (1/2)\sum_{i=0}^{n-1}((x_i + x_{i+1})(y_{i+1} - y_i))$. If the dimension of the polygon is higher than two, $A = (1/2)\left|N \cdot \sum_{i=0}^{n-1}(v_i \times v_{i+1})\right|$, where N is a unit vector normal to the plane. The area of a polygon can also be computed, without loosing generality, by subtracting the area under its lower edges by that of its upper edges.

A vector normal to a plane is simply $n = v_{12} \times v_{13}$.

The ordering of vertices on a face of a polygon is done for the purpose of drawing it or for finding vertice pairs which form edges. This can be done in two ways. One is to get inside the polygon and look at all the vertices around comparing their angles relative to one another. Another one is to look at the polygon from a distance and compare their angles as before, as well as their distance from the viewing point. The angle is $\theta = \arccos[(v_1 \cdot v_2)/(|v_1||v_2|)]$, where v_1 and v_2 are vectors to vertices from the distant point, and θ the angle between them.

To find the convex hull in three dimensions, one may use Algorithm 1.10.

Algorithm 1.10 *Convex hull in three dimensions.*

```
sort $s$ by $x_1$ such that $x_i(p_i) < x_i(p_j)$ if and only if $i < j$;
if $|s \leq k|$ then
  construct $c(s)$;
else
  $s_1 \leftarrow \{p_1, \ldots, p_{\lfloor n/2 \rfloor}\}$;
  $s_2 \leftarrow \{p_{\lfloor n/2 \rfloor}, \ldots, p_n\}$;
  $p_1 \leftarrow c(s_1)$;
  $p_2 \leftarrow c(s_2)$;
  $p \leftarrow$ merge $p_1$ and $p_2$;
endif                                                              □
```

Here $c(s)$ is the convex hull of s. Two convex hulls are merged with each other by first constructing a cylindrical triangulation T which supports p_1 and p_2 along two circuits e_1 and e_2 respectively, then remove from both p_1 and p_2 the portions which have been obscured by T.

The following Jarvis's march algorithm, Algorithm 1.11, finds a convex hull in two dimensions.

Algorithm 1.11 *Convex hull in two dimensions*

```
$p_1 \leftarrow$ the lowest point in $s$;
$q_1 \leftarrow$ the highest point in $s$;
while next point $\neq q_1$ do
  find $p_i \in s$, $i = 2, 3, \ldots$, with an increasing order of the polar angles
    with respect to $p_1$;
endwhile
```

```
while next point $\neq p_1$ do
  find $q_i \in s$, $i = 2, 3, \ldots$, with an increasing order of the polar angles
    with respect to $q_1$ and the negative $x$ axis;
endwhile                                                              □
```

The polar angle is an angle with respect to the positive x-axis. The lowest and the highest points are on the convex hull.

Algorithm 1.12 is the quick hull algorithm.

Algorithm 1.12 *Quick hull algorithm.*

```
$l \leftarrow (x_0, y_0)$;
$r \leftarrow (x_0, y_0 + \epsilon)$;
if $s = \{l, r\}$ then
  return $(l, r)$;
else
  find $k \in s$ that gives $\max A_{\triangle klr}$ or ($\max A_{\triangle klr}$ and $\max \angle klr$);
  $s^1 \leftarrow p \in s$, such that $p$ is on the left of $\overrightarrow{lh}$;
  $s^2 \leftarrow q \in s$, such that $q$ is on the left of $\overrightarrow{hr}$;
  $\{h\} \leftarrow (s^1; l, h)$;
  $\{h\} \leftarrow (s^2; h, r) - h$;
endif                                                                 □
```

The convex hull is $\{h\}$. The points l and r are with respectively the smallest and the largest abscissa. In other words they are the left-most and the right-most points. And k is the furthest point with respect to l and r.

Let $p = \{p_1, p_2, \ldots, p_n\}$ be a set of n generators in ν-dimensional space, the coordinates of which are (x_{ij}), $i = 1$ to d and $j = 1$ to n. Then the Delaunay tessellation in d dimensions which spans p is generated by,

```
for $j = 1$ to $n$ do
  $\{q\} \leftarrow q_j = (x_{ij}, \sum_j x_{ij}^2)$;
endfor
$h \leftarrow c(q)$;
project all the lower $n$-faces of $c(q)$ parallel to the $d^{\text{th}}$ axis on to the original $n$-d space;
                                                                      □
```

Okabe *et al* (1992) give a good review of algorithms for generating VT's. Given the set of generator points $\{p_i\}$, $i = 1$ to n, a brute force albeit simple method generates for all i and j from 1 to n the $(n-1)$ half planes $h(p_i, p_j)$, $1 \leq j \leq n$, $i \neq j$, and then proceeds to construct all $\mathcal{V}(p_i)$ of the VT from their common intersections.

On the other hand, the following Algorithm 1.13 is the quaternary incremental method whose inputs comprise the $(n-3)$ generators p_i, $i = 4$ to n, where all the p_i are in $s = \{(x, y) | 0 \leq x, y \leq 1\}$, and three additional generators $p_1 = (0.5, 0.5(1 + 3\sqrt{2}))$, $p_3 = (0.25(2 - 3\sqrt{6}), 0.25(2 - 3\sqrt{2}))$ and $p_3 = (0.25(2 + 3\sqrt{6}), 0.25(2 - 3\sqrt{2}))$.

Algorithm 1.13 *Quaternary incremental method.*

```
$k \leftarrow \min(k_i)$ such that $k_i \in \mathrm{I}^+$ and $n < 4^{k_i}$;
$s_{ij} \leftarrow (i-1, i/2^k, j-1, j/2^k)$;
construct a quaternary tree;
scan leave buckets from left to right, top to bottom, put generators in buckets;
$\{p_i\}$ do quaternary reordering on the generators;
$\mathcal{V} \leftarrow$ construct Voronoi for $p_1$, $p_2$ and $p_3$;
for $i = 4$ to $n$ do
  repeat $\leftarrow 1$;
  while repeat do
    find $p_\alpha$ such that $\mathrm{d}(p_\alpha, p_\ell) = \min_j \mathrm{d}(p_j, p_\ell)$;
    if $\mathrm{d}(p_m, p_\ell) < \mathrm{d}(p_\alpha, p_\ell)$ then
      repeat $\leftarrow \neg$repeat;
    elseif $p_m \leftarrow p_\alpha$ do nothing;
    else
      repeat $\leftarrow \neg$repeat;
    endif
    endwhile
    $\{w_{i1}, w_{i2}\} \leftarrow$ the intersections between the perpendicular
      bisector of $p_ip_l$ and $\Omega(\mathcal{V}(p_i))$, $1 \le i \le \ell - 1$;
    $\Omega \leftarrow$ construct the boundary of $\mathcal{V}(p_i)$ formed by these $\overline{w_{i1}w_{i2}}$;
    $\{\mathcal{V}_\ell\} \leftarrow (\{\mathcal{V}_{\ell-1}\} \bigcup \Omega) - \{\mathcal{V} | \mathcal{V} \in \{\mathcal{V}_{\ell-1}\}, \mathcal{V}$ is within $\Omega\}$;
endfor
$\{\mathcal{V}\} \leftarrow \{\mathcal{V}_n\}$;
```

□

The construction of s_{ij} is such that $s_{ij} = (i-1, i/2^k] \times (j-1, j/2^k]$, where $i, j \in \mathrm{I}^+$, i and j are 1, 2, … up to 2^k. The nearest neighbour search finds from ℓ generators, $p_1, \ldots, p_\ell$, and $\{\mathcal{V}_{\ell-1}\}$, the generator point p_m such that $\mathrm{d}(p_m, p_\ell) < \mathrm{d}(p_m, p_i)$ for all $i = 1$ to ℓ, $i \neq m$ and $i \neq \ell$; p_j are all generators adjacent to $\mathcal{V}(p_i)$. The boundary growing procedure gives the sequence of boundaries $\{\Omega\}$. The additional generators mentioned in the procedure is graphically shown in Figure 1.16.

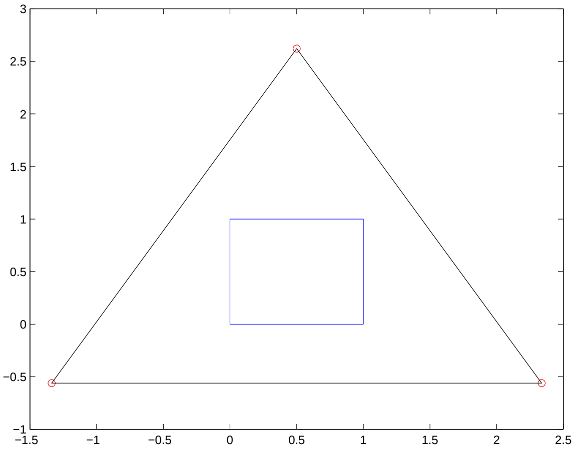


Figure 1.16 Additional generators for the incremental method.

If we draw a horizontal line through each point in two dimensions, or a horizontal plane each point in three dimensions, we divide the space into slabs the line segments within which do not intersect one another. In three dimensions, and for a polygonal model of porous media, we can for instance divide the model into such slabs, then find the effective cross sectional area of each slab, and then determine where the bottle neck to the flow occurs within the media. Whether this would produce the correct determination of the pressure of flow across the material is another matter because flows through porous media may be governed by the combination of the various tortuous paths through the pores, the interrelationship of which can be complicated.

These slabs provide another method in finding the area, or in three dimensions the volume, of each cell of the tessellation. Here the cell is divided into slabs, and then the area or volume of each section calculated.

In three dimensions, rotation around the x-axis is done by $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$, around y-axis by $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$ and around z-axis by $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

§ 1.14 Geometric algorithms

The programme `findfarea.m` in § A.26 finds the area and plane parameters of a face from a matrix containing the list of the coordinates of the ordered vertices. Its synopsis is `(a,p)=findfarea(v)`, where `a` is the face area, `p` the list of the plane parameters and `v` the coordinate matrix.

Real problems are algebraical or analytical whereas computer simulations are arithmetical and numerical (*cf* Knuth, 1997). In between these two lie computer programmes. Therefore the latter are mapping from the analytical to the arithmetical world. There are different ways to solve a problem analytically, and there are different ways numerically. Therefore numerical study by simulation is an endless pursuit. Several programmes listed in § A do the same job, but each one does it differently.

Hamiltonian's are essentially a function that maps N bodies pairwisely to account for all pairs of their mutual interactions. They occur in most fields where there are particles interacting with each other. In quantum mechanics, for example, the time independent Schrödinger equation for a system of N particles interacting via the Coulomb interaction is $H\Psi = E\Psi$, where the Hamiltonian is

$$H = \sum_{i=1}^{N} \left(-\frac{\hbar}{2m_i} \nabla_i^2 \right) + \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \frac{z_i z_j}{4\pi\epsilon_0 |r_i - r_j|}, \tag{12$_\mathrm{i}$}$$

where Ψ is the N-body wavefunction, z the charges of the individual particles and E the energy of either the ground or an excited state of the system. Similarly from Rushbrooke and Morgan (1961), using their notations, the Hamiltonian of the Ising problem is

$$\mathcal{H} = -2J \sum_{\langle i,j \rangle} s_3^{(i)} s_3^{(j)} - g\beta H \sum_{(i)} s_3^{(i)}, \qquad (\mathbf{13})_{\mathrm{i}}$$

where β is the Bohr magneton, g the gyromagnetic ratio and J the magnitude of the exchange interaction.

When simulating such systems, the number of pairwise summation terms can be reduced by half because they represents a symmetric matrix. The total number of terms is thus reduced from $N(N-1)$ to $N(N-1)/2$ (*cf* Wray *et al*, 1983). Because $i \neq j$, all the diagonal components of the matrix are excluded, which makes the number of pairs $n^2 - n = n(n-1)$. In a programme, this is equivalent to two *if* statements, one embedded within the other in the form $i(j(\cdot))$, where the index i runs from 1 to $(n-1)$ and j from $(i+1)$ to n; this has been discovered from experience, as can be seen by comparing the present work with the earlier one (Tiyapan, 1995, KNT3(iii)).

§ B.5 TEXnicalities

LaTEX is a macro which runs on TEX. It gives one convenience but not without a tradeoff in understanding. Also one may not have much freedom in writing macros on LaTEX. Tiyapan used TEX for his Work Notes dated 12th February 2001 .

LaTEX(Lamport, 1985) is written by Leslie Lamport. Newer versions of it has come up at a regular interval. Unlike most other macros which run on TEX, her source code is free for none but herself. Moreover, having used TEX to do what it wants, LaTEX thereby castrates her progenitor in such a way that it is impossible for her users to define new macros efficiently by using the `\def` command. With `\def` disabled, the lion has lost its fangs and users become as docile and dependent as a lobotomised patient. There can be no doubt that with the TEX users having such idea as this, sooner or later LaTEX will have to change in these respects. But this is the way things are at present.

One of the first macros written is the code to change the date format. The algorithm first sets $x = date$, then it assigns the ordinate endings *st, nd, rd* or *th* depending on the value of *ordinate*, which is calculated from

```
if $x > 30$ then $ordinate = 1$ else
  if $x > 20$ then $ordinate = x - 20$ else
    $ordinate = date$
  endif
endif
```

The macros which are either newly written or adapted from elsewhere, mainly from the *manmac* macro by Knuth, are listed in § B.5. Apart from these, this thesis uses the *plain* and the *manmac* macros. Another set of macros developed here is that which deals with languages. The definition of language here is quite wide. It contains many languages among which are those which are used here, for example the languages for Chemistry, Chinese, Czech, German, French, grammatical jargons, Japanese, Lanna, Latin, latin grammar, Mathematics, Pali, Physics, Russian, Sanskrit, and Thai (Daiy), *etc.* Only parts of this set of macros are useful for the writing of this thesis, not least so those which are used for writing the dedication page.

In the original account of his, Tiyapan (2003, KNT8(iii)) wrote,

> When I first started using TEX instead of LaTEX, I only used the macro `plain.tex`. Then in my first book typeset with TEX (Tiyapan, 2001, KNT8(ii)), I used in addition to the plain TEX `manmac.tex` and `epsf.tex`. Now to my amazement, I have discovered many other excellent macros, for instance `rotate.tex`, and found that I could understand how they work when I read them. This is one of the benefits that comes with talking in TEX instead of, for instance, LaTEX. I also know now the difference between the primitive TEX and plain TEX, and that the latter is only one of the infinitely many possible implementations of TEX. However, since all TEX gurus I know use plain TEX as a basis, there is no reason why I should be too proud to follow the practice. Having said that, my next plan is to improvise on the primitive TEX without any direct reference to the plain TEX macro.

The citation programme BibTEX was intended to be used with LaTEX. Karl Berry and Oren Patashnik have written `btxmac.tex` which makes BibTEX usable from plain TEX. But for the present purpose I merely use my own macros, which are much simpler, and do not need BibTEX. Ultimately such database programme as BibTEX would have been extremely useful. But I wish to develop something similar to it on my own.

There are still some unsolved problems in the TEX macros, for example the page references which are embedded within groups are sometimes slightly wrong, that is they may appear to be one more or one less than their actual position. Since publishing macros play but a minor part here compared with mathematics and physics, this problem has been systematically minimised and then tolerated. The solution and explanation of it will be dealt with and published elsewhere.

When lines of text appear beside a picture there are macros which make the latter always stay next to the outer rim of the page. These work satisfactorily well, and those cases in the results which appear to say otherwise are in fact the result of some other more primitive macros earlier written.

2

§ 2. Division of space

A stochastic system which is stochastically invariant under arbitrary translation, that is under the transformation $x \to a + x$, is homogeneous. Homogeneity lets us use the ergodic theory, namely $\dot{F}_{Z,q}(z) \to \dot{F}_Z(z)$ with probability one as q approaches infinity for all z in Z, in other words the empiric- implies the ergodic distribution function of the characteristic Z for the cell C in $Q(q)$. Empirical moments, $\dot{E}_q(z) = \int z\mathrm{d}\dot{F}_{Z,q}$, almost surely converge to the corresponding ergodic moments, $E(z) = \int z\mathrm{d}F_Z(z)$. The study of random division of space aims at defining classes of parameter dependent random divisions and determining their important ergodic distribution. The most important ergodic distribution is the volume (Miles, 1972). But $E(V)$ itself is not particularly informative because it only shows the scale of the model. The best representative of the nature of a random division is then the *coefficient of variation*, $\mathrm{cv}(V)$, which is the ratio of the standard deviation to the mean. The mean 1-projection or mean caliper diameter, M_1, of a domain is the mean length of its orthogonal projection onto an isotropic random line, whereas the mean 2-projection is the mean area of its orthogonal projection onto the same. For a polyhedron, $M_1(4\pi)^{-1}\sum d_e^i(\pi - \theta_i)$ where θ_i is the dihedral angles in radians. The i-facets of a polyhedron, $i = 0, 1, 2$, are its vertices, edges and faces respectively. The seven values of the basic integral geometric polyhedral quantities are V, A, M_1, d_e, n_c^f, n_c^e and n_c^v. If a polyhedron is simple, *i.e.* each vertex is in three and only three faces, then $3n_c^v = 2n_c^e$ and from Euler's formula, $n_c^v - n_c^e + n_c^f = 2$, then it follows that the value of any one of n_c^v, n_c^e and n_c^f determines the value of the other two. The mean value of n_c^f give a good idea of the overall interface structure, and so is only next in importance to the $\mathrm{cv}(V)$. The homogeneous Poisson s-flats in R^d, $\mathcal{P}(s, d)$, underlies most forms of random divisions of space (*cf* Miles, 1972), in particular $\mathcal{P}(0, d)$ where $d = 2$ or 3. Under this notation the standard Poisson process is $\mathcal{P}(0, 1)$. The basic properties of $\mathcal{P}(0, 3)$ are that

$$\mathrm{P}(\text{there are } n \text{ particles in } X \subset V) = e^{-\rho V}(\rho V)^n/n! \qquad (\mathbf{14})_{\mathrm{ii}}$$

where $n = 0, 1, \cdots$, that is the probability is a Poisson (ρV) distribution, and that the numbers of particles in disjoint domains are mutually independent. Miles (1972) gives descriptions of various kinds of random tessellation, for example the box tessellation and particularly the generalised Johnson-Mehl model, described here as Algorithm ℵ, which includes both the standard Johnson-Mehl and the Voronoi tessellation in three dimensions as special cases.

Algorithm 2.1 *Generalised Johnson-Mehl model, Miles (1972)*

```
for each time step do
  while more nuclei to be born do
    borns a nucleus;
    if the new nucleus would occupied a site already occupied then
      remove the nucleus;
    endif
  endwhile
  for all growing nuclei do
    for all its rays still growing do
      nucleus radiates with speed $v$;
    endfor
  endfor
  for all those nuclei which has just grown do
    for all its rays just grown do
      if it has met with a ray of another nucleus then
        label both rays as grown;
      endif
    endfor
    if all its rays are grown then
      label the nucleus as fully grown;
    endif
  endfor
endfor
```

□

Similar to the Minkowski space, the stochastic nucleus birth process is in (3+1)-dimensional space, $(x, y, z; t)$-space where $t \geq 0$ which can be denoted by R^4_+. This process is homogeneous in (x, y, z) but not necessarily so in t because in the actual development process it is always inhomogeneous with respect to time. Then this birth process is a Poisson point process in R^4_+, which is inhomogeneous in t and with the intensity $\alpha(t)$. For the standard Johnson-Mehl, $\alpha(t) = \alpha$ is a constant, whereas for the Voronoi $\mathcal{V}(3,3)$, $\alpha(t) = \rho\delta(t)$, where $\delta(\cdot)$ is the Dirac δ function. This model is thus a general one which can be fitted to a wide range of data by choosing some appropriate $\alpha(\cdot)$.

Each i-facet of the Voronoi tessellation $\mathcal{V}$, created from $\mathcal{P}(0,3)$, is an i-facet of $4-i$ members of $\mathcal{V}$, with each point on the i-facet being equidistant from the corresponding $4-i$ nuclei where $i = 0, 1, 2$. The exact values of the moments of the 3-d Voronoi tessellation in the notation used by Miles (1972) are: for first order moments, $\mathrm{E}(V) = \rho^{-1}$, $\mathrm{E}(S) = (256\pi/3)^{1/3}\Gamma(5/3)\rho^{-2/3} = 5.821\rho^{-2/3}$, $\mathrm{E}(M_1) = 4^{2/3}\pi^{5/3}\Gamma(1/3)\rho^{-1/3}/(3^{5/3}5) = 1.458\rho^{-1/3}$, $\mathrm{E}(L_1) = 12\,\mathrm{E}(M_1) = 17.50\rho^{-1/3}$, $\mathrm{E}(N_2) = (48\pi^2/35) + 2 = 15.54$, $\mathrm{E}(N_2^*) = 8$, $\mathrm{E}(N_1) = 144\pi^2/35 = 40.61$ and $\mathrm{E}(N_0) = 96\pi^2/35 = 27.07$; for a second moment, $\mathrm{E}(V^2) = 1.180\rho^{-2}$; for the aggregate of edges, $\mathrm{E}(L) = \mathrm{E}(L_1)/\,\mathrm{E}(N_1) = 0.4309\rho^{-1/3}$; for the aggregate of faces, $\mathrm{E}(A) = \mathrm{E}(S)/\,\mathrm{E}(N_2) = 0.3746\rho^{-2/3}$, $\mathrm{E}(B) = 2\,\mathrm{E}(L_1)/\,\mathrm{E}(N_2) = 2.252\rho^{-1/3}$ and $\mathrm{E}(N) = 3\,\mathrm{E}(N_0)/\,\mathrm{E}(N_2) = 5.228$; for the plane section, $\mathrm{E}_2(A) = 1/\rho\,\mathrm{E}(M_1) = 0.6859\rho^{-2/3}$, $\mathrm{E}_2(B) = (6/\pi)^{1/3}5\Gamma(2/3)\rho^{-1/3} = 3.136\rho^{-1/3}$, $\mathrm{E}(N) = 6$ and $\mathrm{E}(A^2) = 0.698\rho^{-4/3}$; and for the line section, $\mathrm{E}_1(L) = 4/\rho\,\mathrm{E}(S) = 0.6872\rho^{-1/3}$, $\mathrm{E}_1(L^2) = 0.682\rho^{-4/3}$, $\mathrm{E}_1(L^3) = 0.668\rho^{-1}$ and $\mathrm{E}_1(L^4) = 0.774\rho^{-4/3}$. In our terminology, ρ, S and L_1 are respectively ρ_c, A and d_e, N_0, N_1, N_2 and B respectively n_c^v, n_c^e, n_c^f and s. The Delaunay tessellation, on the other hand, is better understood since we know that

$$\mathrm{E}(V^k) = \frac{35\pi^{1/2}(k+2)!(2k+4)!}{256\Gamma\left(\frac{1}{2}k+2\right)\Gamma\left(\frac{1}{2}(3k+9)\right)(8\pi\rho)^k}. \qquad (\mathbf{15})_{\mathrm{ii}}$$

Shape is the most fundamental geometrical property. Shape and size are all the geometrical information that remain when location and rotational effects are filtered out from an object (Dryden and Mardia, 2002), and between these two you can take the size away so that only shape remains for further analysis. When we talk about particle sizes in simulation, it is usually the case that we have already assumed some kind of particle shape. This is because the definition of size is only meaningful if you have some idea about the shape. Shape analysis works with landmarks, which are also known as anchor-, control-, design-, key-, model-, profile-, or sampling points, facets, markers, nodes, sites, fiducial markers, *etc.* Dryden and Mardia (*ibid.*) work with three types of landmarks, *viz.* anatomical-, mathematical- and pseudo landmarks. Their work could become very interesting if combined with another problem of object location (*cf* Tiyapan, 1996, KNT4(v) and KNT4(vi)). A landmark can be unlabelled or labelled with a name or number. A particular member of the shape set which is used as a representative of all other members is the icon of that set. For the shape analysis in two dimensions the thin-plate spline is a convenient tool which is bijective and is analogous to the monotone cubic spline.

§ 2.1 Stereographic projection

Stereographic projection (*cf* Phillips, 1949) is used to represent a three-dimensional figure in two dimensions. It is useful in the study of symmetry of crystals. The stereographic projection of a cube is shown in Figure 2.1.

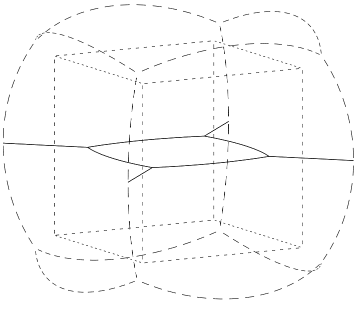


Figure 2.1 *Stereographic projection* of a cube. A stereographic projection is drawn by first projecting each vertex on to the surface of a sphere encircling the polyhedron along the line coming from the origin of the sphere. The projected point on the sphere surface is then projected on to the plane $z = 0$ (ie. $(x, y, 0)$) along the line which originates from it and goes towards the nadir point relative to the half-sphere surrounding it that rests on the $z = 0$ plane.

With this kind of projection, points that are symmetrical to each other with respect to the plane $z = 0$ are projected on to the same point on that plane. Therefore it is a normal practice to distinguish points of the two hemispheres by drawing those in one of them as dots, while drawing the rest as circles. Continuous lines going from one hemisphere to another will become discontinuous on the projected plane. In Figure 2.1, the top and the bottom squares of the cube is projected on top of each other, while the four edges parallel to the z-axis go to the circumference of the great circle of the projected plane first, then retrace their ways back to their vertices.

Figure 2.1 shows that, in effect, what the stereographical projection does is to bloat a polyhedron out into a spherical balloon, and then project the image obtained on the balloon on to the horizontal plane. What the second projection does is to look at the hemisphere above it with a 90° wide-angled lens from the nadir position. In order to compare the shape of the original polyhedron with the top-view of the balloon as well as the stereographical image, Figure 2.2 draws our cube rotated one radian around the vector $(0.3, 0.4, 0.866)$. Such rotation can be done by using the transformation matrix

$$M = \begin{bmatrix} 1-2(y^2+z^2) & 2(xy-wz) & 2(xz+wy) \\ 2(xy+wz) & 1-2(x^2+z^2) & 2(yz-wx) \\ 2(xz-wy) & 2(yz+wx) & 1-2(x^2+y^2) \end{bmatrix},$$

where $[(x,y,z),w] = q$, a quaternion, $(x,y,z) = u \sin\alpha$ and $w = \cos\alpha$. Here $u = (a,b,c)$ is the direction cosine vector of the axis of rotation and $\theta = 2\alpha$ is the angle of rotation.

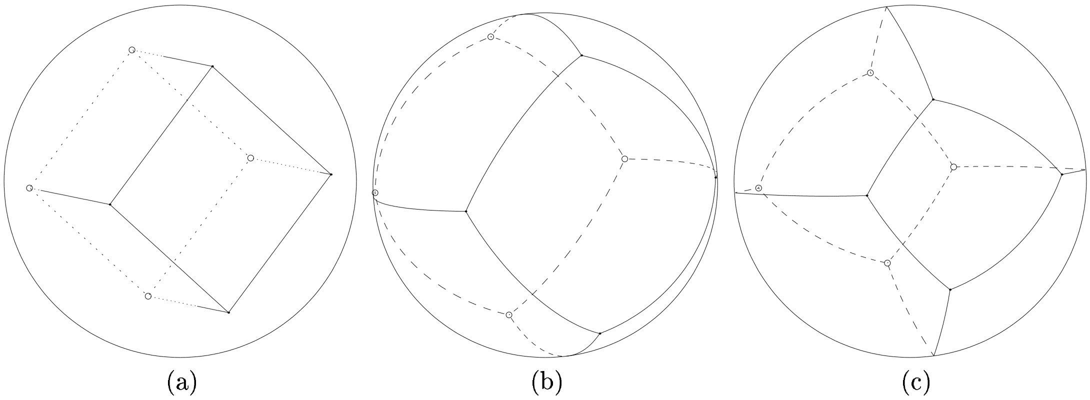

(a) (b) (c)

Figure 2.2 *Rotated cube, (a) top-view, (b) top-view of its bloated sphere and (c) the stereographic projection. While (c) shows the same amount of symmetry information as the other two, it incorporates more information regarding the z-coordinate than the others, (b) being the least informative in this respect.*

The information regarding symmetries of the polyhedron is preserved by the projection. In practice, when it is applicable and possible, the sphere is positioned such that its centre coincides with the centre of symmetry of the polyhedron, the plane $z = 0$ on one of the symmetry planes, and the planes $y = 0$ and $x = 0$ on two others. Thus the projection of Figure 2.1 clearly shows the symmetry around the centre as well as the reflection symmetries with respect to the planes $z = 0$, $y = 0$, $x = 0$, $x = y$ and $x = -y$. Figure 2.2, on the other hand, only shows the symmetry around the centre.

The stereographic projection is an example of a homeomorphism, that is to say, a mapping of one figure onto another that is both continuous and one to one.

A stereographic net is called the Wulff net. It comprises a family of great circles at 2° intervals and a family of small circles. The great circles are equivalent to the meridians of longitude while the small circles the parallel latitudes (*cf* Cox *et al*, 1974).

The stereogram which is made up of two 2-d pictures, one for each eye, can also be used for visualising a crystal structure in three dimensions. It works by tricking or causing brain to see the virtual 3-d image from the input prepared for both eyes. Unlike the stereoscope which requires a viewing device, an autostereogram requires none and only a little practice. During the 1990's there has been a boom in businesses related to a certain type of stereogram referred to as SIRDS, Single Image Random Dot Stereogram, which was only a new name for the autostereogram. The boom of this business had the origin in Japan and was fueled by consumers in the east-asian countries, including Thailand. In this kind of autostereogram there are no two separate images. The two images appear as dot patterns embedded within a random pattern of dots in the background. This makes one see the 3-d images arising out of the blue amidst a seemingly chaotic random mixing of dots. SIRDS sometimes appears as another variant called Single Image Stereogram, SIS, which uses patterns in place of dots.

Stereograms and Autostereograms have no depth of field. This gives it a peculiar sharpness because all points from the highest to the deepest appear in focus at once, whereas in viewing the real world our eyes focus on one distance at a time. The reason behind the surge in the popularity of SIRDS's mentioned must be that they help relax the eyes. Because there is no special gadgets required, one needs to look at the image wide-eyed, that is to say, with one eye focusing parallel to the other, and this is very relaxing to the muscles of the eyes. By contrast, cross-eyed viewing can induce headache and eyestrain. One procedure for drawing a stereogram from a crystal or crystal model is shown in Algorithm 2.2.

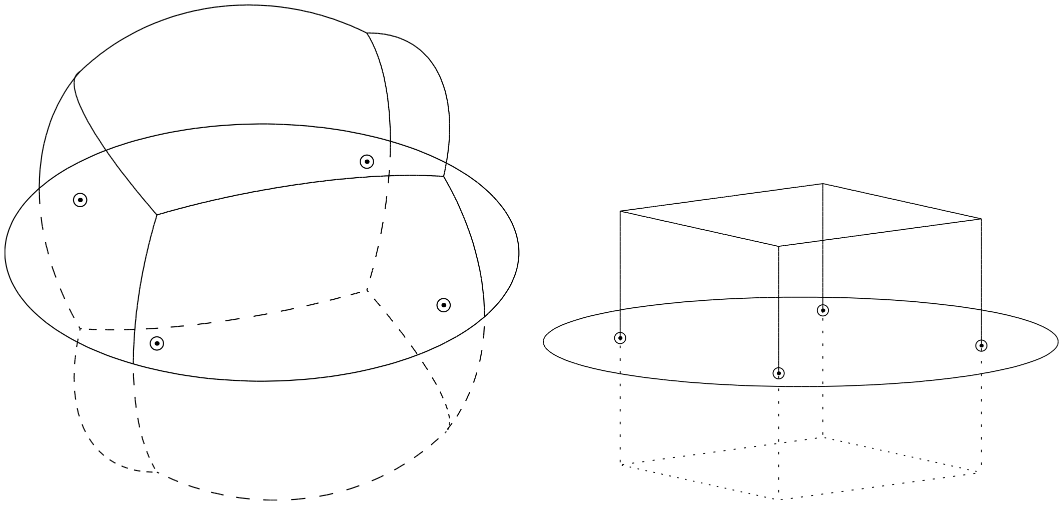
Algorithm 2.2 *A procedure for drawing stereograms from crystal models.*

```
make drawings of the crystal;
label all the different faces;
select prominent zones;
for every zone do
    measure all the interfacial angles;
endfor
plot the prominent zone in the primitive circle;
mark the centre of the stereogram;
locate $a$ arbitrarily;
mark the remaining face poles;
```

□

The interfacial angles are measured using a contact goniometre. The sum of the interfacial angles in a complete zone is 360°. Face poles are marks obtained from intersections of small circles with primitive circles.

The programmes on stereographic projection are collectively called `stp.m` and listed in § A.32. Some of the pictures drawn during its developmental stages are given in Figure 2.3 (a) and (b) while those for future developments in Figure 2.3 (c) and (d).


(a) (b)

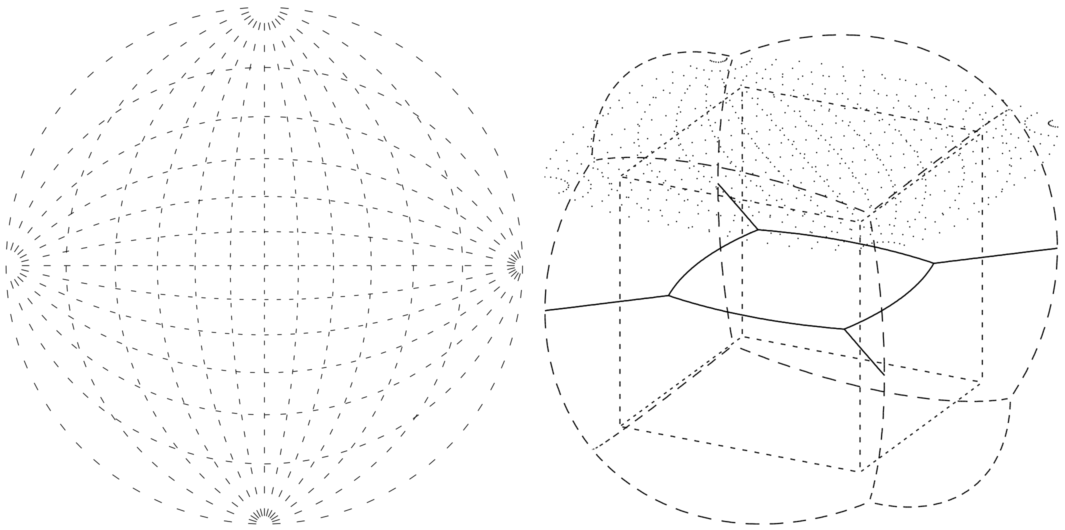

(c) (d)

Figure 2.3 *Developmental stages of programmes for the stereographic projection;* (a) *and* (b) *for the cube,* (c) *and* (d) *the great circles.*

The parameters of any two faces of a crystal are always rationally related to each other. One way of uniquely presenting the relationship between angles of the faces and those of the crystallographic axes is by using the Miller indices. These idices are obtained by first finding intersections between a face and a-, b and c axes. Then divide by the b-axis intersection to get parameters of the face, and then divide these by the respective values of the parametral plane chosen. What is obtained after this stage are the ratio of the parameters to those of the parametral plane, the reciprocal of which gives the Miller indices.

§ 2.2 Covering lattices

A covering lattice of any two-dimensional lattice is the lattice obtained by joining midpoints of consecutive edges together. The code in § A.7 finds covering lattices up to the eighth one and computes the total area of the cells for each case.

The square lattice is the only regular covering lattice, that is it is both the dual and the covering lattices of itself. But all polygonal tilings and tessellations can have a covering lattice, or in fact an infinite orders of covering lattices. Coverings of some lattices can be seen in § 3 (*cf* Tiyapan, 2001, KNT8(ii)).

Coverings can be generalised to a general dimension d. In two dimensions they are lines, *i.e.* having two vertices, straight lines each of which join two lines across a corner. For three-dimensional polyhedral tessellations they are planes with three vertices, triangles each of which join three planes across a corner, in other words a coign. Then in four dimensions they may be polyhedra with four vertices, tetrahedra each of which joins four 4-d polytopes across a four-edged corner in four dimensions, and so on. In d dimensions, then, perhaps they are polytopes with d vertices, three-cornered $(d-1)$-polytopes each of which joins d d-dimensional polytopes across a d-edged corner in d-dimensions.

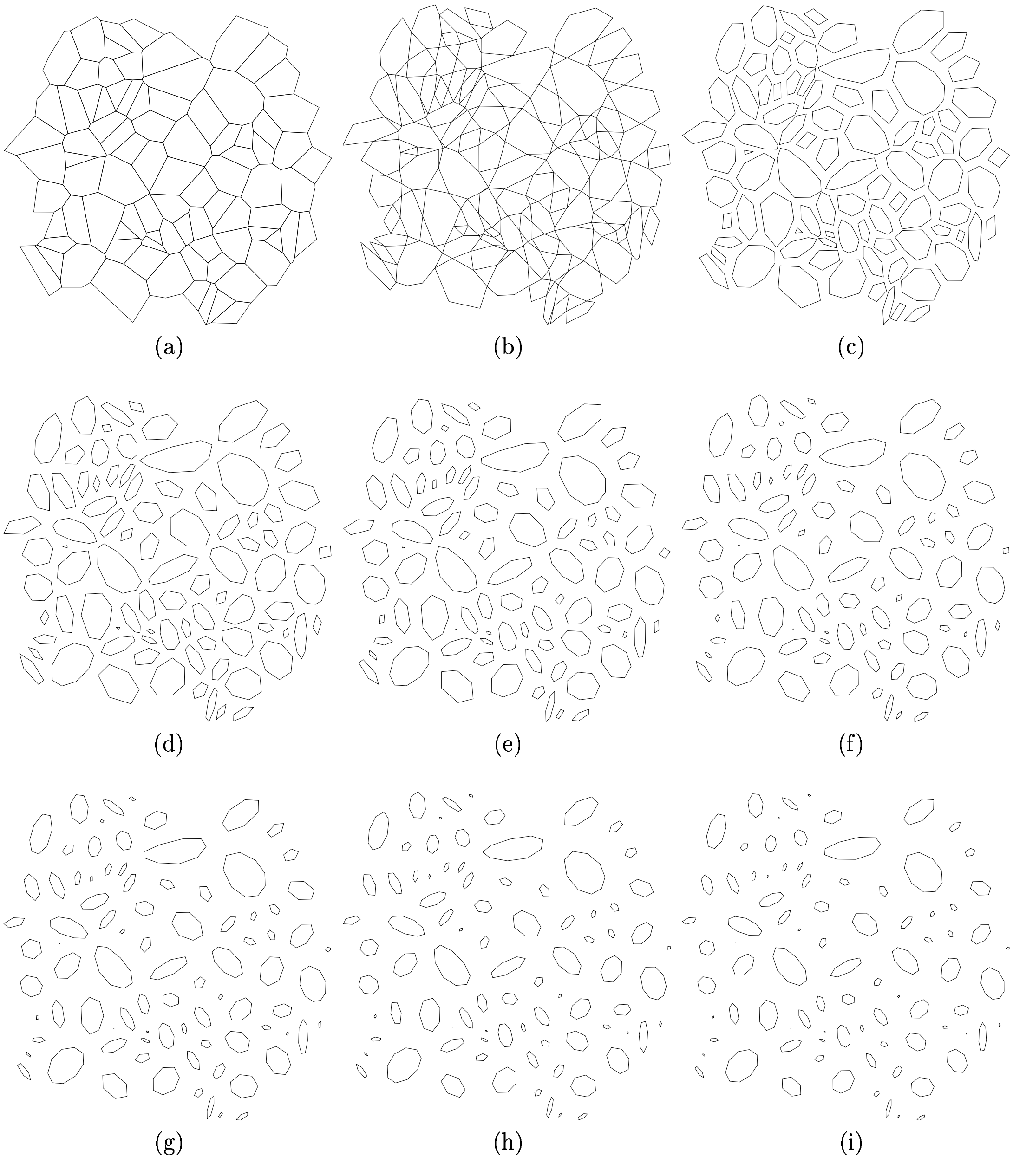
One interesting property of covering lattices is that they leave the voids intact while reducing only their size. Thus the structure and complexity of the original tessellation remain unchanged. This can be useful when we want to exclude some of the volume. In filtering membrane studies, for instance, this is ideal since all the voids still remain in the same position.

The process of covering is similar to that of shrinkage in cell in that there is a retreat away from corners. This could be because corners are hard to maintain. Surface tension is high there, and like a nook, recess, or remote, unstrategic parts of a country the cost of maintaining and governing is high. In the case of a country, conflict in such parts is analogous to the high surface tension in corners of cells.

When an empire or a metropolitan city declines, the process is similar. Such far corners are the first parts to fall into chaos. Law, order and security shrink away from them. The Roman Empire is an excellent example of this. In its heyday it reached out to every corner of Europe, however far. When it came to the decline, it literally *pulled itself together*, though it never could pull itself together again after that. It drew away first from those far corners, England for instance, and then towards its nucleus and died.

Manchester is another interesting example. After the industrial revolution, and under the governance of the Conservative party, the city declined. And as it did, all the different nuclei became prominent, if only because the distance between them became more so. Thus Bolton, Altrincham and Stockport, for example, shrunk towards their nuclei, leaving behind dangerous districts where mugging, murder and crime are rampant like the Moss Side decades ago until shortly after the IRA bombing of the city centre. When an urban area fades away it does not do so suddenly but like the plant cell subject to a dewatering process or a polygonal tessellation to a covering one.

Random fluctuation can create areas of irregularity within a homogeneous and isotropic universe. These irregularities become nuclei, and from duality of the structure fine partitions start to develop around them which become Voronoi facets. Gradually but steadily the gas shrinks to form the galaxies of our present universe.


(a) (b) (c)

(d) (e) (f)

(g) (h) (i)

Figure 2.4 *Covering lattices, stone pavements, or galaxies in the forming?* (a) $\mathcal{V}$, (b) $C^1(\mathcal{V})$, (c) $C^2(\mathcal{V})$, (d) $C^3(\mathcal{V})$, (e) $C^4(\mathcal{V})$, (f) $C^5(\mathcal{V})$, (g) $C^6(\mathcal{V})$, (h) $C^7(\mathcal{V})$, (i) $C^8(\mathcal{V})$.

Figure 2.4 (a) to (i), which are the results of the covering operator applied to a Voronoi graph eight times in succession, represent this situation. Figure 2.5 shows the area of multiply covered cells $\mathrm{C}^n(\mathcal{V})$, $n = 0$ to 8, where $\mathrm{C}^0(\mathcal{V})$ is the Voronoi graph $\mathcal{V}$. Circles are the per cent total area, and the curve is $y = 10^{-0.8x+2}$. The code `gxy.m` to find the covering contours above is given in § A.8. The area of the cells decreases from the covering operator, not linearly but with deceleration as Figure 2.5 shows.

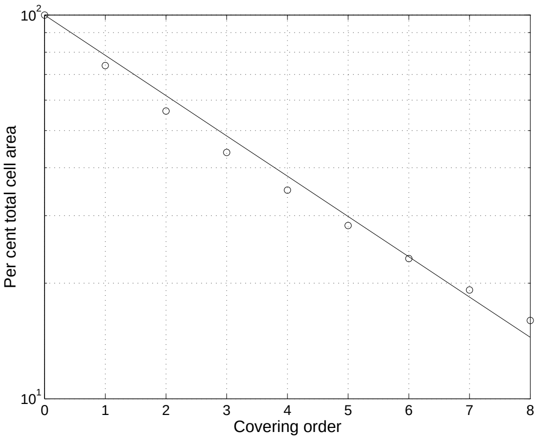


Figure 2.5 The area of multiply covered cells.

On taking a closer look at Figure 2.4, one can interestingly notice that even though all pores shrink from the application of the covering operator, some does so much quicker than others. In particular, round pores shrink but slower. The more corners a pore has the less acute are the angles, which makes it the more stable and thus able to maintain its original size.

Geometrically, one can see that the most unfortunate of all polygons is the triangle. The area of a triangles of any shape reduces by 75 per cent upon being covered. The circle is the most fortunate in this matter since it has no corners and therefore it is impossible for these to be cut. This is in accordance with our argument that corners are unstable region.

§ 2.3 Viscous fingers

Viscous fingering is a happening which occurs when a low viscosity fluid with high pressure penetrates the border of a higher viscosity fluid in a form of thin branching fingers resembling a flash of lightnings in shape. An experimental account as well as descriptive pictures and diagrams are given by Nittmann et al (1985). However, a reference made to one previous work (Hele-Shaw, 1898) was inaccurate in at least three ways, one of which is in the initials of its author, while the rest in the actual contents. The first one can easily be verified with a little research, is that the initials of the author of the paper is H. S., not J. S. S. as given his full name being Henry Selby Hele-Shaw (1854–1941). For the remaining two, firstly the paper by Hele-Shaw is about fluid flow pass free boundary of solid, *not* viscous fingering instability. Secondly, Hele-Shaw was in fact an engineer who has been a professor at University college in Liverpool for 17 years, whose interest was in layer parallel motion (laminar) and sinuous (turbulent) flows pass various bodies of uniform cross section as well as in flows through channels of varying cross section. Born at Billericay, Essex, and a holder of Whitworth scholarships while a student at University college, Bristol, he was elected to the Royal Society in 1899 because of his experiments done on streamline flow. He has successfully introduced the use of air bubbles in experiments to help portray the stream lines of the flow; his cited work mentions neither the physics of nor the application to petroleum science as the context would have us believe. However, his other and subsequent work (Hele-Shaw, 1899) does mention briefly about the importance of viscosity in nature and the difficulty of modelling the motion of viscous fluids mathematically, but nowhere in either one of these two papers did he consider the interaction between fluids of different viscosity.

The simulation for the viscous fingering problem may be related to that for the propagation through rivulets or channels that we study here in § 6.11. It may also be related to the front tracking simulation of dendritic growth. Front tracking has been to solve various problems numerically, for example shocks, flame-, chemical reaction and solidification fronts. In this method a dynamical problem in n dimensions is transformed into a set of partial differential or integro-differential equations which are then solved numerically at a finite set of points on the front or interface (Srolovitz, 1990).

Dendritic growths, on the other hand, are characterised by their rich patterns and share one property that is similar to percolation, which is that it is very difficult, if not impossible, to completely describe the mathematics of their dynamic evolution.

As in other microstructural simulations, the underlying structure where these fingerings occur can be simplified by the mean-field method, which replaces the microstructure by a typical grain, or by making a simple geometrical model to represent it, for example a Voronoi tessellation. The computer simulation can be done using Monte Carlo methods, which describes the energy of the structure in terms of the location of defects and is based on a kinetic rate, or by numerically integrating the equations of motion that describe the evolution of the defects.

§ 2.4 Crystals, quasicrystals, and polycrystals

Crystals, quasicrystals, and polycrystals

The shape of individual grains in polycrystal and the inter granular surface can be found by stereoscopic microradiography, but using random plane section is more convenient in practice (Aboav and Langdon, 1969). The grains of polycrystals are not arranged at random but in a characteristic way which can be expressed in simple terms and seems to be scale-free (Aboav, 1970). The average number of sides of neighbours of a grain is $n_n^e = 5+8/n_e$ where n_e is the number of sides of the grain. Aboav (1970) studies grains of polycrystalline magnesium oxide and finds $n_e = 5.85$. He also finds that most of the time $n_n^e > n_e$ which seems like a contradiction but is because the probability that a point lies in a grain of a particular shape depends on the size and abundance of such grains. He also finds the average number of n-sided grain $n_c^{n-e} \propto (n-2)$. Grains of a polycrystal are different from cells of a soap foam in that they possess a stable grain diameter, which only depends on the temperature.

This stable diameter of grain, d_e, is the average grain diameter at which the growth ceases, and is $d_e^{1/2} = c(T - T_0)$ (Aboav, 1971). For cadmium $T_0 = -53°\text{C}$, $0°\text{C} < T < 170°\text{C}$. The distribution of grain size is $z = z_m \exp\left\{-\alpha^2\left[(x/x_m)^{1/2} - 1\right]^2\right\}$ where z is the number of grain sections in a plane section, α a constant, x the diameter of grain section and x_m the value of x at z_m.

There are similar patterns of grains in a polycrystalline ceramic, magnesium oxide, cadmium, *etc.* The moments of distribution above $n = 6$ is $\mu_m = \sum_n (n-6)^N f_n$ where f_n is the fraction of cells with n sides. The second moment $\mu_2 > 0$ unless all cells have six sides, and we have a purely topological relation $\sum_n n_n^e n_e f_n = \mu_2 + 36$. If $n_n^e \propto 1/n_e$, then $n_n^e = (6-a+b\mu^2/6)+(6a+(1-b)\mu_2)/n$. If $a = 1$ and $b = 0$, this equation is reduced to $n_n^e = 5 + (6+\mu_2)/n$ and furthermore if $\mu_2 = 0$, $n_n^e = 5 + 8n_e$. In a polycrystal the distribution of n_e does not usually vary as the grains grow. Typically $\mu_2 = 2.4$. Soap foams resemble a polycrystal (Aboav, 1980), and $n_n^e = A + B/n_e$, $n_n^e = (6-a) + (6a+\mu_2)/n$, $a = 1.2$.

The growth process of both crystals and quasicrystals are nonlocal in nature, but that of polycrystals is of a multigrain growth. Crystals with a very large unit cell exists. Quasicrystals cannot grow in the same manner as playing a jig-saw puzzle (Penrose, 1989) because no matter how many steps one looks ahead there will come a point where there is a gap that none of the available basic building blocks can fit in. A *legal* tiling contains no gaps or overlaps; if it can be extended to cover the whole plane it is also a *correct* tiling. A *mistake* occurs when a tile added to a correct tiling renders it a legal but incorrect one. By using the basic units to build larger units having the same structure as these basic units and then recursively repeat the process, one can in the end reach a correct tiling. But this is also a nonlocal process since one still requires the ability to look ahead in order to make sure that the units are always structurally the same.

The fact that zeolites are crystalline seems to rule out the randomness assumption that one may otherwise use to model their growths as random continuum percolation. One needs to resort to the methods and tools used by the geologists if one wants to study zeolites, synthetic as well as natural ones.

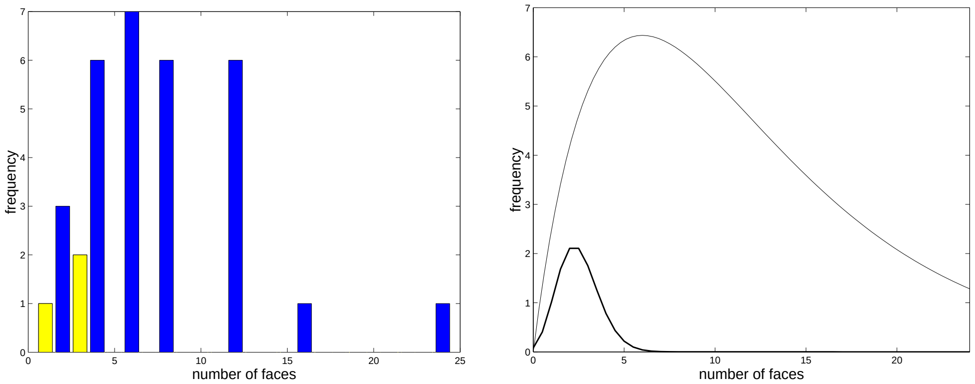
Krýstallos is the Greek word for ice. Crystallography is nowaday the science of the crystalline state which, apart from the crystals themselves, includes such seemingly unlikely materials as plastics, rubber, silk, wool, liquids and gases (*cf* Phillips, 1949). There are four kinds of symmetry axes in crystal models, corresponding to $n = 2$ to 6 in the formula for the angles $\frac{360°}{n}$ by the rotation of which a crystal will repeat itself. These axes are respectively called *diad*, *triad*, *tetrad*, and *hexad*.

Crystals	*centres of symmetry*	*planes of symmetry*	*diad*	*triad*	*tetrad*	*hexad*
Cube	1	9	6	4	3	
Rhombohedron	1	3	3	1		

Table 2.1 *Symmetries of crystals*

The symmetries of the seven crystal systems, the Triclinic contains no axes of symmetry, Monoclinic one diad axis, Orthorhombic three diad axes, Tetragonal one tetrad ais, Cubic four triad axes, Trigonal one triad axis, and Hexagonal one hexad axis. A *monoclinic* crystal has all three axes unequal and one oblique intersection while an *anorthic*, aka *triclinic*, crystal has all three axes unequal and intersecting at oblique angles. The relative development of crystals in different forms give rise to the *habit* of crystal. Thus if one gradually truncates the coigns of a cube one will come to an octahedron, and vice versa. All the continuous transformations between the cube and the octahedron are called *cubo-octahedron.* Crystal models of different habits readily yield on inspection the same symmetry group, but real crystals often look irregular and determination of the symmetry is based on the law of consistency of angle and uses goniometers.

The number of faces in crystals are usually even numbers. Take for example the 33 non isometric crystal forms (Klein and Hurlbut, 1993). Here only three have their faces in odd number, *viz.* pedion which has only one face, and trigonal prism and trigonal pyramid which have three faces each. For the rest, those which have two faces are pinacoid, dome and sphenoid; four faces rhombic prism, tetragonal prism, rhombic pyramid, tetragonal pyramid, rhombic disphenoid and tetragonal disphenoid; six faces ditrigonal prism, hexagonal prism, ditrigonal pyramid, hexagonal pyramid, trigonal dipyramid, trigonal trapezohedron and rhombohedron; eight faces ditetragonal prism, ditetragonal pyramid, rhombic dipyramid, tetragonal dipyramid, tetragonal trapezohedron and tetragonal scalenohedron; twelve faces dihexagonal prism, dihexagonal pyramid, ditrigonal dipyramid, hexagonal dipyramid, hexagonal trapezohedron and hexagonal scalenohedron; sixteen faces ditetragonal dipyramid; and twenty-four faces dihexagonal dipyramid. Among these, there is none which has its faces in a number of 5, 7, 9–11, 13–15, 23, or 25 and above. In Figure 2.6 the distribution curve for even numbers of faces is a scaled Chi-square distribution $y = K \cdot (x/3)^{(\nu-2)/2} \exp(-(x/3)/2)/(2^{\nu/2}\Gamma(\nu/2))$, where the degree of freedom ν is four and $K = 35$, while the curve for odd numbers of faces is the contour of a scaled Poisson distribution $y = K \cdot \lambda^{2x} \exp(-\lambda)/(2x)!$, where $K = 12$, $\lambda = 5$, and $2x$ is any positive integer or zero.



(a) (b)

Figure 2.6 *(a) number of faces of the thirty-three non isometric crystal forms; (b) approximation curves, the heavy line encompasses* $y = 12\exp(-5) \cdot 5^{2x}/(2x!)$ *while the lighter line* $y = (35/12)x\exp(-x/6)$.

Miller indices define the shape of crystals in terms of their faces by numbering their axial intercepts. These indices are $h = a/X$, $k = b/Y$ and $l = c/Z$ where X, Y and Z are respectively the x-, y- and z-intercepts. First, decide the three crystallographic axes. Then choose as the standard or parametral plane a plane that inclines against all these three axes. The shape of crystal, which according to the proportion of each crystal face is called the crystal habit, is governed by the slowest growing face and is affected by the presence of the additive solvent or impurities. Stereoisomerism is the difference in the spatial arrangements of the same atoms and functional groups in a molecule. Enantiomers occur in pairs. They are also known as optical isomers and are mirror images of each other. Polymorphs are substances which can crystallise into different forms the chemical formulae of which are similar to one another. For example, carbon can occur as graphite, diamond, or fullerenes. Isomorphous crystals always appear in one form. In pharmaceutical industries it is important to know the different properties of enantiomorphs of a drug, and to be able to grow each of them separately from the other. Crystalline materials have a tendency to cake or bind together during storage. The dimension of the particles is characterised in one of the following ways: as spheres where $V = \pi d^3/6$ and $A = \pi d^2$; as cubes, $A = 6L^2$, $V = L^3$ and $d = \sqrt{3}L$; as the maximum characteristic chord length of an irregularly shaped particle; as an equivalent diameter or characteristic length. The shape factor for the surface is $f_s = A/L^2$ while for the volume it is $f_v = V/L^3$. Their values are subjected to an inequality constraint, namely $f_s \geq \pi$ and $f_v \geq \pi/6$ where the equality is for the case of a sphere. The ratio $r_f = f_s/f_v$ called the specific surface shape factor, whereas the surface area ratio between that of the sphere and that of the particle of equal volume is called the sphericity, Ψ. The specific surface is $\alpha = A/V$ and consequently $r_f = \alpha L$. The specific surface of particles is the surface area of particles per volume of particles, $\alpha_p = r_f/\bar{d}_s$, while the specific surface of bed is the surface area of particles per total volume of bed, $\alpha_b = (1-\varepsilon)\alpha_p$. The particle size distribution is usually described by the cumulative mass fraction which is the fraction of mass of each particle size, the differential mass fraction $x = \mathrm{d}m/\mathrm{d}d$ or $\int_0^\infty x\mathrm{d}d = 1$, or the size increments. The number of particles in mass fraction x_i is $n_i = x_i/(\rho_s f_v d_i^3)$, where x_i is the ratio of the mass in sizes in the i^{th} interval to the total mass of all particles, and n_i is the product between the number, the volume, and the density of particles. The weight- or volume mean size is

$$\bar{d}_w = \frac{\sum_i x_i d_i}{\sum_i x_i} = \sum_i x_i d_i = \frac{\sum_i n_i d_i^4}{\sum_i n_i d_i^3}, \qquad (\mathbf{16})_{\mathrm{ii}}$$

where $\sum_i x_i = 1$. The mean weight- or volume size, $d_{\bar{w}}$, is the size of each particle of a mono-disperse powder such that $f_v d_{\bar{w}}^3 \sum_i n_i = f_v \sum_i (n_i d_i^3)$, which gives

$$d_{\bar{w}} = \left(\frac{\sum_i (n_i d_i^3)}{\sum_i n_i}\right)^{1/3} = \left(\frac{1}{\sum_i (x_i/d_i^3)}\right)^{1/3}. \qquad (\mathbf{17})_{\mathrm{ii}}$$

The surface mean size† is

$$\bar{d}_s = \frac{\sum_i V_i}{\sum_i S_i} = \frac{\sum_i d_i S_i}{\sum_i S_i} = \frac{\sum_i n_i d_i^3}{\sum_i n_i d_i^2} = \frac{\sum_i x_i}{\sum_i (x_i/d_i)} = \frac{1}{\sum_i (x_i/d_i)}. \qquad (\mathbf{18})_{\mathrm{ii}}$$

The mean surface size is the uniform size of mono-disperse particles which makes the surface area of the particles equal to the surface area of the actual powder. In other words, $\sum_i n_i f_s d_s r^2 = \sum_i n_i f_s d_i^2$, which gives $d_s r = [\sum_i n_i d_i^2 / \sum_i n_i]^{1/2} = [\sum_i (x_i/d_i) / \sum_i (x_i/d_i^3)]$. The specific surface is the surface area of a powder per unit mass or volume, $\alpha = f_s \bar{d}_s^2 / \rho_s f_v \bar{d}_s^3 = r_f/\rho_s \bar{d}_s$. The coefficient of variation (cv) describes the spread of the distribution about the mean, $\mathrm{cv} = \sigma/d$ where d is the mean size and σ standard deviation. The void fraction or voidage, $0 < \varepsilon < 1$, is the ratio of the particulate void volume to the total bed volume which comprises of the volume of voids and solids. Sieve test is used to find the distributions of particles. The mesh number of a sieve is the number of apertures per unit length of sieve, $N = 1/(L+W)$ where L is the aperture size and W the wire width. Crystal defects include point defects, edge- and screw dislocations.

The electrical double layer is $V_d = B' r \exp(-kh)$ where B' is related to the surface charge. The interaction energy from electric field of two charged particles has two minima, $L_1 < L_2$ where L_1 is the primary minimum which occurs at $d_1 < d_2$.

At L_1 the coagulation is rapid whereas at L_2 it is slow. Added polymers may stabilise a colloidal system when the charged adsorbed polymer layers repel each other, or they may destabilise it by making the particles more susceptible to salts or by forming polymer bridges which flocculate the particles.

The superficial velocity, u, is the flowrate per cross-sectional area of bed, $u = Q/A$, and is known as the *velocity flux* or the volumetric flowrate per unit cross-sectional area. The interstitial velocity, v, is the true linear velocity of the fluid. It is the flowrate per unit cross-sectional area of voids, *viz.* $v = Q/(\varepsilon A) = u/\varepsilon$. An ideal particle moves in a fluid with the velocity v as a function of μ, ρ, ρ_s and d. The momentum equation in one dimension for such particle is $\rho_s(\partial/\partial t + v\partial v/\partial z) = B + F - \partial p/\mathrm{d}z$, where B is all the body forces and F all the surface forces acting on the solid phase that is not included in the pressure gradient $-\partial p/\partial z$. The drag force per unit area on a single lone particle is $F/A = c_\infty \rho v^2/2$ where F is force, A the projected area, and c_∞ the single particle drag coefficient at infinite dilution. For sphere, $F = c_\infty \pi d^2 \rho v^2/8$. The force balance equation is $W - B - F = \dot{M}$ or $mg - m'g - F = m\mathrm{d}v/\mathrm{d}t$ where m and m' are the mass of particle and fluid displaced, and thus the second term represents the buoyancy. Balancing the accelerative and the resistive forces leads to $(\pi/6)d^3(\rho_s - \rho)g = c_\infty \pi d^2 \rho v_t^2/8$, where v_t is the terminal velocity, *i.e.* when $\mathrm{d}v/\mathrm{d}t = 0$, which gives $v_t = [4d(\rho_s - \rho)g/3\rho c_\infty]^{1/2}$. The Reynolds number of the particle is $\mathrm{Re} = \rho v d/\mu$. When $\mathrm{Re} < 0.2$ the flow is laminar and Stokes' law applies and the force balance equation becomes $(\pi/6)d^3(\rho_s - \rho)g = 3\pi\mu d v_t = F$, which gives $v_t = d^2 g(\rho_s - \rho)/18\mu$, $c_\infty = 24/\mathrm{Re}$ and the Stokes diameter $d_s t = [18\mu v_t/g(\rho_s - \rho)]^{1/2}$.

The mass flux is defined as the multiplicative product of concentration and velocity. In laminar flow, Darcy's law applies, that is $u = \mathrm{d}V/A\mathrm{d}t = K_1 \Delta P/H$, where K_1 is the bed permeability, a measure of the total drag force. The inverse linear relationship between the viscosity and permeability is $u = B\Delta P/\mu H$, where $B = K_1\mu$ is the permeability coefficient in m^2. Let the pore diameter be d_1, its length H_1 and velocity v. Assume that the fluid in pore velocity is $v = u/\varepsilon$, the hydraulic pore diameter is δ is related to $d_1 = v_b\varepsilon/v_b s_b = \varepsilon/s_b$, and the pore length H_1 depends on the bed depth H. For laminar flow through a pipe with viscous drag the Hagen-Poisseuille equation applies, that is $v = d_1^2\Delta P/(32\mu H_1)$. For the capillary model, $v = u/\varepsilon = \varepsilon^2\Delta P/(K_2 s_b^2 \mu H)$. The bed specific surface area, s_b, and the particle specific surface area, s_p, are related to each other by $s_b = s_p(1-\varepsilon)$, and thus $u = v\varepsilon = \varepsilon^3\Delta P/(K_2 s_p^2(1-\varepsilon)^2\mu H)$, the Carmen-Kozeny equation, from where $B = \varepsilon^3/(K_2 s_p^2(1-\varepsilon)^2)$, where the Kozeny constant K_2 lies between 3.5 and 5.5 and is normally taken to be 5. In compressible beds voidage is a function of P or ΔP and the Carman-Kozeny equation for $\mathrm{d}x$ is $-\mathrm{d}p/\mathrm{d}x = K_2\mu(1-\varepsilon)^2 s_p^2 u/\varepsilon^3$. For turbulent flow $\Delta P/\Delta H = K'(1-\varepsilon)\rho u^2/(\varepsilon^3 d)$, while a linear sum of the laminar and turbulent flow is $\Delta P/\Delta H = K^3(1-\varepsilon)^2\mu u/(\varepsilon^3 d^2) + K_4(1-\varepsilon)\rho u^2/(\varepsilon^3 d)$ where $K_3 = 150$ and $K_4 = 1.75$ (Jones, 2002). Across a filter cake the modified Darcy equation is $u = \mathrm{d}V/A\mathrm{d}t = \Delta p_c/(r\mu H)$ where V is the volume of filtrate which passed through the cake. If the specific cake volume, *i.e.* the volume of cake per unit volume of filtrate, is v_c, then the total volume of cake is $Vv_c = HA$ where H is the thickness or height of the cake. Then $\mathrm{d}V/\mathrm{d}t = A^2\Delta p_c/(r\mu v_c V)$. This, together with the Carman-Kozeny equation for packed beds $u = \varepsilon^3\Delta p_c/[5(1-\varepsilon)^2 s_p^2 \mu H]$, gives the specific cake resistance $r = 5(1-\varepsilon)^2 s_p^2/\varepsilon^3$ There are two modes of filtration, the constant rate- and the constant pressure modes. In the former $\mathrm{d}V/\mathrm{d}t$ is constant and, from $V/t = A^2\Delta p_c/(r\mu v_c V)$ which is also constant, it follows that $\Delta p_c/V$ is also constant. In the latter Δp_c is constant and equals to Δp_{max}. It follows that $V^2 = 2A^2\Delta p_c t/(r\mu v_c) = k_1 t$ and $t/V = r\mu v_c V/(2A^2\Delta p_c) = k_2 V$ When the effect of the filter medium is considered Δp will comprise of two components, from the cloth and from the cake. The first one includes also the particles held in the filter. The resistance is $R = r(H + L)$ where L is the equivalent cake thickness of the filter. The filtration equation becomes $\mathrm{d}V/A\mathrm{d}t = \Delta p/[r\mu(H + L)]$, in other words $\mathrm{d}V/\mathrm{d}t = A^2\Delta p/[r\mu v_c(V + LA/v_c)]$ For the constant rate filtration $V_1^2 + (LA/v_c)V_1 = (A^2\Delta p/r\mu v_c)t_1$, while for the constant pressure filtration $(t - t_1)/(V - V_1) = [r\mu v_c/(2A^2\Delta p)](V - V_1) + r\mu v_c V_1/(A^2\Delta p) + r\mu L/(A\Delta p)$. Agglomeration of crystals can be on either an individual or collective basis. The first one is sometimes called primary agglomeration examples of which are parallel, dendritic and twin growths. The second one is also known as secondary agglomeration and is essentially those cases where crystals within the solution come together to form clusters. Secondary agglomeration can be perikinetic, from Brownian motion of small particles, or orthokinetic, from velocity gradients in the fluid. Crystals flocculate or coagulate loosely together, then they aggregate by starting to join one another, and then they agglomerate strongly together to become a single poly-crystalline particle. There are four main classes of forces, namely colloidal, stochastic, direct- and indirect systematic forces. Surface adsorbs certain ionic species which in turn attract opposite ions, thus forming electrical double layers which repel each other There are three types of particle interaction, that is van de Waals, electrostatic and steric. These depend on shape and size of the particle, surface charge, solution's pH and ionic strength, temperature and the separation distance between the particles. The van der Waals forces are between molecules having closed shells and do not obey inverse square laws. Three of these are attractive, namely the orientation effect between permanent dipoles, the induction effect between a permanent- and a temporary dipoles, and the dispersion effect or London force between temporary dipoles and induced dipoles. The potential equations is $V_v = -A_{12}\mathrm{f}(h)$ where A_{12} is the Hamaker constant of the material.

---

† aka volume mean or Sauter mean.

§ 2.5 Random lines and percolation

Miles (1964) studies the line system, $p = x\cos\theta + y\sin\theta$, where $-\infty < p < \infty$ and $0 \le \theta < \pi$. Chan (1990) studied numerical properties of such structure of random lines using the same generator and then used it to model the structure of non-woven media. This amounts to the study by simulation of filters the internal structure of which is fibre mat, for example polypropylene. The algorithm he developed deals with mechanical blocking and back-flushing of the fibres. Wilcock's study (1994), on the other hand, deals with woven materials. With them she tried to demistify the myth and mysteries surrounding the rule of thumb techniques in designing wire mesh demisters. Her algorithms first weave the mesh and then apply Monte Carlo to it for the simulation.

The steel and carbon fibres in cement pastes contributes respectively to hole- and electron conductions (Wen and Chung, 2000). This is a problem of percolation of interconnected random lines. On the other hand, we can also study the resistance instead of conductance, which is what Pennetta *et al* (2000) does for a random network of resistors. Interest in researches on random resistor or conductor networks is strong (for example, Pennetta *et al*, 2001). Cheng *et al*(2001) look at NiMH (nickel metal hydride) batteries' electrodes. In their presentation they wrap all the ends of those fibres that stick out from the prototile. This gives the periodicity which they use in their simulation. The NiMH has high porosity, no less than 97 per cent, and it contains two or more phases.

Some recent developments related to polypropylene are those done by Mironi-Harpaz and Narkis (2001), Narkis *et al* (2000), and Zhang *et al* (2001). Those who study the conductivity and resistivity of materials are, for example, Benoit *et al* (2001), Broderix *et al* (2001), Flandin *et al* (2001), Hindermann-Bischoff and Ehrburger-Dolle (2001), Huang *et al* (2001), Hunt (2001), Jevtic *et al* (2001), Nielson (2000), Petrovsky and Rak (2001), and Stenull *et al* (2001). An example of the application to dentistry is that which is reported by Sharp *et al* (2000). Rong *et al* (2001) look at tensile curves and study the resistance to thermal deformation in nanocomposites.

§ 2.6 Convex hull

The set of extreme points E in some superset S is the smallest subset of S such that the convex hulls of both E and S are identical. Extreme points never lie in a triangle.

The Graham's scan algorithm (*cf* Preparata and Shamos, 1985) positions itself in the midst of the points and then scans around in one direction. It determines three points in turn, and rejects a mid point among the three if the angle made there is reflexive, *i.e.* α such that $\alpha \geq \pi$ in the anti-clockwise direction. In other words, an angle is a right turn if it is reflexive; it is a left turn otherwise. The algorithm is shown in Algorithm 2.3. Here $\{h\}$ is a stack which contains the points on convex hull

Algorithm 2.3 *Graham's scan.*

```
$i \leftarrow 1$;
$j \leftarrow 2$;
$k \leftarrow 3$;
while there exist unprocessed points do
  if $\angle p_i p_j p_k \geq \pi$ then
    $j \leftarrow i$;
    $i \leftarrow i - 1$;
  else
    $i \leftarrow j$;
    $j \leftarrow k$;
    $k \leftarrow k + 1$;
  endif
endwhile
```
□

The algorithm starts from a known extreme point. It can also start from two points which are known to be extreme, in which case the space is divide into an upper- and a lower hulls. Similarly to the Graham's scan, Jarvis's march finds one extreme point after another as it wraps a line around the convex hull. The Quickhull algorithm in two dimensions is described in Algorithm 2.4.

Figure 2.4 *Quickhull in two dimensions.*

```
$l \leftarrow$ the point with the smallest abscissa;
$r \leftarrow$ the point with the largest abscissa;
$\{s\} \leftarrow$ all points above $\overline{lr}$;
$\{s\} \leftarrow$ all points below $\overline{lr}$;
while there remains an unprocessed $s_i$ in $\{s\}$ do
  $h \leftarrow$ point in $s_i$ which maximises the area of $\triangle hlr$;
  reject points bound by $\triangle hlr$;
  $\{s\} \leftarrow$ all points outwards from $\overline{lh}$;
  $\{s\} \leftarrow$ all points outwards from $\overline{rh}$;
endfor
```
□

The divide-and-conquer algorithms divides a problem into subproblems, finds the convex hull for each one of them and then merges these together by finding the convex hull of convex hulls. Algorithm 2.5 finds the convex hull of two convex hulls.

Algorithm 2.5 *Convex hull of convex hulls*

```
$p \leftarrow$ one point in $h_1$;
```

```
if $p$ is also in $h_2$ then
    $h \leftarrow$ scan around from $p$, and merge $h_1$ and $h_2$;
else
    $(u, v) \leftarrow$ points on $h_2$ such that $\angle upv$ is maximised;
    $c \leftarrow$ the chain from $u$ to $v$ which is furthest away from $p$;
    $h \leftarrow$ merge $c$ and $h_1$;
endif                                                                  □
```

There are also dynamic algorithms for finding the convex hulls. In this case the input is online and one can not look ahead at the input. This kind of algorithms may be useful for online applications, for example the traffic control in real time. Two things can happen in such dynamic algorithms; points are inserted or points are deleted.

The gift-wrapping methods, which are similar to the Jarvis's march, can be extended to the general d dimensions. Here a point is beneath a facet if it is on the same side of it as the hull, otherwise it is beyond it. Also, with respect to a point p, if p is beneath all facets that contain v then v is concave; if p is beyond the same then v is reflex; otherwise v is supporting.

§ 2.7 From convex hull to the Voronoi tessellation

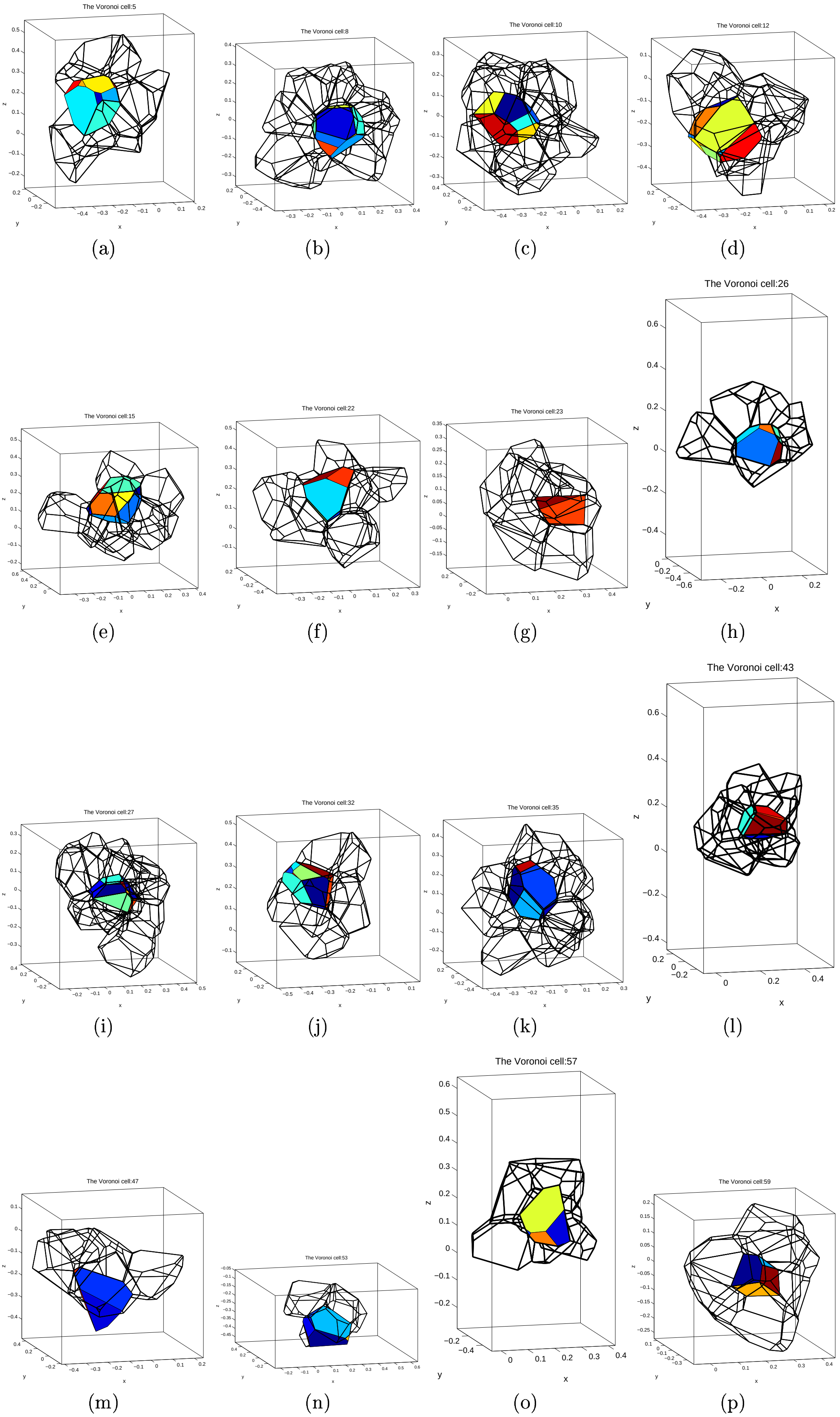
The earlier studies of the Voronoi structure used the programme `qhull` to create the structures. Programmes were developed on Matlab to manipulate the data and find the statistics. These programmes were tested on small structures first. Contrary to what the literature says, these very small networks turned out to have all the statistics in close agreement with those much larger ones. This confirms an earlier report (Tiyapan, 1995, KNT3(iii)) that the average values of the statistics stays the same over a wide range of network sizes. So a Voronoi is a Voronoi, as one could have said, almost like an Englishman.

Voronoi tessellation is the solution of a proximity problem, namely the division of the space into n partitions around n particles, such that all points within the i^{th} partition is closest to the i^{th} particle than any other particle. There are a host of proximity problems which, in the end, are related to one another and to the Voronoi problem. Some example of these are the problems concerning the nearest neighbour, the closest pair and the Euclidean minimum spanning tree. The minimum spanning tree always contains the shortest edge of the graph.

Given a convex hull containing n points, one can join all the points together by straight line segments such that the whole region inside the convex hull is tessellated by the triangles formed by them. This problem is related to the nearest neighbour problem, since among all straight lines connecting to each point there is one which joins it to its nearest neighbour. Moreover, the problem is related to a problem of spatial proximity the solution of which is the Voronoi tessellation. The solutions of these two problems are dual to each other. The triangular tessellation is called the Delaunay triangulation.

Descartes was the first person to draw a picture of a Voronoi tessellation (Descartes, 1644). In his essay, he imagine vorticities surrounding heavenly bodies. The path of an object through space, he says, passes along edges and vertices of what is now known as the Voronoi tessellation. The idea he introduced was original but the discourse philosophical, which is perhaps why his name was never associated with the tessellation which could easily have borne his name instead of that of Voronoi. Even though we regard Philosophy very highly as the mother of all sciences, as is probably the reason why we call our highest formal education 'Doctor of Philosophy' or Ph.D., but from our experience we could see that philosophy in our dictum generally means only one thing, that is mathematics. Therefore the tessellation is named after Voronoi because he was the first person to have written a *substantial* amount of mathematics on it. In short, Descartes has provided the idea and philosophy, Dirichlet the geometrical description and Voronoi the mathematics. Voronoi could easily have claimed having written the most amount of mathematics on the tessellation which now bears his name, than any other person to date. To see the contest between Voronoi and Dirichlet, for instance, compare their seminal papers (Dirichlet, 1850; Voronoi, 1908, 1909) (*cf* Tiyapan, 2001, KNT8(i)). Ironically it was Descartes' philosophy that all knowledges must be based on mathematics, so he should not have minded.

The pictures of all Voronoi cells in a group of 71 cells are given in Figure 2.7. These are all the inner cells of a larger group of 200, the rest of which are cells on the boundary.

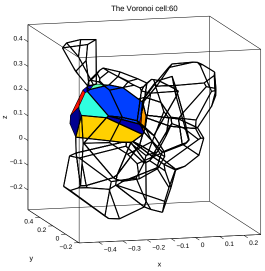


(a) (b) (c) (d)

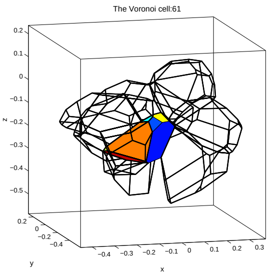
(e) (f) (g) (h)

(i) (j) (k) (l)

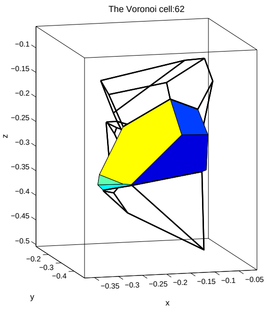
(m) (n) (o) (p)



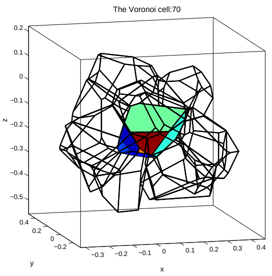
(q)



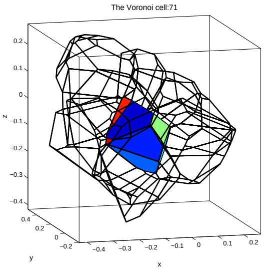
(r)



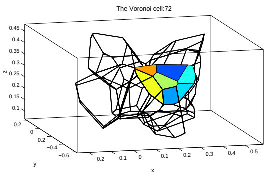
(s)



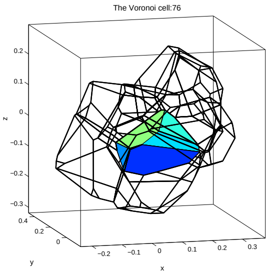
(t)



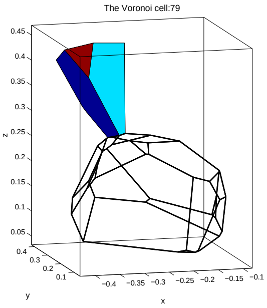
(u)



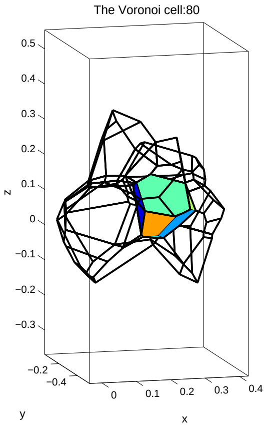
(v)



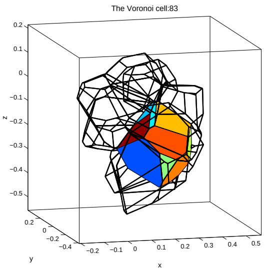
(w)



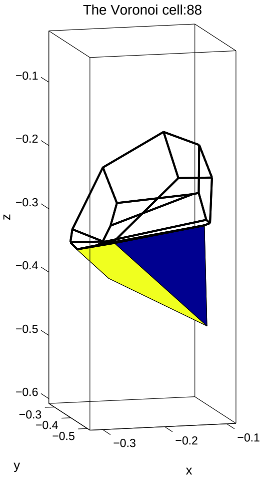
(x)



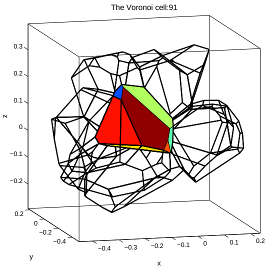
(y)



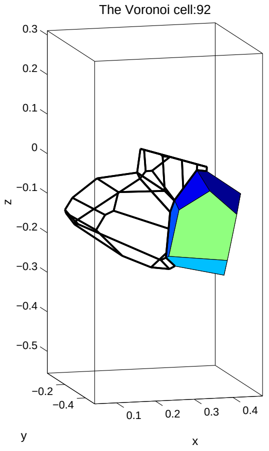
(z)



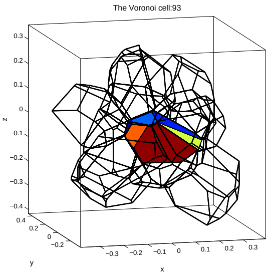
(A)



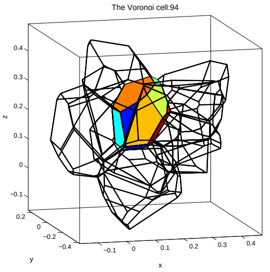
(B)



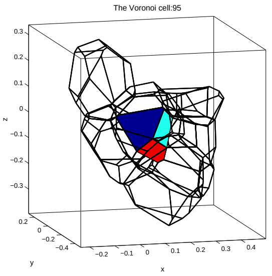
(C)



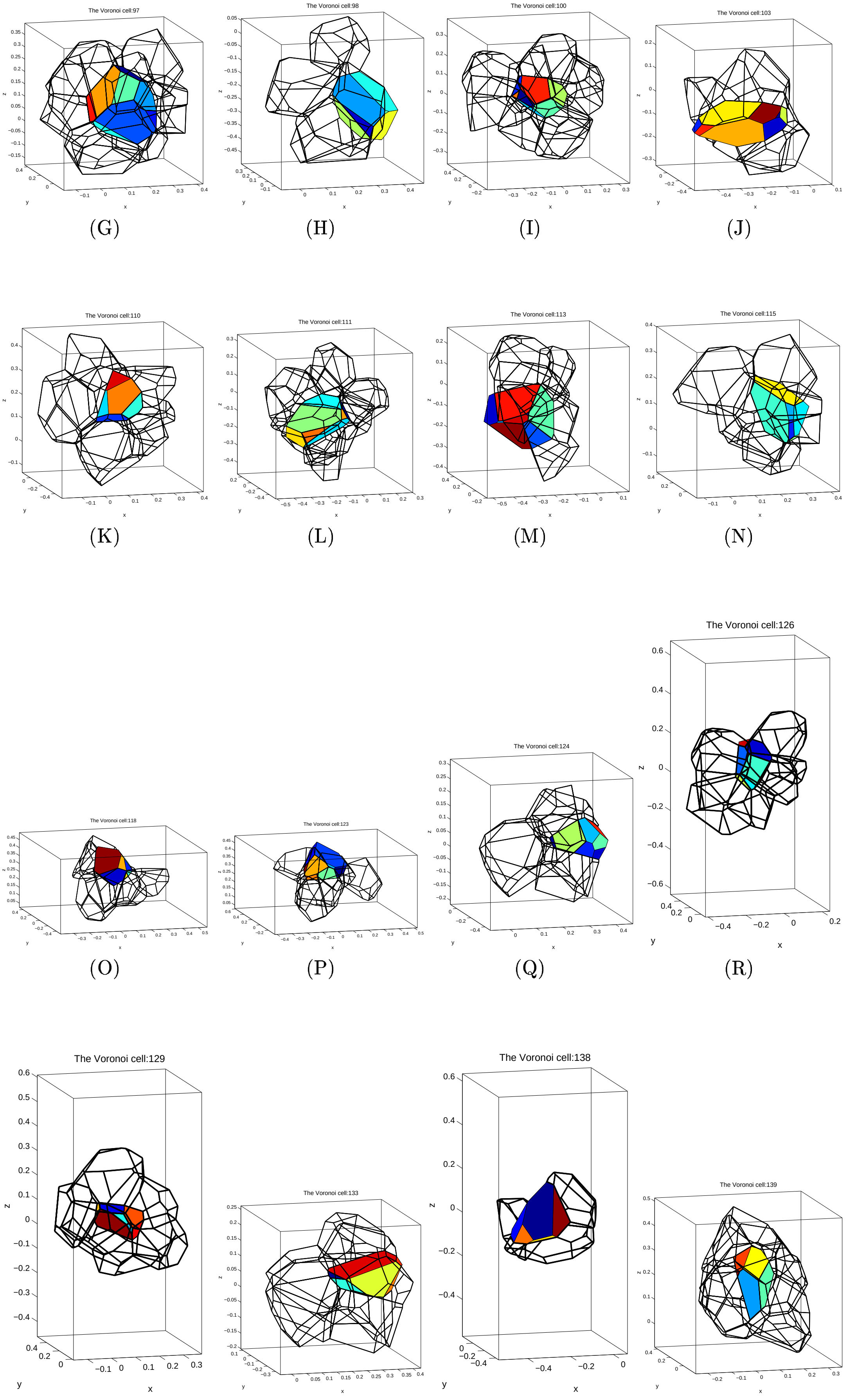
(D)



(E)



(F)

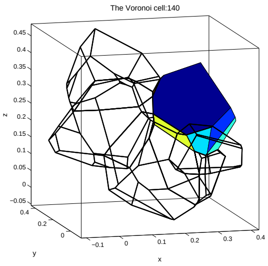


(G) (H) (I) (J)

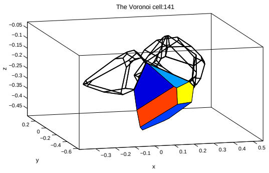
(K) (L) (M) (N)

(O) (P) (Q) (R)

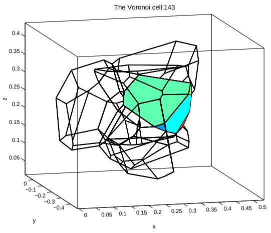
(S) (T) (U) (V)



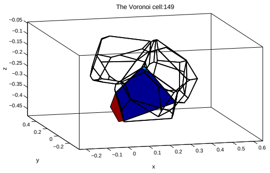
(W)



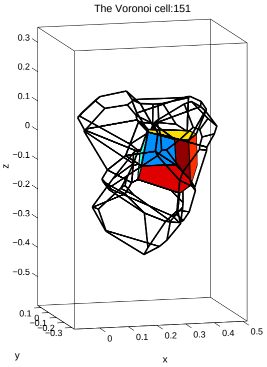
(X)



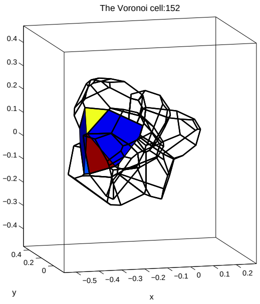
(Y)



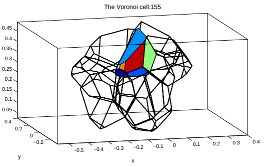
(Z)



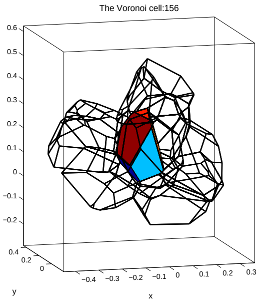
(aa)



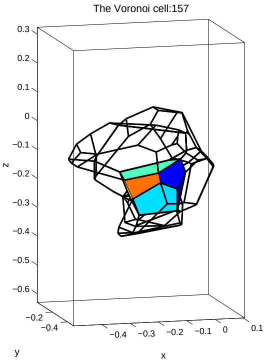
(ab)



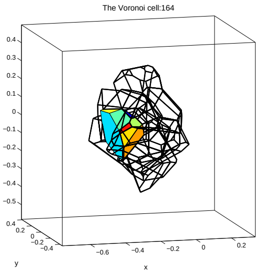
(ac)



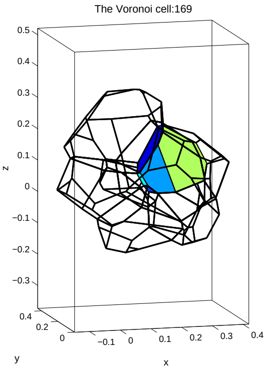
(ad)



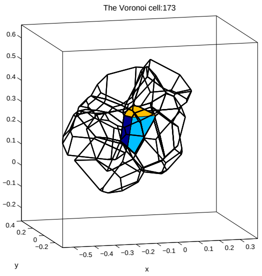
(ae)



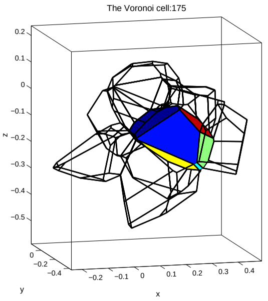
(af)



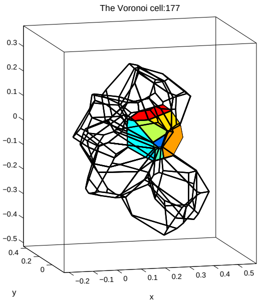
(ag)



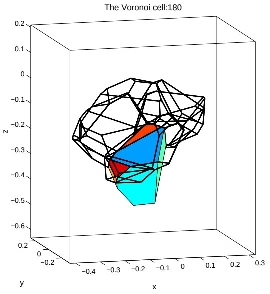
(ah)



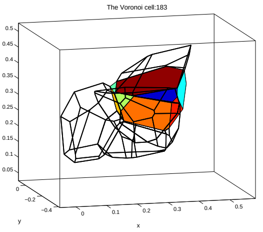
(ai)



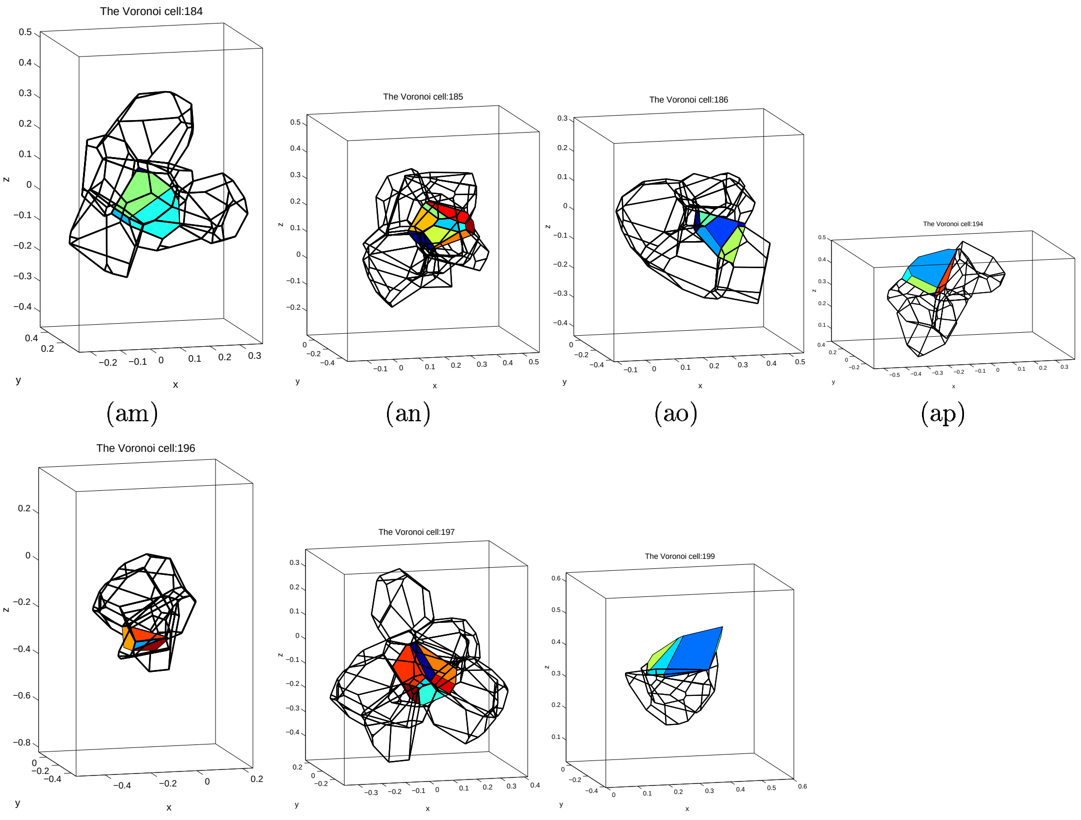
(aj)



(ak)



(al)



(am) (an) (ao) (ap)

(ar) (as) (at)

Figure 2.7 *The 71 Voronoi cells in an aggregation.*

The statistics of these 71 cells are the following. There are originally 200 cells, but these statistics represent 71 inner cells among these. The rest, 129 cells, are on the boundary and are not considered. The abbreviation 'mn' stands for the mean-, while 'cb' the cube normal.

Number of faces per inbound cell

Mean: 14.2113
Variance: 11.0262
Minimum: 6
Maximum: 23
Second moment: 10.8709
Third moment: 14.0161
Kurtosis: 3.109624
Geometric mean: 13.8246
Harmonic mean: 13.4227
Median: 13
Mean absolute deviation: 2.7169

Number of vertices per inbound cell

Mean: 24.4225
Variance: 44.1046
Minimum: 8
Maximum: 42
Second moment: 43.4834
Third moment: 112.1287
Kurtosis: 3.109624
Geometric mean: 23.5039
Harmonic mean: 22.5058
Median: 22
Mean absolute deviation: 5.4338

Area of surface of inner cell

Mean: 0.1599
Variance: 0.0024
Minimum: 0.0628
Maximum: 0.2660
Second moment: 0.0023
Third moment: 7×10^{-6}
Kurtosis: 2.3015
Geometric mean: 0.1520
Harmonic mean: 0.1435
Median: 0.1602
Mean absolute deviation: 0.0398

Area of surface of inner cell (mn)

Mean: 1.000000
Variance: 0.0926
Minimum: 0.3928
Maximum: 1.6634
Second moment: 0.0913
Third moment: 0.0017
Kurtosis: 2.3015
Geometric mean: 0.9505
Harmonic mean: 0.8972
Median: 1.0018
Mean absolute deviation: 0.2489

Area of surface of inner cell (cb)

Mean: 0.3031

Variance: 0.0085
Minimum: 0.1191
Maximum: 0.5042
Second moment: 0.0084
Third moment: 4.8×10^{-5}
Kurtosis: 2.3015
Geometric mean: 0.2882
Harmonic mean: 0.2720
Median: 0.3037
Mean absolute deviation: 0.0754

Volume of inbound cell
Mean: 0.0261
Variance: 0.0018
Minimum: 5.5944×10^{-3}
Maximum: 0.3315
Second moment: 0.0017
Third moment: 0.0004
Kurtosis: 42.2383
Geometric mean: 0.0183
Harmonic mean: 0.0153
Median: 0.0167
Mean absolute deviation: 0.0177

Volume of inbound cell (mn)
Mean: 1.0000
Variance: 2.5764
Minimum: 0.2146
Maximum: 12.7188
Second moment: 2.5402
Third moment: 24.1689
Kurtosis: 42.2383
Geometric mean: 0.7012
Harmonic mean: 0.5879
Median: 0.6408
Mean absolute deviation: 0.6795

Volume of inbound cell (cb)
Mean: 1.0000
Variance: 2.5764
Minimum: 0.2146
Maximum: 12.719
Second moment: 2.5402
Third moment: 24.1689
Kurtosis: 42.2383
Geometric mean: 0.7012
Harmonic mean: 0.5879
Median: 0.6408
Mean absolute deviation: 0.6795

Number of vertices per inbound face
Mean: 5.1114
Variance: 2.1650
Minimum: 3
Maximum: 10
Second moment: 2.1619
Third moment: 1.8902
Kurtosis: 3.0234
Geometric mean: 4.9065
Harmonic mean: 4.7086
Median: 5
Mean absolute deviation: 1.1655

Perimeter of inner face
Mean: 0.3899
Variance: 0.0434
Minimum: 0.0017
Maximum: 1.0067
Second moment: 0.0433
Third moment: 9.88×10^{-4}
Kurtosis: 2.3090
Geometric mean: 0.3071
Harmonic mean: 0.1497
Median: 0.3940
Mean absolute deviation: 0.1720

Perimeter of inner face (mn)
Mean: 1.0000
Variance: 0.2854
Minimum: 0.0043
Maximum: 2.5819
Second moment: 0.2850
Third moment: 0.0167
Kurtosis: 2.3090
Geometric mean: 0.7877
Harmonic mean: 0.3839
Median: 1.010419
Mean absolute deviation: 0.4410

Perimeter of inner face (cb)
Mean: 0.3288
Variance: 0.0308
Minimum: 0.0014
Maximum: 0.8488
Second moment: 0.0308
Third moment: 0.0006
Kurtosis: 2.3090
Geometric mean: 0.2590
Harmonic mean: 0.1262
Median: 0.3322
Mean absolute deviation: 0.1450

Area of faces of inner cell
Mean: 0.0110
Variance: 0.0001
Minimum: 4.0594×10^{-8}
Maximum: 7.2892×10^{-2}
Second moment: 1.22×10^{-4}
Third moment: 2×10^{-6}
Kurtosis: 5.2926
Geometric mean: 0.0045
Harmonic mean: 2.2×10^{-5}
Median: 0.0074
Mean absolute deviation: 0.0087

Area of faces of inner cell (mn)
Mean: 1.0000
Variance: 1.0085
Minimum: 3.6840×10^{-6}
Maximum: 6.6151
Second moment: 1.0071
Third moment: 1.4244
Kurtosis: 5.2926

Geometric mean: 0.4086
Harmonic mean: 0.0020
Median: 0.6715
Mean absolute deviation: 0.7933

Area of faces of inner cell (cb)
Mean: 0.1253
Variance: 0.0158
Minimum: 4.6179×10^{-7}
Maximum: 0.8292
Second moment: 0.0158
Third moment: 0.0028
Kurtosis: 5.2926
Geometric mean: 0.0512
Harmonic mean: 0.0003
Median: 0.0842
Mean absolute deviation: 0.0994

§ 6.10 The first part, suspended particles

All two dimensional lattices are tilings. The *kagome* lattice deserves some mention here since it is a first-order covering lattice of the honeycomb lattice, which in turn is dual to the triangular lattice. Some physicists conjecture that the name originates from a name of a person. Apart from a few exceptions from the reason that the word is written in the wrong script, those scientists who are japanese should know that the word from their own language which means, according to the dictionary by Nelson (1962), *basket interstices* or *woven bamboo pattern.* A better translation is simply *basket pattern* (*cf* Tiyapan, 2001, KNT8(i)), *kago* (*rad*: *take* and *tatsu*) meaning *basket* and *me* a radical word for *eye* which also means *pattern.*

Those who do not know follow the established practice and write it as *kagomé* (*cf*, for example, Tiyapan, 1995, KNT3(iii)) with the *l'accent aigue* (acute accent) as if it were a french word. For the case of two dimensions, Algorithm 2.6 gives the algorithm for producing patterns in general. It is used for producing the 2-homohedral tilings in § 4.9 and is the basis of the programme in § A.6. In this algorithm pseudo-prototiles are produced which fill the space. We shall call these our unit tiles. There are four types of these unit tiles corresponding to the four sections or groups as shown in Figure 2.8. The first group contains only one tile, *i.e.* the one at the origin, the second one contains those unit tiles to be put at the bottom row, the third one yet those to be put at the left-most column. The rest and majority of tiles belong to the fourth group.

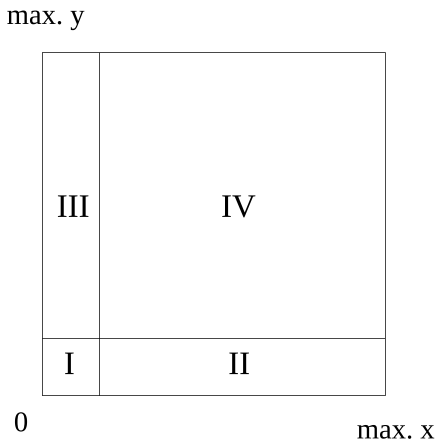


Figure 2.8 Four groups of tiles.

Vertices in each unit tile are divided into five groups, namely one for the boundary in each direction, *i.e.* north, east, west and south, and the fifth those in the midst of the unit tile. This is essentially in order to avoid creating a vertex twice, which would have resulted in duplicates. Those unit tiles in the fourth group take their left-most vertices from the unit tile to their left and their bottom-most vertices from the unit tile immediately below it. Unit tiles in the third group create their own left-most vertices but still take from the unit tile below them their bottom-most ones. On the other hand, unit tiles in the second group create their bottom-most vertices while taking their left-most ones from their neighbour who lives on their left. Lastly the only unit tile of the first group creates all its vertices, which include the bottom-most and left-most ones.

There are three types of edges, namely internal-, left and bottom edges, the latter two of which link respectively to vertices in their left and bottom neighbours. And then, there are four types of cells, namely middle, south, west and south-west cells, the latter three of which contains the other half of its edges in their neighbour in the corresponding direction. Each unit tile is divided into grids by lines parallel to x and y axes. All vertices lie on some intersection of these grids.

Algorithm 2.6 *Tilings in two dimensions.*

```
for every unit tile do
```

```
    decide its type;
    assign all vertices according to its type;
    connect bonds according to the connection rules;
    define cells by their vertices;
endfor
```
□

When faced with an unfamiliar tiling, we first decide upon their pseudo-prototile, that is their unit tile. Then list the coordinates of their vertices. And then list the bonds, that is the numbers of the two vertices that define each of them. We also have to divide the bonds into groups as mentioned, and include this information in our input.

The programme keeps the two lists of coordinates, one for each axis. Coordinates of vertices are then referred to in grid numbers instead of the actual lengths concerned. This helps reading and using the programme and procedure simpler. It marks the division between delight and despair. Also, the group information can then be represented in the programme as a mapping of border vertices from a previous unit to the present one.

A cross-border bond normally has one end on the border. However there are those neither ends of which are on the border. These bonds make up a separate group of their own, or rather they make up two groups corresponding to the two groups, II and III.

This programme when tried on a few well known regular lattices gives satisfactory results. Because these lattices are small, their coordination numbers x differ from the true values. But since this must be because of the jagged borders of the networks created the discrepancy is systematic and so there is nothing to worry about. Also there is nothing one can do about it unless one makes changes to the programme. The problem lessens but never goes away when the size of the network becomes larger. As the four sides that make up the border are in total $4d/d^2$, *i.e.* $4/d$ parts of the area, increasing the network size by one hundred times, *i.e.* $d = 10$, probably would reduce the error by 60 per cent.

It is possible to improve the programme as regarding to this problem, and this is the plan for the future work. At present, this shortcoming will have some effect on the values of the six percolation probabilities obtained from each system simulated.

For the first test, the programme was run on a square lattice. The six representative networks produced, which subsequently become the ground for the corresponding six values of percolation thresholds, have their statistics as follows:

$n_C = 100$, $x_C = 6.8400$, $n_B = 342$, $x_B = 12.4211$, $n_c = 100$, $x_c = 3.6000$, $n_b = 180$, $x_b = 5.3778$, $n_v = 121$, $x_v = 3.3058$, $n_e = 200$ and $x_e = 5.4100$.

Then from 2×5 runs, *i.e.* two runs for each of the five permuted list of blockages, the values of percolation thresholds are

$p_C = 0.3920 \pm 0.0627$, $p_B = 0.2459 \pm 0.0373$, $p_c = 0.5610 \pm 0.1272$, $p_b = 0.4806 \pm 0.0750$, $p_v = 0.5942 \pm 0.0802$ and $p_e = 0.5040 \pm 0.0740$.

To explain the results, the capital C and B in the subscript mean respectively cells and bonds when neighbours mean that they share at least one vertex. When in lower case letters c and b mean respectively cells and bonds when neighbourhood means sharing at least one edge. Neither the vertices nor the edges, respectively v and e, is ambiguous since the former has zero dimension while the latter has only one. In this case, as it is in general, p_C and p_B are nothing that one normally talks about, while p_c and p_b should be the same as p_v and p_e in that order. Because all connections end at the boundary, it is to be expected that the values of all x's are lower than their exact values for an infinite network. The results above show that, for this case at least, x_c falls short of its exact value by 10 per cent, while similarly x_b by 10.37, x_v by 17.35 and x_e by 9.83 per cent. Because c and b naturally form one pair while v and e another, it is interesting to note that x_c should fare better than x_v while on the other hand x_e is more accurate than x_b. For the probability values, p_c is off the mark by (*cf* § 4) -5.36 per cent and p_b by -3.9 per cent, while p_v does so by 2.5 per cent and p_e by a mere 0.8 per cent. The first one of these pairs seems to be on the lower side while the second one, on the other hand, is on the higher end and more accurate than the first. This first test gives a result in accordance with our expectation that for vertices and edges the results should be more accurate than those from cells and bonds since the first two come from the input, while the last two are secondary values derived up from them by the programme. Figure 2.9 shows the networks simulated for this test.

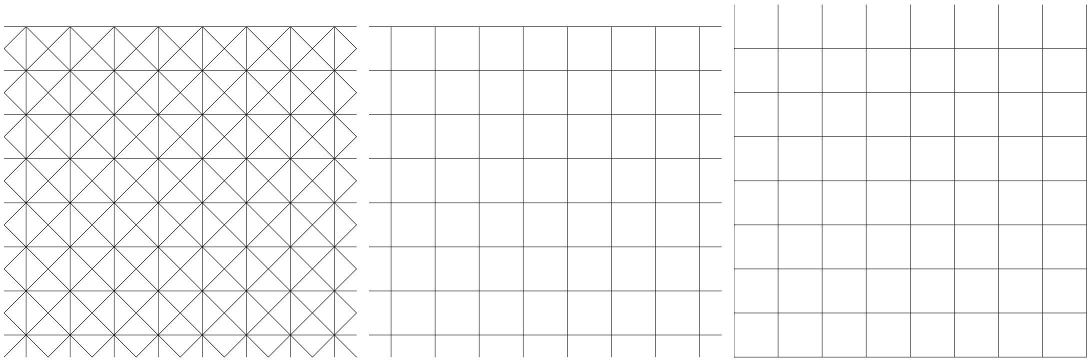

(a) (b) (c)

Figure 2.9 *Percolation of networks related to the square lattice. Networks comprising of* (a) *Cells and Bonds,* (b) *cells and bonds, and* (c) *vertices and edges.*

The next test is on the honeycomb lattice. Here the results obtained from the simulation is $n_C = n_c$ 202, $x_C = x_c$ 5.4554, $n_B = n_b$ 551, $x_B = x_b$ 9.2777, $n_v = 479$, $x_v = 2.8058$, $n_e = 672$ and $x_e = 3.8244$. For the probabilities of percolation $p_C = p_c$ 0.4889 ± 0.0704, $p_B = p_b$ 0.3342 ± 0.0385, $p_v = 0.6833 \pm 0.0369$ and $p_e = 0.6382 \pm 0.0410$. These results are obtained from 2×10 simulations in the case of statistics on cells and bonds, whereas in the case of vertices and edges they are obtained from 2×5 runs. Their pictures are shown in Figure 2.10.

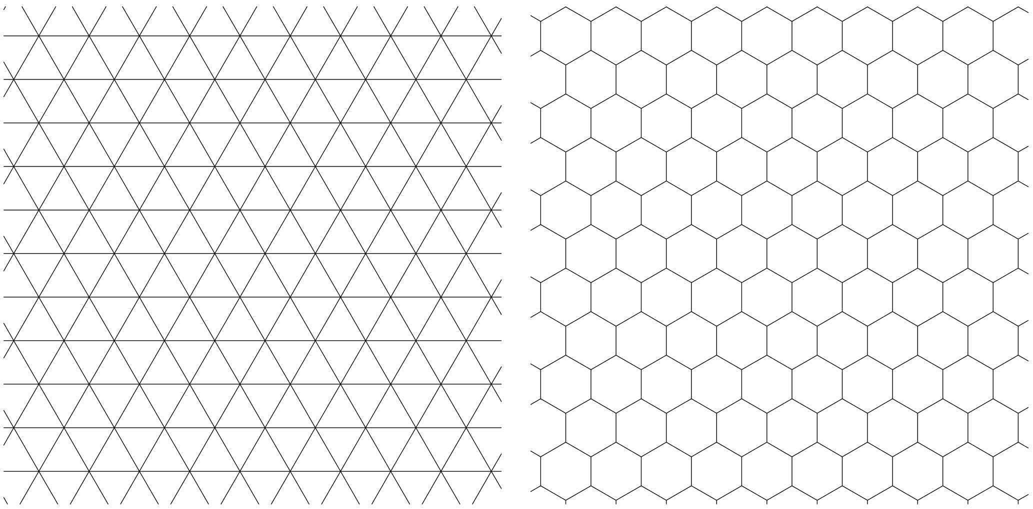

(a) (b)

Figure 2.9 *Percolation of networks related to the honeycomb lattice. Networks are made up of* (a) *Cells and Bonds, or cells and bonds,* (b) *vertices and edges.*

For the honeycomb lattice, the errors are for p_c -2.22, for p_b -3.77, for p_v -1.85 and for p_e -2.22 per cent. The errors for the network statistics are for $x_c = -9.08$, $x_b = -7.22$, $x_v = -6.47$ and $x_e = -4.39$ per cent.

The Kagome lattice is the next test. Here the statistics obtained for the networks are

$n_C = 230$, $x_C = 6.4087$, $n_B = 737$, $x_B = 13.1452$, $n_c = 230$, $x_c = 3.2783$, $n_b = 377$, $x_b = 5.2944$, $n_v = 316$, $x_v = 3.7342$, $n_e = 590$ and $x_e = 5.7017$.

The percolation probabilities are

$p_C = 0.4222 \pm 0.0734$, $p_B = 0.2463 \pm 0.0482$, $p_c = 0.6548 \pm 0.0787$, $p_b = 0.5332 \pm 0.0532$, $p_v = 0.6760 \pm 0.0288$ and $p_e = 0.5309 \pm 0.0651$.

Here comparison with literature is already limited as published data begins to be rare. We can compare with exact values and say that x_v obtained contains an error of -6.65 per cent, p_v of 3.68 per cent and p_e of 1.70 per cent. Figure 2.11 shows the networks created and simulated.

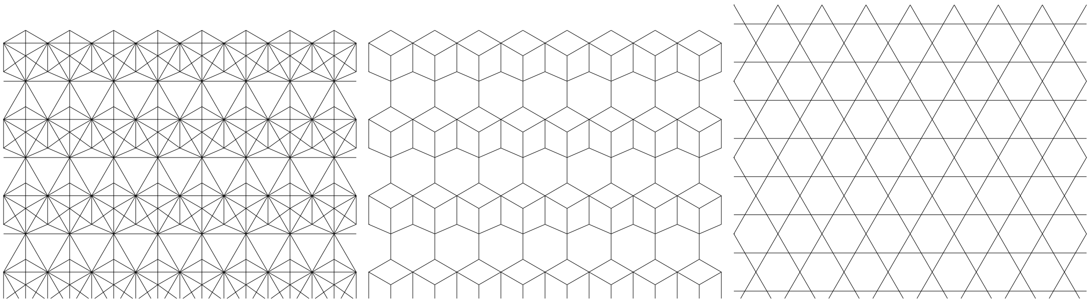

(a) (b) (c)

Figure 2.11 *Percolation of networks related to the basket pattern, kagome lattice. The networks of* (a) *Cells and Bonds,* (b) *cells and bonds, and* (c) *vertices and edges.*

From these tests we can see that, even for a rather small size, our programmes for creating the network and finding percolation show approximately five per cent error.

3

§ 3. Voronoi tessellation

A tessellation is a tiling similar to mosaics but in any number of dimensions. A Voronoi tessellation is one that expands like bubbles from nuclei and so press against one another when they meet that their originally spherical surfaces mutually form flat walls and straight edges in the process. Since Creation is a percolation, any other percolation that follows it must necessarily be a percolation within percolation. This is evidently the case, as Van de Weygaert and Icke (1989) points out our Universe being structurally a 3-d Voronoi Tessellation and we living on the flats, edges and points of it, any percolation process occurring in the material world is consequently a percolation within Voronoi Tessellation. The galaxy formation, evolution and life are all by-products of this Cosmic Voronoi Percolation.

In a puzzle of Figure 3.1 there are five rooms with doors in the position as shown. The problem is whether one can walk through every door only once and the answer according to the graph theory is *no*, because there are more than two rooms which has an odd number of doors. The proof of Theorem 3.1 was from Gomsan Bajåraṿåṇijẙa around 1989. From this, when one plays a puzzle like that of Figure 3.1 one always starts off from a room with an odd number of doors and ends in another such room.

This is the same thing as saying that one starts and ends outside rooms with an even number of doors. Therefore the number of the latter is of no consequence, but that of the former is crucial for the existence of a solution and must never be any number other than two. Puzzle of five rooms with doors. In Figure 3.1 there are three rooms with five doors, two with four, and one with nine. There are here four rooms with an odd number of doors. Starting off from one of these four one can only end up in one of the other three, which leaves the remaining two rooms unaccounted for. In other words at least two doors will necessarily remain unvisited.

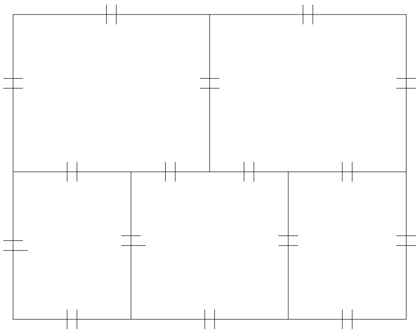
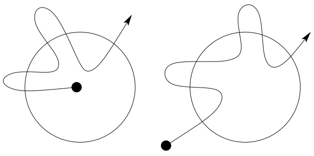


Figure 3.1 Rooms with doors puzzle.

Theorem 3.1. *A travel along a network can only starts and ends at nodes which have an odd coordination number.*



Proof. Looking at Figure 3.2, if one starts from inside a room with an odd number of doors one always ends up outside it. On the other hand one always ends up inside a room with an even number of doors if one starts off from it. In the second picture such a room is all the area outside the circle. □

The following Corollaries 3.1[1], 3.1[2] and 3.1[3] assume nondegeneracy of the Voronoi network. Such a path as mentioned in these corollaries is also called *self-avoiding.*

Figure 3.2 Departure and arrival rooms.

Corollary 3.1[1]. *There can be no path which traverses all edges of a Voronoi graph only once.*

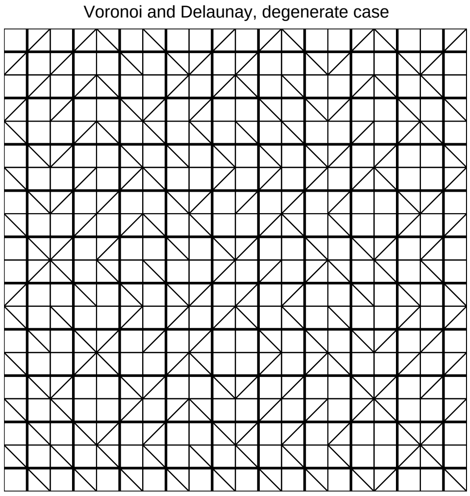
Proof. This follows from Theorem 3.1 because a two dimensional Voronoi network has a coordination number three. □

Corollary 3.1[2]. *On a three dimensional Voronoi structure there always exists a path that runs through every edge once and only once.*

Proof. This also readily follows from Theorem 3.1 since a three dimensional Voronoi network has a coordination number of four. □

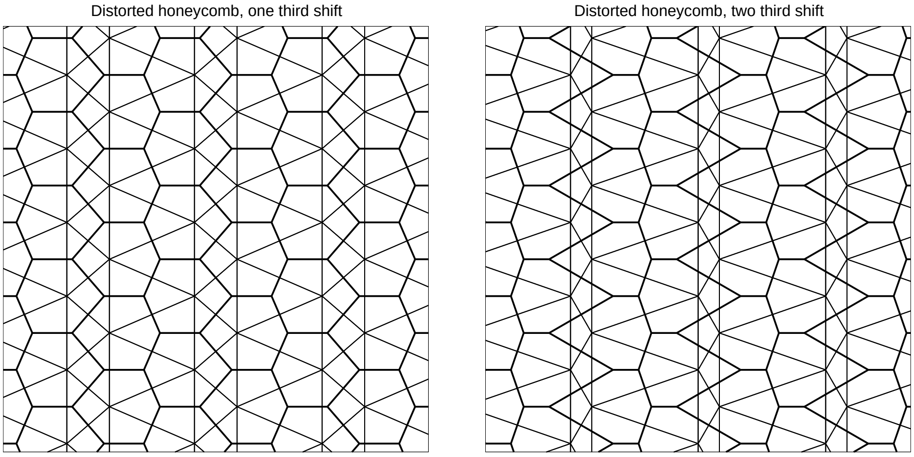
Corollary 3.1[3]. *Take any Voronoi cell of the three dimensional network, it is impossible to walk through all its edges without repeating some of them.*

Proof. Again from Theorem 3.1 and because the surface of a Voronoi polyhedron is a two dimensional network of polygons which has the coordination number three. □



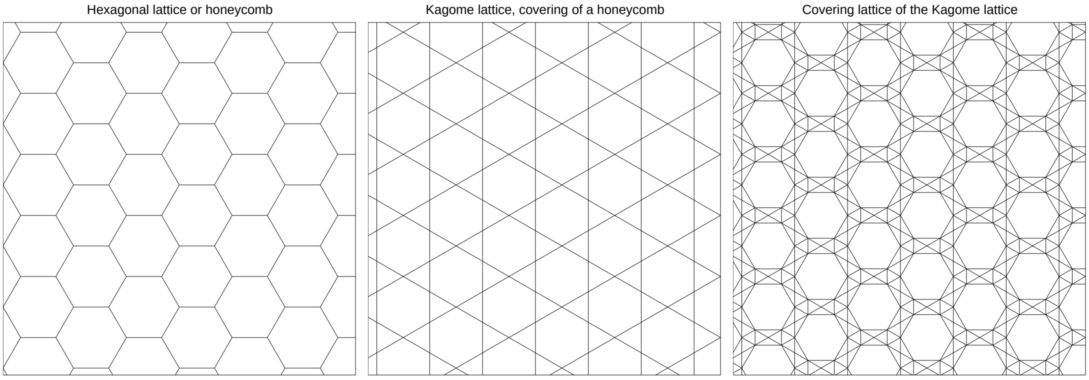
Jerauld *et al* (1984) compared the Voronoi, with the triangular networks and found that the bond percolation probability of the former is 4.3% or 0.015 less than that of the latter, small site clusters more, and small bond clusters less likely. When a square lattice was fed to *voronoi* and *delaunay* in Matlab, by the programme `degen.m` in § A.15, there were error messages saying that points were collinear and possibly triangulation is incorrect. This case, Figure 3.3, is degenerative.

Figure 3.3 Voronoi from degenerative data.


Distorted honeycomb, one third shift
Distorted honeycomb, two third shift

(a) (b)

Figure 3.4 *The honeycomb or hexagonal lattice whose alternate y-plane has been shifted (a) one-third, and (b) two-third respectively. Triangulation is shown with thinner lines. The programme used is* `honey.m`


Hexagonal lattice or honeycomb
Kagome lattice, covering of a honeycomb
Covering lattice of the Kagome lattice

(a) (b) (c)

Figure 3.5 (a) *The hexagonal lattice,* (b) *its covering lattice (Kagome), and* (c) *the covering lattice of its covering lattice (i.e. covering of Kagome). (*`cover.m`*,* `covers.m`*, and* `coverss.m`*)*

If we indicate by $C_v^n(x)$ the n^{th}-order covering lattice of a lattice x, then the first picture is Hexagonal, the second one Kagome or C_v^1(Hexagonal) and the third one C_v^2(Hexagonal) or in other words C_v^1(Kagome). Figure 3.6 is the next iteration, a C_v^3(Hexagonal) or C_v^2(Kagome). Now let us look at the Voronoi graph and its covering lattices. Pictures in Figure 3.7 are drawn by first creating and cropping a Voronoi graph with the help of the programme `crop.m`, then use the recursive procedure described above to find up to the third covering lattice.

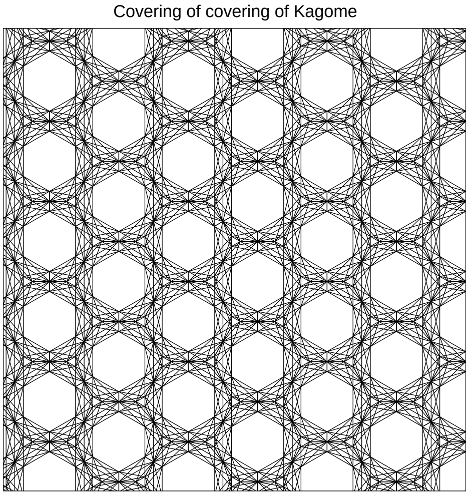
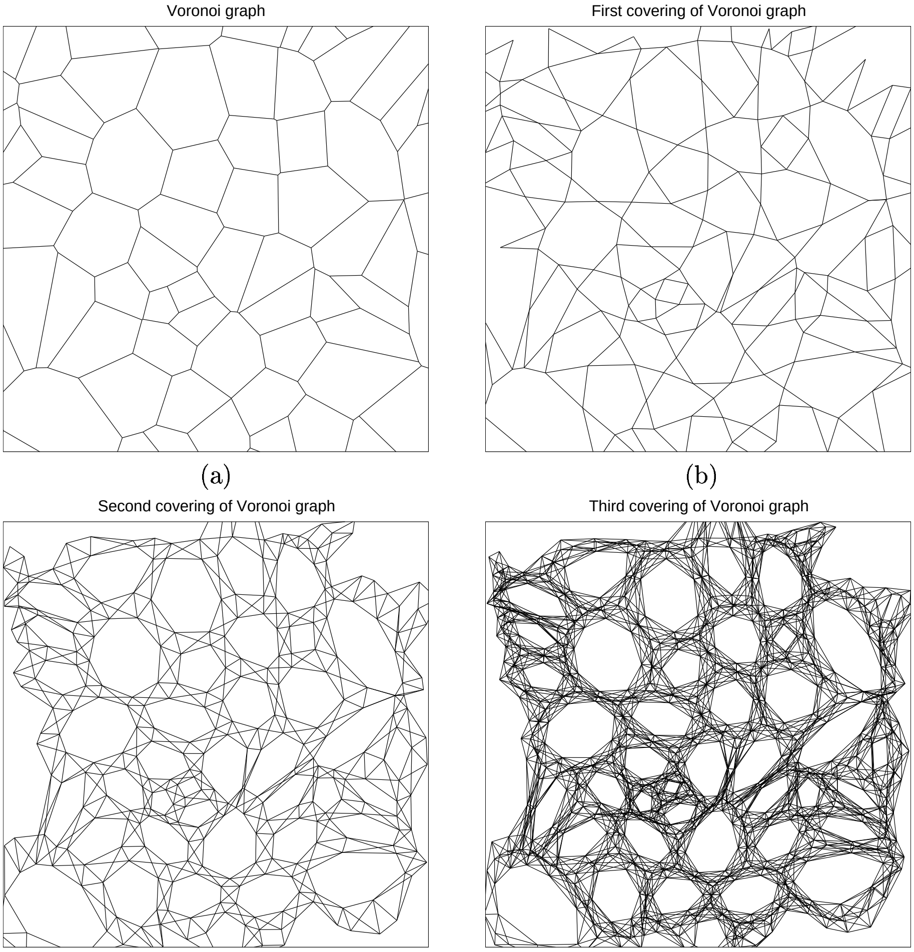


Figure 3.6 The next covering lattice.


Voronoi graph
First covering of Voronoi graph
Second covering of Voronoi graph
Third covering of Voronoi graph

(a) (b)

(c) (d)

Figure 3.7 (a) *Voronoi graph (V.g.)*, (b) $C_v^1(\mathbf{V.g.})$, (c) $C_v^2(\mathbf{V.g.})$, (d) $C_v^3(\mathbf{V.g.})$.

Notice that the covering lattices retain the skeleton structure of the original Voronoi graph. Those of higher orders represent closer the structures of nature where walls have thickness.

§ 3.1 Quadratic equations

Quadratic equations are equations of binary quadratic forms. Around 400 BC Barbilonia had algorithmic equivalences of quadratic equations which are based on the method of completing the square and where all answers are unsigned, *i.e.* positive, lengths. Because there was no notion for zero, Diophantus considered three types of quadratic equations $ax^2 + bx = c$, $ax^2 = bx + c$, and $ax^2 + c = bx$. Euclid, *circa* 300 BC, used geometric equivalences or quadratic equations whose roots are also lengths. Brahmagupta allowed negative quantities, which he called debts, and used abbreviations for the unknown. Al-Khwarizmi classified quadratics into six types, namely squares equals roots, squares equals numbers, roots equal numbers, squares and roots equal number, squares and numbers equal roots, and roots and numbers equal squares. In his book *Liber embadorum*, published in 1145, Abraham bar Hiyya Ha-Nasi (aka Savasorda) gives the complete solution of quadratic equations. Luca Pacioli published *Summa de arithmetica, geometrica, proportioni et proportionalita* (or Summa) in 1494. He also applied quadratic methods to quartics of the form $x^4 = a + bx^2$. Scipione del Ferro solved the cubic equations of the form $x^3 + mx = n$.

§ 3.2 Quadratic forms

The theory of quadratic forms and the theory of matrix are inseparable though the history of these two subjects are somewhat fragmentary. A bilinear form in the sets x_i and y_i, $i = 1 \ldots n$, is $\sum_{i,j} x_i y_j$, or $\mathbf{x}^{\mathrm{T}}\mathbf{A}\mathbf{y}$. If $x_i = y_i$ for all i, then the form is quadratic in x_i. In other words, a quadratic form is a general expression which contains second order terms.

A quadratic form in the variables x_i is the homogeneous quadratic polynomial $\sum_{i,j=1}^{n} a_{ij} x_i x_j$, where a_{ij} are arbitrary scalars. The set of all quadratic forms in x_i with coefficients in a field $\mathcal{F}$ is a vector space over $\mathcal{F}$. A bilinear form in the variables x_i and y_i is the homogeneous polynomial $\sum_{j=1}^{m}\sum_{i=1}^{n} a_{ij}x_iy_j$, where a_{ij} are arbitrary scalars. The set of all bilinear forms with coefficients in a field $\mathcal{F}$ is a vector space over $\mathcal{F}$ (*cf* Hohn, 1958). A quadratic form can also be represented in a matrix form as $X^{\mathrm{T}}AX$. Distinct $n \times n$ matrices A_1 and A_2 have the same quadratic polynomial if all corresponding $a_{ij} + a_{ji}$ are equal. Given the quadratic form, it is not possible to identify the corresponding matrix A by inspection. This ambiguity is eliminated by replacing each pair of the coefficients a_{ij} and a_{ji} by their mean $(a_{ij}+a_{ji})/2$, which amounts to replacing A by $(A+A^{\mathrm{T}})/2$ which is symmetric. If x_i are independent variables, the rank and the determinant of A are respectively the rank and the discriminant of the form. Two quadratic forms $X^{\mathrm{T}}A_1X$ and $\widetilde{X}^{\mathrm{T}}A_2\widetilde{X}$ are equivalent if and only if there is a nonsingular transformation $X = B\widetilde{X}$ such that $X^{\mathrm{T}}A_1X = \widetilde{X}^{\mathrm{T}}B^{\mathrm{T}}A_1B\widetilde{X} = \widetilde{X}^{\mathrm{T}}A_2\widetilde{X}$, that is if and only if for a suitable nonsingular matrix B, $A_2 = B^{\mathrm{T}}A_1B$. Then A_1 and A_2 have the same rank and are said to be congruent to each other.

Lattice can be represented by quadratic forms and vice versa, therefore the classification of quadratic forms is also the classification of lattices.

§ 3.3 Voronoi algorithms

Green and Sibson (1978) gives the algorithm that produces Voronoi graphs, as summarised in Algorithm 3.1. They call a Voronoi tessellation 'Dirichlet tessellation'. The points outside the window are ignored, and points on the periphery has tile bounded in part by sequences of effective constraints instead of by inter-tile edges. Contiguous tiles have a boundary sequences in common. A degenerate vertex, that is a vertex where four or more tiles meet, causes an incorrect record of diagonal contiguity. From the Euler-Poincaré formula, $f - e + v = 2$, where in the plane the infinite region is counted as a face, gives rise to the total number of contiguities to be recorded 4× `effective constraints` +6× `accepted points` −6. A square lattice has all its vertices degenerate.

Algorithm 3.1 *Voronoi 2-d algorithm, Green and Sibson (1978)*

```
define window as a set of linear inequality constraints $ax + by + c < 0$;
do
  add a new point $n$ randomly within the window;
  do
    find the line $b_i$ joining $n$ and its nearest neighbour $p_i$;
    find $l_i$ the perpendicular bisector of $b_i$;
    find the intersection between $l_i$ and the edge $e_i$ of its nearest tile,
      going clockwise with respect to $n$;
    add $l_1$ to the object- and contiguity lists;
  until $2\pi$ radian around $n$ traversed
until no more new points added enddo                                    □
```

Moore and Angell developed a algorithm which can cope with degenerate cases. Because their programme could find them, they are no longer considered degenerate and they fittingly called these *vertices of* (>3)*-hedral valency.*

§ 3.6 Voronoi section

Van de Weygaert (1994) studies linear section of three dimensional Voronoi network where the mean length is

$$\langle\lambda\rangle = L = \frac{d\Gamma(d-1/2)\Gamma((d+1)/2)^2}{(d-1)!\rho_c^{1/d}2\Gamma(d/2+1)^{2-(1/d)}\Gamma(2-1/d)}. \qquad (\mathbf{19})_{\mathrm{iii}}$$

For two dimensions $\langle\lambda\rangle = \pi/(4\sqrt{\rho})$, while for three dimensions $\langle\lambda\rangle = [81/(32\pi\rho)]^{1/3}/\Gamma(2/3)$. Consequently for two dimensions $\langle\lambda\rangle = 0.7854\rho^{-1/2}$ while for three dimensions $\langle\lambda\rangle = 0.6872\rho^{-1/3}$, $\langle\lambda^2\rangle = 0.632\rho^{-2/3}$, $0.668\rho^{-1}$ and $\langle\lambda^4\rangle = 0.774\rho^{-4/3}$. For 3-d Poisson-Voronoi tessellation the exact values for the moments are shown in Table 3.1.

ρ_v	$\frac{24}{35}\pi^2\rho_c$	$6.763\rho_c$	number density of vertices	J_0
ρ_e	$\frac{48}{35}\pi^2\rho_c$	$13.535\rho_c$	number density of edges	J_1
ρ_f	$\left(\frac{24}{35}\pi^2+1\right)\rho_c$	$7.768\rho_c$	number density of faces	J_2
ρ_c	ρ_c	ρ_c	number density of cells	J_3
n_c^v	$\frac{96}{35}\pi^2$	27.07	number of vertices per cell	$\mathrm{E}(N_0)$
n_c^e	$\frac{144}{35}\pi^2$	40.61	number of edges per cell	$\mathrm{E}(N_1)$
n_c^f	$\frac{48}{35}\pi^2+2$	15.54	number of faces per cell	$\mathrm{E}(N_2)$
n_f^v or n_f^e	$\frac{144\pi^2}{24\pi^2+35}$	5.228	number of vertices or edges per face	$E_2(N_0)$ or $E_2(N_1)$
V_c	$\frac{1}{\rho_c}$	$\frac{1}{\rho_c}$	volume of cell	$\mathrm{E}(V)$
	$\frac{1.180}{\rho_c^2}$	$\frac{1.180}{\rho_c^2}$		$\mathrm{E}(V^2)$
A_c	$\left[\frac{256\pi}{3\rho_c^2}\right]^{1/3}\Gamma\left(\frac{5}{3}\right)$	$5.821\rho_c^{-2/3}$	surface area of cell	$\mathrm{E}(A_c)$
s_c	$\frac{(4\pi)^{5/3}\Gamma(1/3)}{5(9\rho_c)^{1/3}}$	$17.496\rho_c^{-1/3}$	perimeter of cell	$\mathrm{E}(s_c)$
A_f	$\frac{35(2^{8/3})\pi^{1/3}\Gamma(2/3)}{(9\rho_c)^{2/3}(24\pi^2+35)}$	$0.3747\rho_c^{-2/3}$	area of face	$\mathrm{E}(A_w)$
s_f	$\frac{7(2^{10/3})\pi^{5/3}\Gamma(1/3)}{(9\rho_c)^{1/3}(24\pi^2+35)}$	$2.252\rho_c^{-1/3}$	perimeter of face	$\mathrm{E}(s_w)$
d_e	$\frac{7\Gamma(1/3)}{9(36\pi\rho_c)^{1/3}}$	$0.4309\rho_c^{-1/3}$	length of edge	$\mathrm{E}(L)$

Table 3.1 *Moments of 3-d Poisson-Voronoi tessellation, cf van de Weygaert (1994).*

The moments of 2-d section of 3-d Voronoi tessellation are shown in Table 3.2.

ρ_v	$\frac{2\Gamma\left(\frac{1}{3}\right)\left(\frac{16\pi^5\rho^2}{9}\right)^{1/3}}{15}$	$2.9159\rho_c^{2/3}$	number density of vertices	J_0
ρ_e	$\frac{3}{2}\rho_v$	$4.3739\rho_c^{2/3}$	number density of edges	J_1
ρ_c	$\frac{1}{2}\rho_v$	$1.4530\rho^{2/3}$	number density of cells	J_2
n_c^v or n_c^e	6	6	number of vertices or edges per cell	$\mathrm{E}(N_0)$ or $\mathrm{E}(N_1)$
A_c	$\frac{15}{\Gamma\left(\frac{1}{3}\right)\left(\frac{16\pi^5\rho_c^2}{9}\right)^{1/3}}$	$0.6859\rho^{-2/3}$		$\mathrm{E}(A)$
		$0.698\rho_c^{-4/3}$		$\mathrm{E}(A^2)$
s_c	$\frac{30\Gamma\left(\frac{2}{3}\right)}{(36\pi\rho_c)^{1/3}\Gamma\left(\frac{1}{3}\right)}$	$3.1356\rho_c^{-1/3}$	perimeter of cell	$\mathrm{E}(S)$
d_e	$\frac{5\Gamma\left(\frac{2}{3}\right)}{(36\pi\rho_c)^{1/3}\Gamma\left(\frac{1}{3}\right)}$	$0.5226\rho_c^{-1/3}$	length of edge	$\mathrm{E}(L)$

Table 3.2 *Moments of 2-d section of 3-d Voronoi, cf van de Weygaert (1994).*

The form factor of a cell, $f_c = 36\pi V_c^2/A_c^3$, is a dimensionless parameter which partially describes the shape of the cell. The form factor of a face is $f_f = 4\pi A_f/s_f^2$, which is the unity when the face is a circle. Regularly shaped cells have their shape approaching that of the sphere where $f_f = 1$. Statistics of 3-d VT which van de Weygaert (1994) gives include those listed here in Table 3.3.

	$\mu(\cdot)$	$\sigma(\cdot)$	$\gamma_1(\cdot)$	$\gamma_2(\cdot)$	
ρ_v	6.747 ± 0.014				J_0
ρ_e	13.493 ± 0.028				J_1
ρ_f	7.747 ± 0.014				J_2
ρ_c	1.000 ± 0.000				ρ
n_c^v	26.986 ± 0.055	6.61 ± 0.18	0.346 ± 0.029	-0.029 ± 0.030	N_0
n_c^e	40.479 ± 0.083	9.92 ± 0.27	0.346 ± 0.029	-0.029 ± 0.030	N_1
n_c^f	15.493 ± 0.028	3.305 ± 0.091	0.346 ± 0.029	-0.029 ± 0.030	N_2
V_c	1.000 ± 0.000	0.418 ± 0.009	0.73 ± 0.11	0.70 ± 0.39	V_{cell}
A_c	5.801 ± 0.018	1.461 ± 0.043	0.28 ± 0.11	-0.01 ± 0.17	A_{cell}
s_c	17.443 ± 0.054	3.655 ± 0.094	0.305 ± 0.035	0.00 ± 0.13	S_{cell}
f_c	0.540 ± 0.006	0.082 ± 0.003	-0.565 ± 0.032	0.36 ± 0.11	F_{cell}
n_f^v	5.2255 ± 0.0014	1.564 ± 0.016	0.582 ± 0.016	0.058 ± 0.024	N_0^w
A_f	0.3744 ± 0.0015	0.3722 ± 0.0037	1.266 ± 0.047	1.40 ± 0.19	A_{wall}
s_f	2.2518 ± 0.0092	1.2009 ± 0.0050	0.089 ± 0.022	-0.796 ± 0.029	S_{wall}
f_f	0.6389 ± 0.0005	0.1635 ± 0.0011	-0.855 ± 0.035	0.40 ± 0.13	F_{wall}
d_{cf}	0.6402 ± 0.0017	0.2092 ± 0.0051	-0.024 ± 0.022	-0.333 ± 0.074	D_{nw}
V_{cf}	0.0645 ± 0.0001	0.0579 ± 0.0009	1.177 ± 0.097	1.52 ± 0.47	V_{nw}
d_e	0.4309 ± 0.0018	0.3216 ± 0.0023	0.829 ± 0.020	0.209 ± 0.044	L
α_{ee}	$111°\text{C}107 \pm 0.018$	$35°\text{C}310 \pm 0.078$	-0.499 ± 0.012	-0.276 ± 0.039	α_{ee}
α_{ff}	$120°\text{C}0 \pm 0.0$	$23°\text{C}53 \pm 0.30$	-0.296 ± 0.014	-0.255 ± 0.038	α_{ww}

Table 3.3 *Statistics of 3-d VT, van de Weygaert (1994).*

His statistics on the planar section of the 3-d Voronoi tessellation is shown here as Table 3.4.

	$\mu(\cdot)$	$\sigma(\cdot)$	$\gamma_1(\cdot)$	$\gamma_2(\cdot)$
ρ_s^c	1.4530	0.0592	0.2871	-1.0099
ρ_s^v	2.9060	0.1134	0.2871	-1.0099
ρ_s^e	4.3590	0.1776	0.2871	-1.0099
n_s^v	6.0000	1.6895	0.3311	-0.1142
A_s^c	0.6882	0.4741	0.4573	-0.3902
s_s^c	3.1418	1.2212	-0.5819	-0.2879
f_s^c	0.7050	0.1433	-1.3425	2.0875
d_s^e	0.5221	0.3631	0.6223	-0.2321
α_s^{ee}	120°C0000	31°C6039	-0.6351	0.1321
α_s^{ff}	63°C1812	18°C5232	-0.6351	-0.3520

Table 3.4 *Statistics of the planar section of the 3-d Voronoi tessellation, van de Weygaert (1994).*

Then Table 3.5 lists the statistics he gave for the line section of 3-d VT.

	$\mu(\cdot)$	$\sigma(\cdot)$	$\gamma_1(\cdot)$	$\gamma_2(\cdot)$
λ	0.6703	0.3942	0.2600	-0.6923
$\sigma(\lambda)$	0.6123	0.1159	0.0643	0.0204
α_l^{fl}	44°C69	19°C5325	0.0067	-0.8080
α_l^{ff}	75°C4123	30°C9389	-0.0185	-0.5829

Table 3.5 *Statistics for the line section of the 3-d Voronoi tessellation, van de Weygaert (1994).*

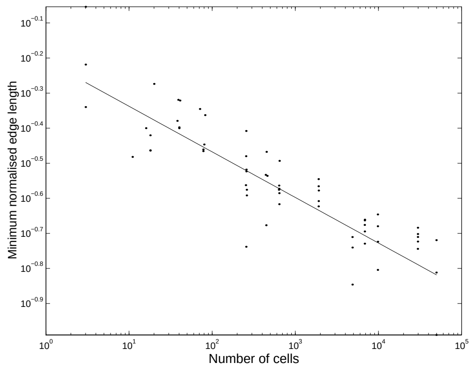
Tables 3.3, 3.4 and 3.5 are all on VT's which are created from $\mathcal{P}(0,3)$. Apart from these, van de Weygaert (*ibid.*) also gives results based on VT's with anticorrelated- and correlated nuclei the statistics of which are not listed here.

Another interesting statistics is the correlation between the number of sides of a grain n_e, and the expected number of sides of its neighbouring grains n_n^e. Aboav (1970), in his investigation of polycrystalline MgO, empirically found it to be $n_n^e = 5 + 8/n_e$ and Weaire (1974) found a theoretical value of $n_n^e = 5 + 6/n_e$.

Later Aboav (1983) published another finding in which he studied a thin film of arsenic-selenium glass, As_2Se_3. This time with the initial assumption that $n_n^e = 6 - a + (6a + \mu_2)/n$ and $\mu_2 = \sum_n (n-6)^2 f_n = \langle (n-6)^2 \rangle$ he empirically finds $n_n^2 = 4.79 + 8.98/n$.

§ 3.5 Voronoi statistics

In two dimensions the statistical descriptions are as follows.

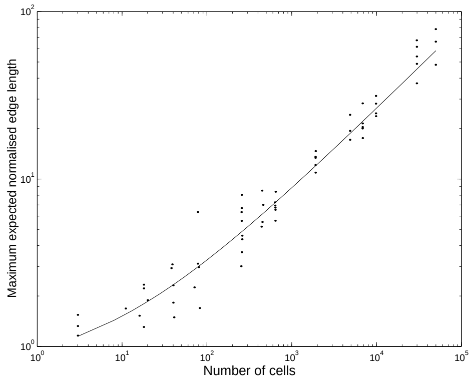


The curve in Figure 3.8 is

$$y = 0.62/n^{0.13}.$$

First the edge lengths are normalised by the edge length of the equivalent or characteristic square . A *equivalent square* is defined as the square figure whose area is equal to the area of the polygon in question, here a Voronoi polygon. Then find the average of the edge lenghts in each cell.

Figure 3.8 The minimum of the average normalised edge length.

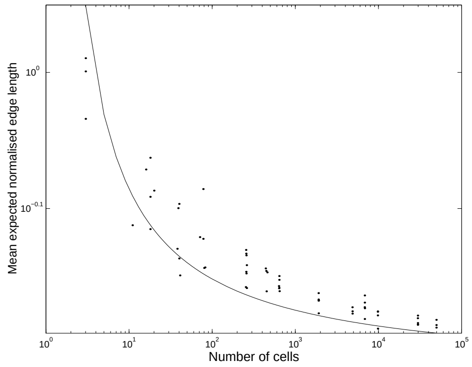


In Figure 3.9 the curve is

$$y = \left|(n/15)^{1/2}\right| + 0.7 \ .$$

Of these cell-averaged normalised edge length obtained from simulations on various sizes of networks the minimum values are plotted in Figure 3.8, the maximum in Figure 3.9, and the expected value in Figure 3.10. Note that the last quantity is the average over the whole structure of all the averages obtained one from each cell.

Figure 3.9 The maximum of the average normalised edge length.



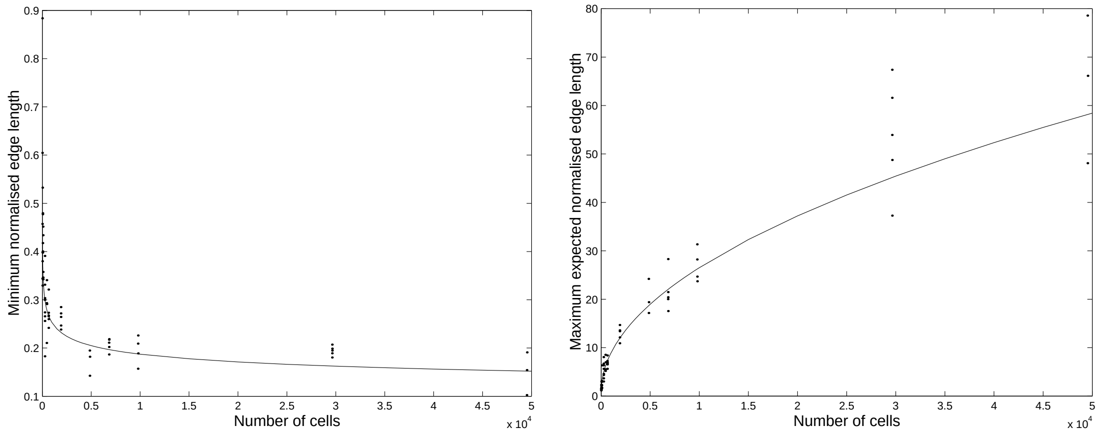
In Figure 3.10 the curve is

$$y = 10^{0.2/\log x} - 0.4$$

To summarise, as the networks gets larger its minimum, maximum, and mean of the edge lenghts when compared with the characteristic length approach constant values. The characteristic length is defined as $l = [\sum_{i=1}^{n} A_i/n]^{1/2}$, where A_i is the area of the i^{th} polygon and n is the number of cells.

Figure 3.10 The mean of the average normalised edge length.

That the three values mentioned become constant may not seem obvious by the look of Figures 3.8 to 3.10 because the scale used there is a logarithmo-logarithmic scale, not a Euclidean one. These figures emphasise the smaller ranges of size. Figure 3.11 below on the other hand is plotted using a normal scale which enables one to see the asymptotic effect more clearly. Here the figures (a), (b), and (c) are respectively Figure 3.8, 3.9, and 3.10. Let the term *representative* stands for 'of the cell-average normalised', and *length* means 'edge length' in this context.



(a) (b)

Then the *minimum representative length* approaches 0.15 from Figure 3.11 (a), the *maximum representative length* is ever increasing, seemingly by a power law of approximately 0.5, while *mean representative length* approaches the value of 0.65. Properties of the Voronoi tessellation can be divided into individual and collective properties. With this in mind the term *length* above represents a property, *representative* means individual, and *minimum*, *maximum* and *mean* show the collective attributes.

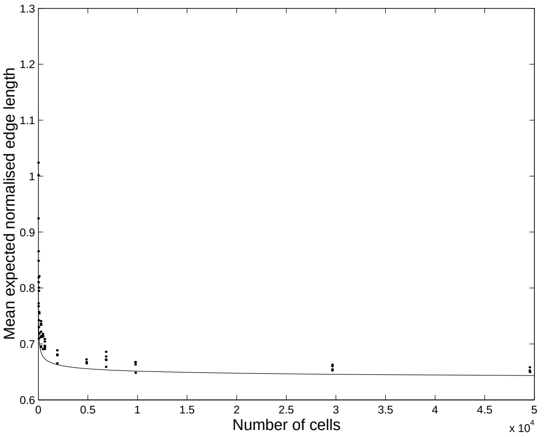


Figure 3.11 Mean, maximum, and mean of the cell-average normalised edge length..

This $\sigma^2(\mathcal{N}_v^c(E(l_e)))$ increases very slowly with the increasing sizes of the networks. The curve shown has the equation

$$y = \left| \left[\frac{x}{8 \times 10^4} \right]^{1/3} \right| + 0.05 \ .$$

The number of vertices per cell increases dramatically as one goes up the dimension ladder. The programme in § A.10 contains the essential part of the code which produces Figure 3.13.

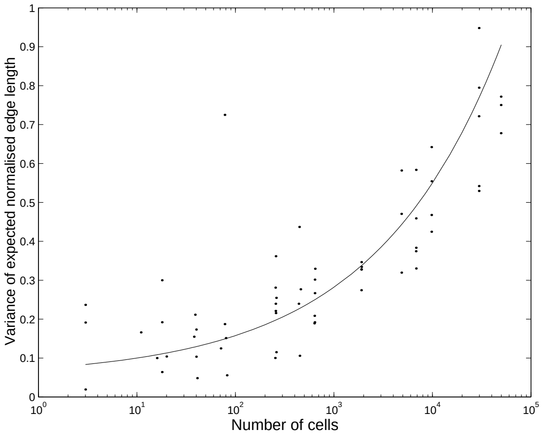
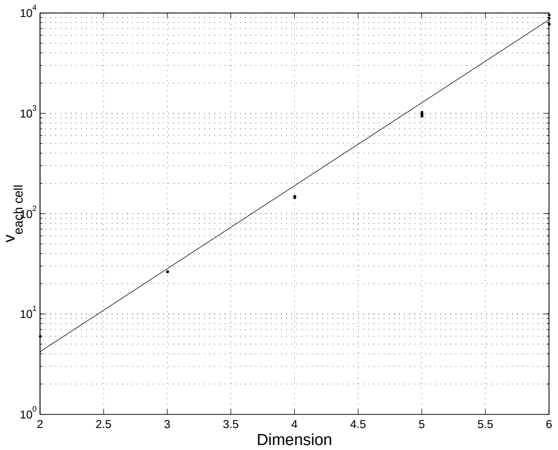
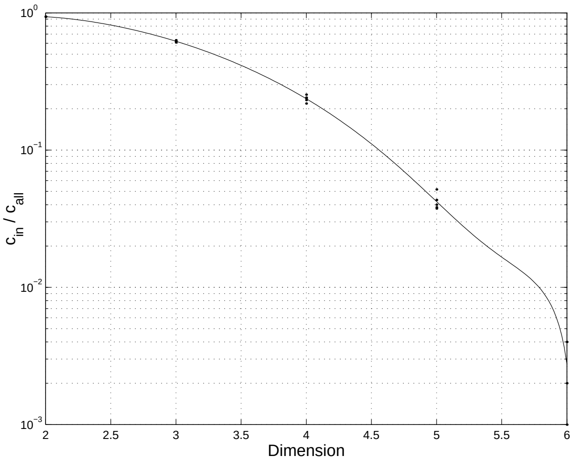


Figure 3.12 The variance of the expected values of the normalised length of edges of a cell.



The straight line shown is $0.093(4+e)^n$. Notice the trend towards a greater rate of increase at dimensions higher than the maximum six shown. Only Voronoi cells which lies within the original domain and the vertices of these are considered. The same programme also gives Figure 3.14. The expanse and the hypervolume of the Voronoi tessellation increases at an enormous rate, which results in the number of cells totally bounded within the original domain decreasing rapidly in Figure 3.14 as the dimension goes up.

Figure 3.13 Number of vertices per cell.



The effect at close to the zero ratio is emphasised by using the log scale for the y-axis. Also with the logarithmic scale the polynomial estimation curves that give negative values of the ratio is automatically excluded. One can fit the polynomial $p(x) = p_1x^n + p_2x^{n-1} + \ldots + p_nx + p_{n+1}$ to the data with a least square algorithm. If $\mathbf{x}$ is the vector containing the data, then the the $(n+1)$ coefficients of the estimated polynomial can be found from $\hat{\mathbf{x}} = (\mathbf{x} - \mathrm{E}(\mathbf{x}))/\sigma(\mathbf{x})$.

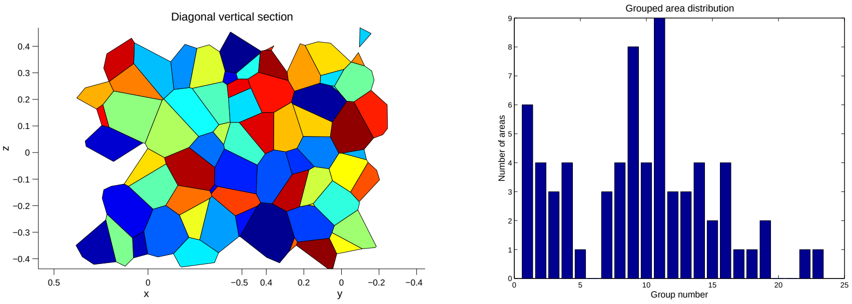
Figure 3.14 Ratio of cells in the original domain.

For data containing independent normal errors with a constant variance, the error bounds contain at least half of the predictions. The curve shown in Figure 3.14 is $p(x) = -0.048x^4 + 0.064x^3 - 0.203x^2 - 0.459x + 0.263$, the average value $\mathrm{E}(\mathbf{x})$ is 3.917, and the standard deviation $\sigma(\mathbf{x})$ is 1.412. The structure of the polynomial fit can be described using the Cholesky factor of the Vandermonde matrix

$$R = \begin{bmatrix} -12.19 & -1.56 & -6.08 & -0.47 & -3.17 \\ 0 & 8.46 & -0.45 & 4.47 & -0.44 \\ 0 & 0 & 1.21 & 0.33 & 2.93 \\ 0 & 0 & 0 & -1.63 & 0.30 \\ 0 & 0 & 0 & 0 & 2.25 \end{bmatrix},$$

the degree of freedom which is 19, and the norm of the residuals which is 0.034 in this case.

§ 3.6 Voronoi section



(a) (b)

Figure 3.15 *(a) Section by the plane $x - y + \epsilon z = \epsilon, \epsilon = 10^{-4}$. (b) Grouped distribution of area, the number of groups is approximately one-third the number of regions.*

	$\mathcal{V}_c$	$\aleph_c$	A_c^{fr}	V_c^{fr}	n_c^e	$E(n_{f,c}^e)$	$\sigma(n_{f,c}^e)$
min	8	6	2.2191×10^{-4}	9.9722×10^{-5}	12	4	0.6030
max	46	25	4.7015×10^{-3}	0.14364	69	5.52	2.5690
μ	26.338	15.169	1.8975×10^{-3}	1.8975×10^{-3}	39.507	5.1712	1.5586
σ^2	40.608	10.152	5.2144×10^{-7}	4.2313×10^{-5}	91.368	0.035983	0.1186
σ	6.3725	3.1862	7.2210×10^{-4}	6.5048×10^{-3}	9.5587	0.18969	0.3444
μ_g	25.543	14.829	1.7572×10^{-3}	1.1575×10^{-3}	38.315	5.1676	1.5169
μ_h	24.703	14.478	1.6026×10^{-3}	8.5548×10^{-4}	37.054	5.1638	1.4700
med	26	15	1.8125×10^{-3}	1.1660×10^{-3}	39	5.2	1.5706
mad	5.0068	2.5034	5.6975×10^{-4}	1.4250×10^{-3}	7.5103	0.14465	0.2715
$\mathcal{M}^2$	40.531	10.133	5.2045×10^{-7}	4.2233×10^{-5}	91.195	0.035915	0.1184
$\mathcal{M}^3$	72	9	2.5644×10^{-10}	5.4435×10^{-6}	243	-8.3404×10^{-3}	-0.0060
$\mathcal{M}^4$	5088.9	318.06	1.0467×10^{-12}	7.6653×10^{-7}	25763	7.9566×10^{-3}	0.041906
$\mathcal{K}$	3.0978	3.0978	3.8644	429.77	3.0978	6.1689	2.9916

Table 3.6 *Simulation uses rbox (1000 random points, seed 234985) and qhull (option v and o); $d = 3$, $n^c = 527$, $n^v = 6357$, CPU time 6,466.99 sec for the counting of statistics, 270.56 sec for finding area of the faces, 4.47 sec for calculating cell volume and 2.8 sec for finding the number of edges.*

§ 3.7 Number of vertices and edges

It has been observed from the simulations that in three dimensions cells always have vertices in even numbers and edges odd ones. This can be explained by the following theorems.

Theorem 3.2. *(cf Miles, 1972) In a simple three dimensional Voronoi tessellation, $3n_c^v = 2n_c^e$.*

Proof. Pick any Voronoi cell of the tessellation. Suppose that it has n^v vertices. Add up the number of edges connected to all vertices. Because every cell is a simple polyhedron, there are exactly three edges connected to each vertex. The number of edges thus counted is therefore $3n^v$. But each of the edges is connected to two vertices, so we have counted every one of them twice. Therefore,

$$2n^e = 3n^v.$$

This is the case for any cell, hence the theorem is proved. □

This theorem gives rise the following two theorems.

Theorem 3.3. *The number of vertices of any cell within a simple three dimensional Voronoi tessellation is an even positive integer.*

Proof. Observe that the term $2n_c^e$ in the theorem above is divisible by 2. This term is equal to $3n_c^v$, therefore the latter is also divisible by 2. Since 2 can not divide into 3, the only term left, n_c^v, must be divisible by 2 and hence even number. □

Theorem 3.4. *The number of edges of any cell within a simple three dimensional Voronoi tessellation is a positive integer divisible by three.*

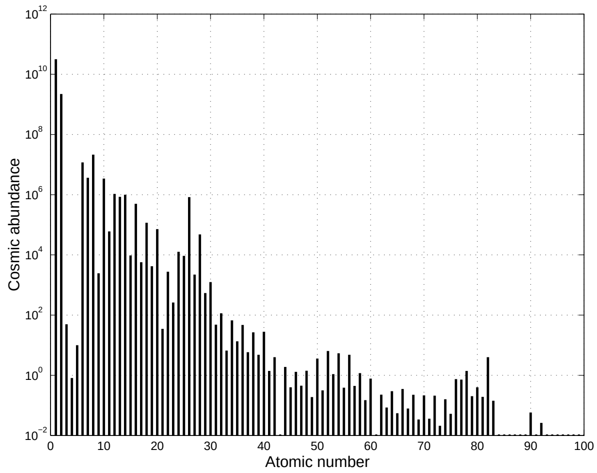
Proof. With the same line of reasoning as above, observe that $3n_c^v$ is divisible by 3. Therefore $2n_c^e$, and hence n_c^e, is also divisible by 3. □

Another proof for both the above theorems is the following.

Proof. For an equality to hold, both sides must have the same factors. By supposing an unknown common factor i and by cross-multiplying the coefficients on both sides, one obtains $2\cdot(3\cdot i) = 3\cdot(2\cdot i)$, where i is a positive integer. Therefore $n_c^e = 3i$ and $n_c^v = 2i$. In other words, n_c^e is divisible by three and n_c^v is even. □

The theorems above assume that every cells are simple. This can not be the case in real situation where edges have dimensions and rather represent tubes than one-dimensional lines. Such case is similar to the so-called *degenerative* case in a computational model of Voronoi tessellation where there exist vertices the number of edges connected to each of which exceeds four. Even in the degenerative case, one would perhaps still expect a tendency for n_c^v to be an even number and for n_c^e to be divisible by three to hold.

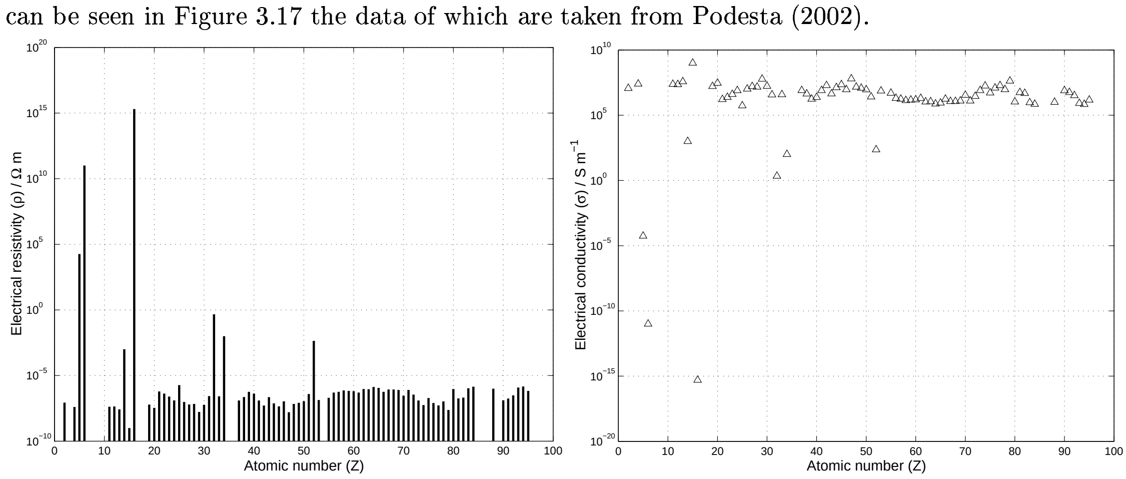
In nature there are things which have a tendency towards even numbers. The following graph shows the the abundance in cosmic materials from the compilation by Cameron (1973). The year of publication of this paper is often misquoted as 1970, due to misprints in a footnote on the first page of the paper. Even the legendary Fred Hoyle had consistently made this mistake and never corrected it all through the whole of his career, that is to say, in two of his books and at least one of his papers, spanning the period of approximately forty years in total (*cf* Hoyle, 1977). Out of a sample of 278 of those papers which cite this work, this misprint resulted in 80% of the total number of errors in the year cited, which in turn amounts to 1.8% of the number of samples.



The abundances of the chemical elements in the universe. They are assumed to be the same as those found in the primitive solar nebula, which had been deduced from data on abundances found in chondritic meteorites and those found in the Sun. All abundances are relative to that of Si which is taken to be 10^6. Missing bars appear where the atomic numbers are unstable. Except for the atomic number 1 of Hydrogen, which is the most universal element, all other elements with even atomic numbers are locally more abundant than those with near-by odd atomic numbers.

Figure 3.16 Abundance of elements.

Of interest are also the electrical resistivity and conductivity of solid matters. The conductivity σ is by definition the reciprocal of the resistivity ρ. Some of the solids, particularly boron, carbon, silicon, sulphur, germanium, selenium and tellurium, have a distinctively higher resistivity than the majority. Interestingly all of these, with only one exception of boron whose atomic number is five, are of an even atomic number, which respectively from carbon are 6, 14, 16, 32, 34, and 52. This can be seen in Figure 3.17 the data of which are taken from Podesta (2002).



(a) (b)

Figure 3.17 *The electrical resistivity, (a), and conductivity, (b), of elements which are solid at the room temperature.*

	n_c^e	$\mathcal{N}_v^c(E(l_c^e))$	$\mathcal{N}_v^c(\wp_c)$	$\mathcal{N}_v^c(A_c)$
min	3	0.20716	0.2749	0.058428
max	11	8.1072	10.134	22.307
μ	5.8973	0.70626	1.0089	1
σ^2	1.8955	0.18607	0.2942	1.4379
σ	1.3768	0.43136	0.54241	1.1991
μ_g	5.742	0.66036	0.94795	0.79225
μ_h	5.5904	0.62941	0.89999	0.61799
med	6	0.65709	0.96614	0.84267
mad	1.0736	0.17995	0.24357	0.49216
$\mathcal{M}^2$	1.8912	0.18565	0.29355	1.4347
$\mathcal{M}^3$	1.6194	0.96371	1.7931	22.703
$\mathcal{M}^4$	12.438	6.8376	15.785	468.03
$\mathcal{K}$	3.4775	198.38	183.18	227.39

Table 3.7 *Neighbour statistics. Here for the normalisation purpose,* $l_{\text{basis}}^e = 0.046662, \wp_{\text{basis}} = 0.18665, A_{\text{basis}} = 0.0021773$*. Simulation uses* `voronoin` *command in Matlab;* $d = 2$*,* $n^c = 448$*,* $n^v = 946$*, CPU time 1.19 seconds.*

Next simulation was done with $d = 2$, $n^c = 3$ to 49551.

Figure 3.18 shows the percentage of space covered by a Voronoi structure. The number of cells is the total number of cells generated. The percent space covered is the volume of the structure after boundary cells, that is cells which extrude the unit volume boundary, have been excluded. The equation of the reference curve is

$$y = -210/\log x + 120,$$

y being the vertical- and x the horizontal axis.

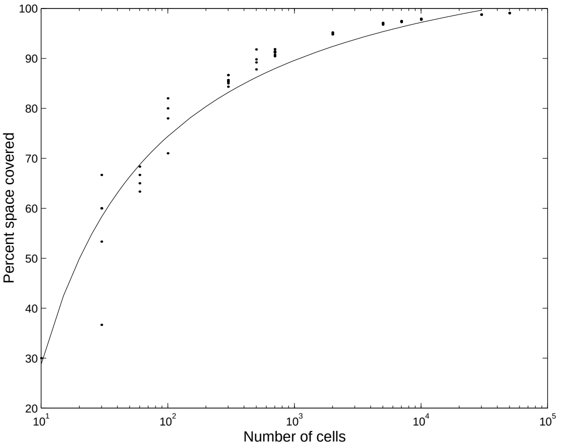


Figure 3.18 Space covered by Voronoi.

The reference line in Figure 3.19 is the linear equation

$$y = 2x + 10.$$

Again, here x is the horizontal-, while y the vertical axis. In other words the relationship between n_c and n_v is linear, which is to be expected in any random tessellation and is a necessity in any regular tessellation. The larger the network the more linear this relationship becomes. Boundary cells have been excluded.

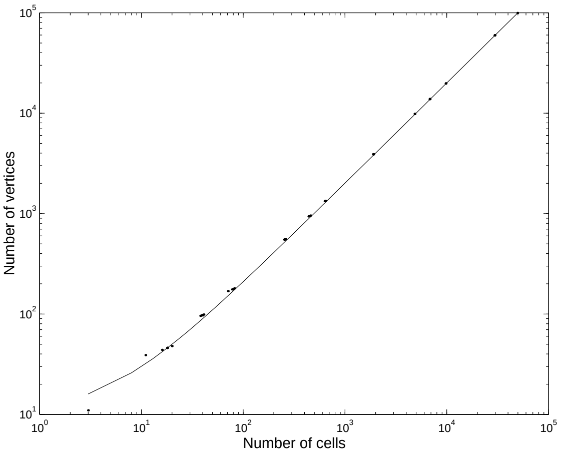
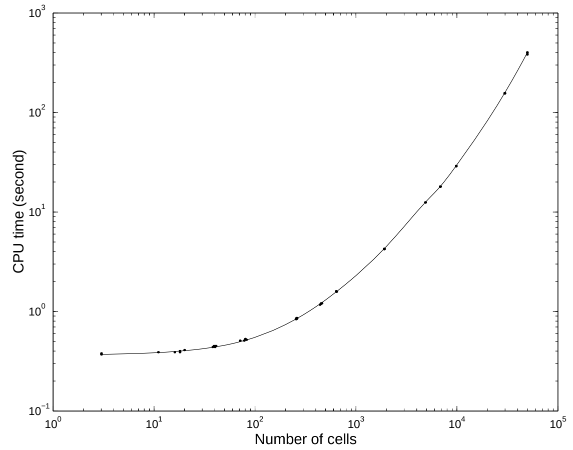
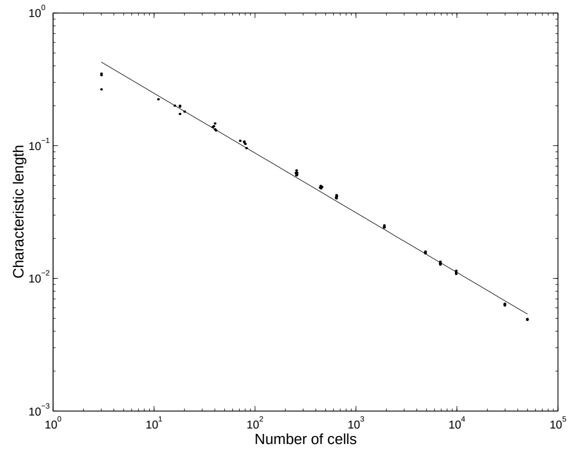


Figure 3.19 Number of vertices versus number of cells.



The curve in Figure 3.20 is the result of curve fitting by cubic spline interpolation which fits a different cubic polynomial between each pair of data points.

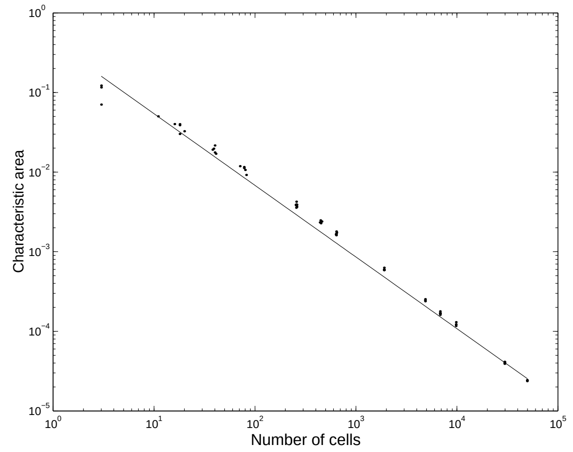
Figure 3.20 The CPU time in seconds.



The characteristic length is the length of the side of the cubic structure having the same number of cells and the same total volume as the Voronoi structure. The characteristic lengths in Figure 3.21 are shown as dots. The reference line is

$$y = 0.7/x^{0.45}.$$

Figure 3.21 Characteristic length versus the number of cells.



The characteristic area is the area of each square in the assembly the total volume and the number of cells of which are the same as those of the Voronoi graph. The reference curve shown in Figure 3.22 is

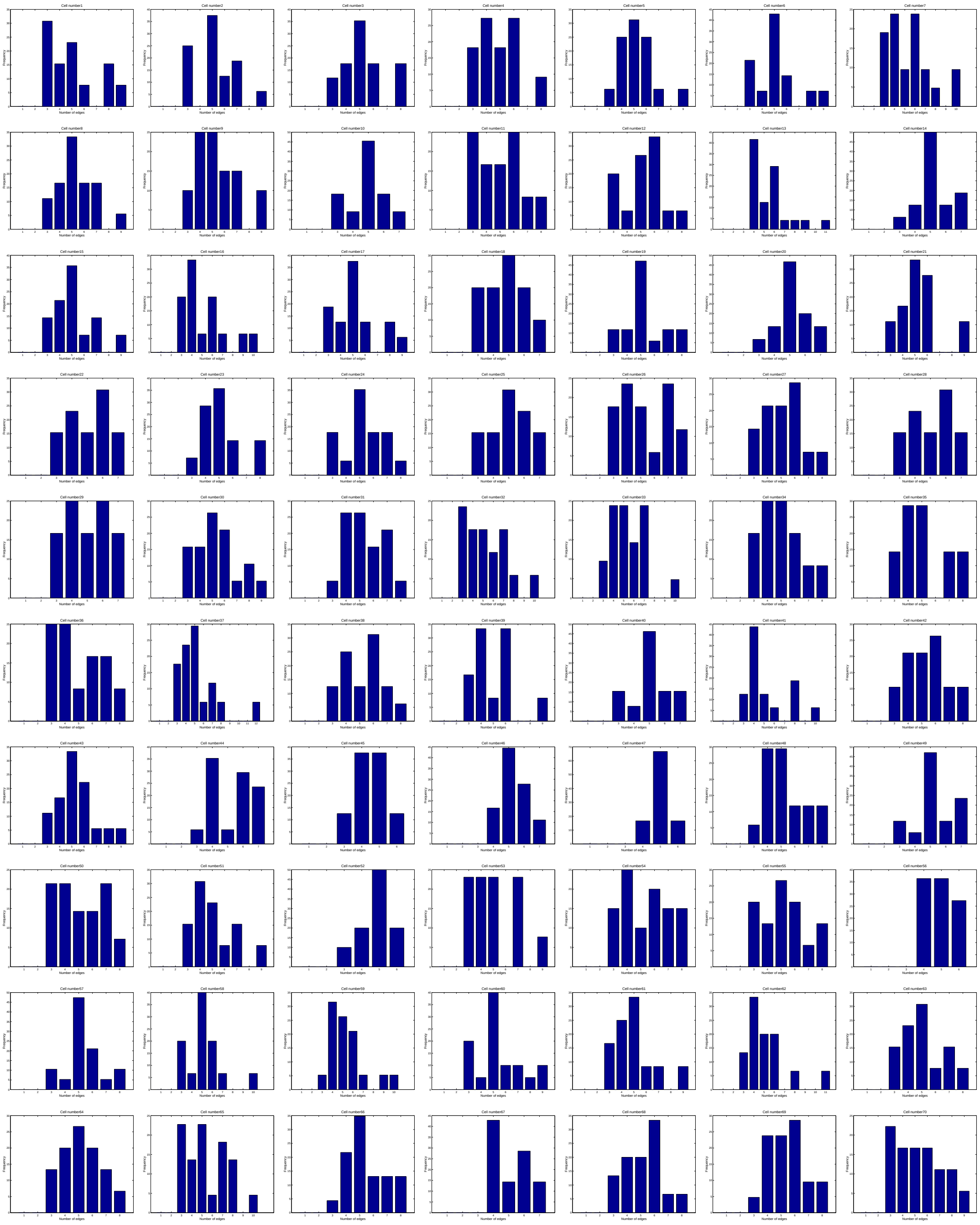
$$y = 0.43/x^{0.9}.$$

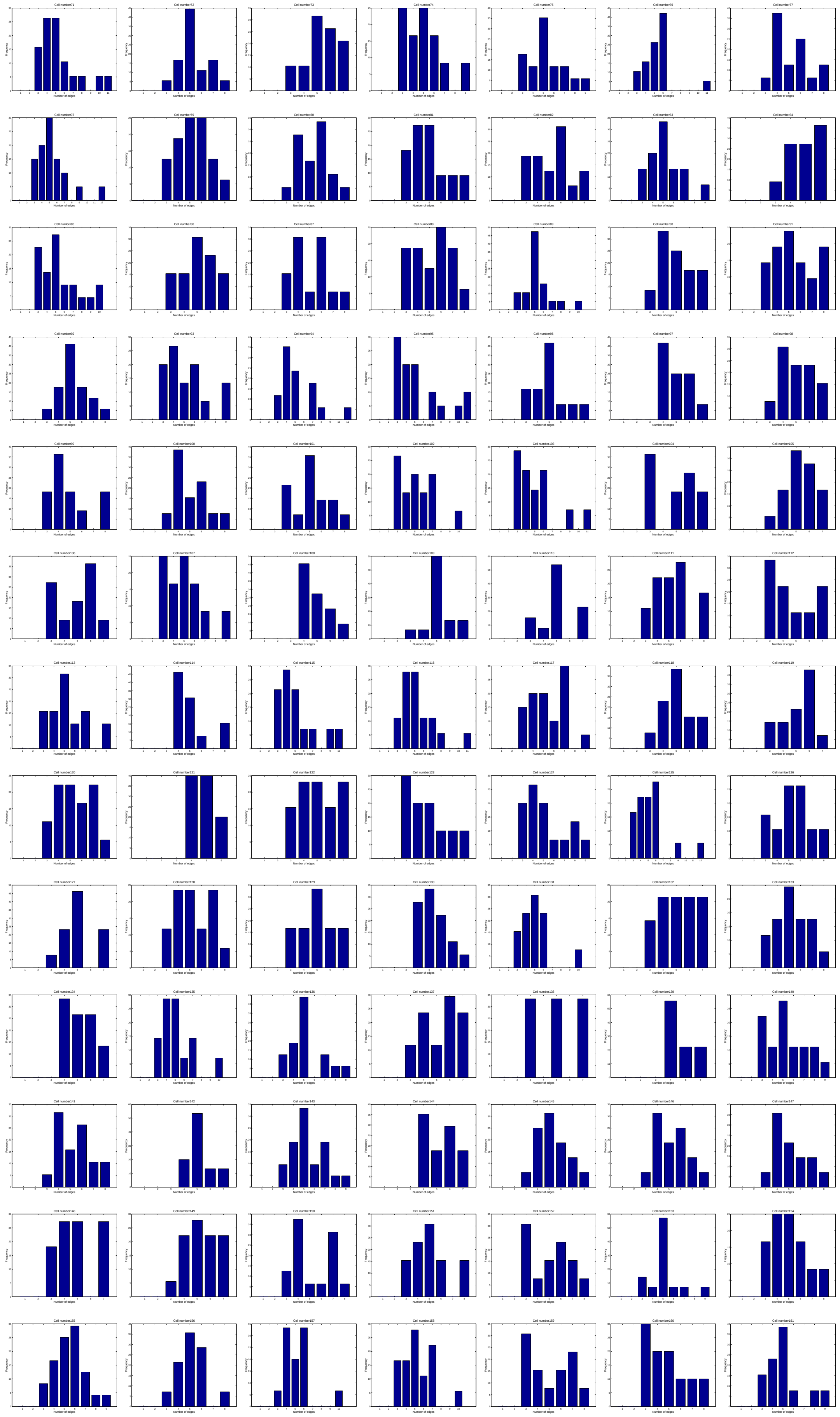
Both Figure 3.21 and Figure 3.22 are hyperbolic.

Figure 3.22 Characteristic area versus the number of cells.

	n_c^v	n_f^v	$\mathcal{N}_v^c(\mathcal{P}_c^f)$	$\mathcal{N}_v^c(A_f)$	$\mathcal{N}_v^c(A_c)$	$\mathcal{N}_v^c(V_c)$	n_c^f
min	8	3	0.001414	4.6179×10^{-7}	0.11908	0.21464	6
max	42	10	0.84885	0.82919	0.50425	12.719	23
μ	24.423	5.1114	0.32876	0.12535	0.30315	1.0000	14.211
σ^2	44.105	2.1650	0.030844	0.015846	0.008511	2.5764	11.026
μ_g	23.504	4.9065	0.25897	0.051216	0.28815	0.70118	13.825
μ_h	22.506	4.7086	0.12620	2.50×10^{-4}	0.27198	0.58794	13.423
med	22	5	0.33219	0.084165	0.30368	0.64077	13
mad	5.4338	1.1655	0.14499	0.099433	0.075445	0.67950	2.7169
$\mathcal{M}^2$	43.483	2.1619	0.030800	0.015823	0.008391	2.5401	10.871
$\mathcal{M}^3$	112.13	1.8902	5.92×10^{-4}	0.002805	4.8×10^{-5}	24.169	14.016
$\mathcal{K}$	3.1096	3.0234	2.3090	5.2926	2.3015	42.238	3.1096

Table 3.8 *From* `rbox` *(200 random points, seed 34565473) and* `qhull` *(option v and o);* $d = 3$, $n^c = 71$.




Cell number71
Cell number72
Cell number73
Cell number74
Cell number75
Cell number76
Cell number77
Cell number78
Cell number79
Cell number80
Cell number81
Cell number82
Cell number83
Cell number84
Cell number85
Cell number86
Cell number87
Cell number88
Cell number89
Cell number90
Cell number91
Cell number92
Cell number93
Cell number94
Cell number95
Cell number96
Cell number97
Cell number98
Cell number99
Cell number100
Cell number101
Cell number102
Cell number103
Cell number104
Cell number105
Cell number106
Cell number107
Cell number108
Cell number109
Cell number110
Cell number111
Cell number112
Cell number113
Cell number114
Cell number115
Cell number116
Cell number117
Cell number118
Cell number119
Cell number120
Cell number121
Cell number122
Cell number123
Cell number124
Cell number125
Cell number126
Cell number127
Cell number128
Cell number129
Cell number130
Cell number131
Cell number132
Cell number133
Cell number134
Cell number135
Cell number136
Cell number137
Cell number138
Cell number139
Cell number140
Cell number141
Cell number142
Cell number143
Cell number144
Cell number145
Cell number146
Cell number147
Cell number148
Cell number149
Cell number150
Cell number151
Cell number152
Cell number153
Cell number154
Cell number155
Cell number156
Cell number157
Cell number158
Cell number159
Cell number160
Cell number161
Frequency
Number of edges

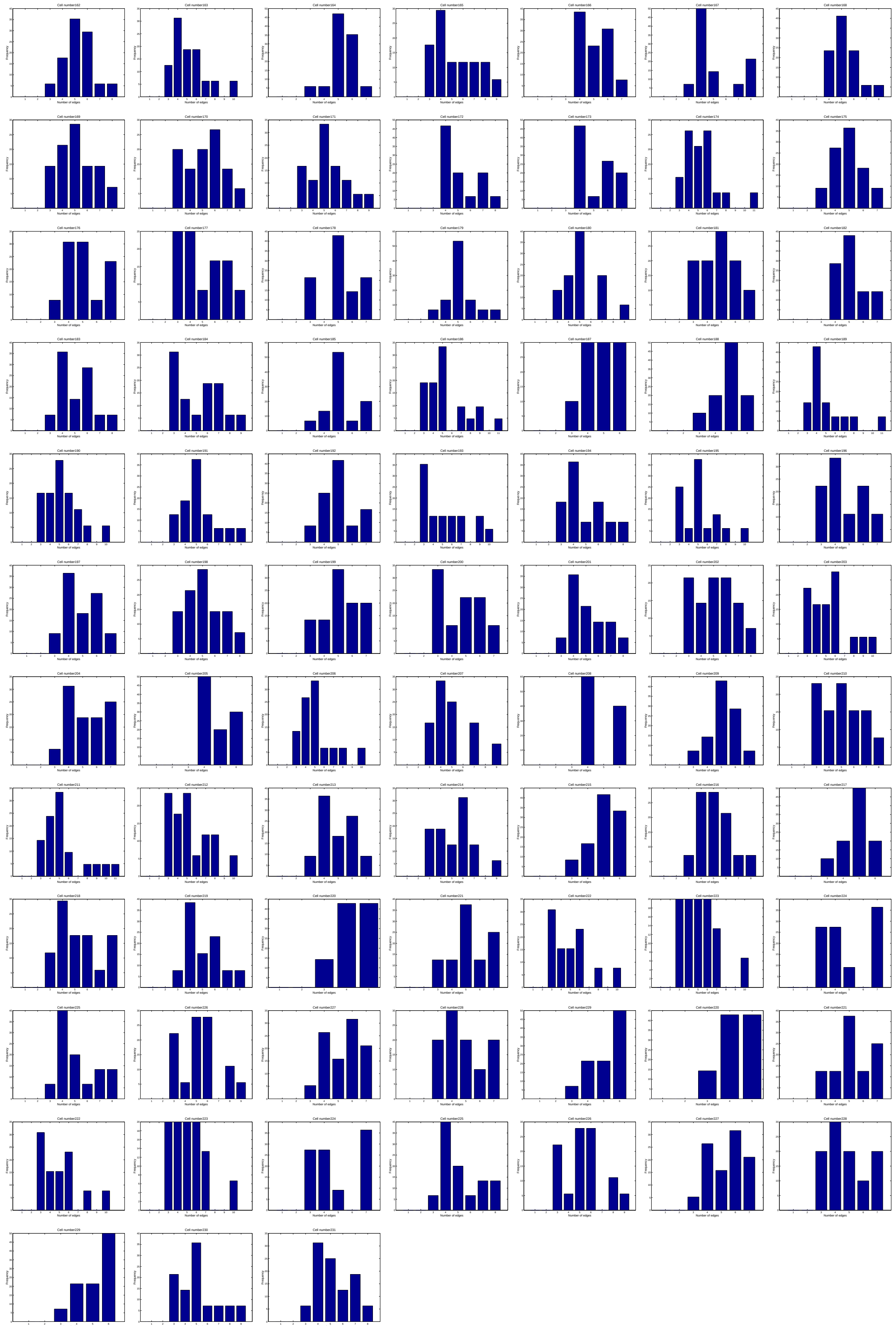


Figure 3.23 *Distribution of the number of edges per face. Each picture is an individual cell. The distribution shows the relative abundance or the number of faces (the vertical axes) having the number of edges as shown by the horizontal axes. The horizontal axis scales are positive integers starting from zero at the origin. Simulation uses* `rbox` *(500 random points, seed 893280) and qhull (option v and o);* $d = 3$, $n^c = 231$, $n^v = 3{,}107$, *CPU time 81.1 sec for finding face area, 2.22 sec for cell volume, 49.59 sec for counting edges and 0.03 sec for finding the number of edges and faces.*

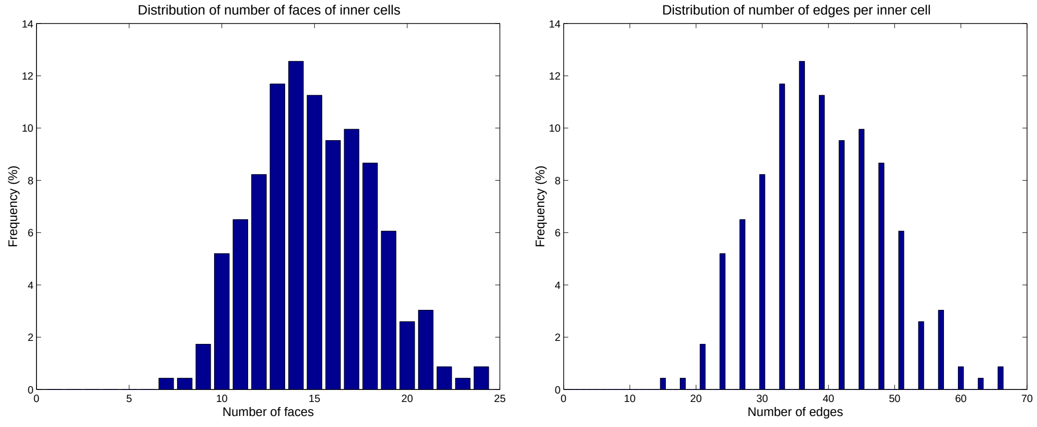
The minimum number of edges for each face is three. This number is the same as the number of vertices of that face. From these figures most of the cells have at least one face with three edges. There are only 19 cells (8.23 per cent) which does not have any three-edged face, and all of them have some four-edged faces. Therefore there is no cell with five as the minimum number of edges per face. The maximum number of edges per face is less clear-cut. There are twelve cells (5.19 per cent) with 11 as the maximum number of edges per face and two (0.87 per cent) with 12.

	n_c^v	$\aleph_c$	A_c^{fr}	V_c^{fr}	α	n_c^e	$E(n^e_{f,c})$	$\sigma(n^e_{f,c})$
min	10	7	1.0989×10^{-3}	2.5790×10^{-4}	0.0406	15	4.2857	0.6030
max	44	24	1.0956×10^{-2}	0.24840	4.2505	66	5.5000	2.7028
μ	25.974	14.987	4.3290×10^{-3}	4.3290×10^{-3}	1.2515	38.961	5.1601	1.5450
σ^2	40.852	10.213	3.3308×10^{-6}	2.9419×10^{-4}	0.3802	91.916	0.0372	0.1243
σ	6.3915	3.1958	1.8250×10^{-3}	1.7152×10^{-2}	0.6166	9.5873	0.1928	0.3526
μ_g	25.166	14.642	3.9633×10^{-3}	1.8465×10^{-3}	1.0816	37.749	5.1564	1.5030
μ_h	24.323	14.288	3.6026×10^{-3}	1.2471×10^{-3}	0.8241	36.484	5.1526	1.4580
med	26	15	4.1467×10^{-3}	1.5715×10^{-3}	1.1915	39	5.2000	1.5315
mad	5.1532	2.5766	1.4064×10^{-3}	4.5864×10^{-3}	0.4720	7.7298	0.1505	0.2794
$\mathcal{M}^2$	40.675	10.169	3.3163×10^{-6}	2.9291×10^{-4}	0.3785	91.518	0.0370	0.1238
$\mathcal{M}^3$	59.377	7.4222	5.7258×10^{-9}	6.3817×10^{-5}	0.2143	2.0040×10^2	-0.0071	0.0058
$\mathcal{M}^4$	4.5398×10^3	2.8374×10^2	4.6684×10^{-11}	1.5395×10^{-5}	0.7390	2.2983×10^4	0.0063	0.0463
$\mathcal{K}$	2.7440	2.7440	4.2448	1.7943×10^2	5.1578	2.7440	4.5730	3.0230

Figure 3.24 *Statistics of a 231 cells Voronoi structure.*

Because the Voronoi tessellation being studied is simple, $\aleph_c^v = \aleph_c^f = n_c^f$. In other words, any two cells having at least one vertex in common are neighbours to each other, and the number of neighbours around any cell in an infinite network is equal to the number of its faces.

Figure 3.25 shows the distributions of the number of faces and edges per cell.



(a) (b)

Figure 3.25 *The distributions of the number of* (a) *faces, and* (b) *edges, per cell.*

The results from 527 Voronoi cells are shown next. Table 3.9 is from a 3-d Voronoi structure generated from 1,000 cells. The creation codes started with a random seed of 234985. They spent 6,466.99 seconds for the counting of statistics, 270.56 for finding area of the faces, 4.47 for calculating cell volume and additional 2.8 seconds to find statistics for the number of edges. There are 6,357 vertices generated, plus another added distant vertice. Boundary cells were discarded, which leaves us with 527 inner cells which are taken into account.

	$v_{c_{in}}$	$\aleph_{v,c_{in}}$	$A_{fr,c_{in}}$	$V_{fr,c_{in}}$	$n_{f,c_{in}}$	$n_{e,c_{in}}$	
Min	8	6	2.2191×10^{-4}	9.9722×10^{-5}	6	12	
Max	46	25	4.7015×10^{-3}	0.14364	25	69	
μ	26.338	15.169	1.8975×10^{-3}	1.8975×10^{-3}	15.169	39.507	
$\sigma[2]$	40.608	10.152	5.2144×10^{-7}	4.2313×10^{-5}	10.152	91.368	
σ	6.3725	3.1862	7.2210×10^{-4}	6.5048×10^{-3}	3.1862	9.5587	
μ_g	25.543	14.829	1.7572×10^{-3}	1.1575×10^{-3}	14.829	38.315	
μ_h	24.703	14.478	1.6026×10^{-3}	8.5548×10^{-4}	14.478	37.054	
Med	26	15	1.8125×10^{-3}	1.1660×10^{-3}	15	39	
δ_μ	5.0068	2.5034	5.6975×10^{-4}	1.4250×10^{-3}	2.5034	7.5103	
$M[2]$	40.531	10.133	5.2045×10^{-7}	4.2233×10^{-5}	10.133	91.195	
$M[3]$	72	9	2.5644×10^{-10}	5.4435×10^{-6}	9	243	
$M[4]$	5088.9	318.06		1.0467×10^{-12}	7.6653×10^{-7}	318.06	25763
κ	3.0978	3.0978	3.8644	429.77	3.0978	3.0978	

Table 3.9 *Statistics from 527 Voronoi cells.*

The mean value of the mean numbers of edges per face obtained was 5.1712. This can be called $\mathrm{E}(\mathrm{E}(n_{e,f}))$, the expected value over all cells of the mean number of edges in a face averaged over all its faces. The variance was 0.035983, minimum value 4, maximum 5.52, standard deviation 0.18969, geometric mean 5.1676, harmonic mean 5.1638, median 5.2, mean absolute deviation 0.14465, 2^{nd} moment 0.035915, 3^{rd} moment -0.0083404, and kurtosis 6.1689. The mean and standard deviation of all faces are given in Appendix B.6. The standard deviation of $\mathrm{E}(n_{e,f})$ averaged over all faces of bounded cells is 1.5586. The rest are $\sigma^2 = 0.1186$, $\sigma = 0.3444$, minimum 0.6030, maximum 2.5690, $\mu_g = 1.5169$, $\mu_h = 1.4700$, median 1.5706, $\delta_\mu = 0.2715$, $M^2 = 0.1184$, $M^3 = -0.0060$, and $\kappa = 2.9916$.

Figure 3.26 shows the ratios between the number of vertices and the number of cells in Voronoi networks of various dimensions. The line shown in Figure 3.26 is

$$y = 0.2410e^n,$$

where n is the dimension of the network and is the horizontal axis; the coefficient of the exponential term is obtained by averaging over the averages in each dimension, each of which in turn comes from five batch runs.

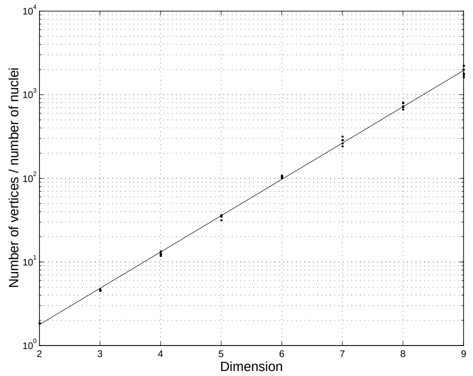


Figure 3.26 v_n/c_n in Voronoi networks of various dimensions.

Figure 3.27 shows the CPU time in creating the Voronoi networks for Figure 3.26. The line shown in Figure 3.27, found manually by trial and error, has the equation

$$y = 4.61 \times 10^{-6}(2+e)^n.$$

In comparison, substituting the average cpu time for each dimension for y in the equation $y = Ae^n$ to obtain A and then find the average again over all dimensions results in $\overline{A} = 1.612 \times 10^{-4}$. The programme which produces both Figure 3.27 and 3.26 is given in § A.9.

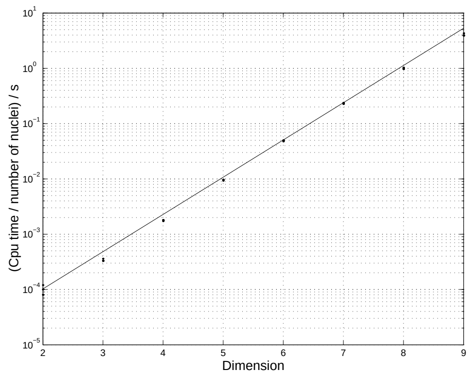
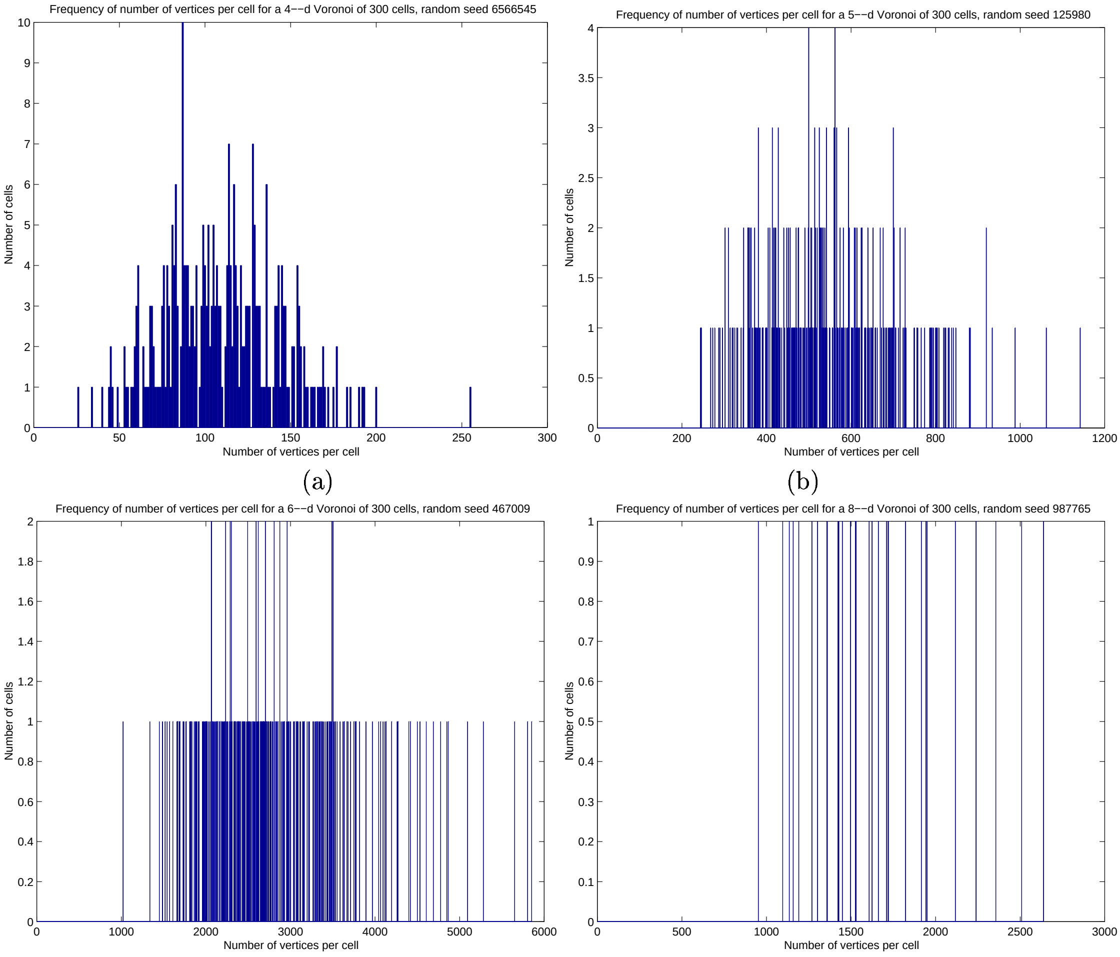


Figure 3.27 CPU time in creating the Voronoi networks.

The number of vertices of higher dimensions is investigated briefly in the following Table 3.10 and Figure 3.28 the simulation of which was carried out by the programme in § A.18.

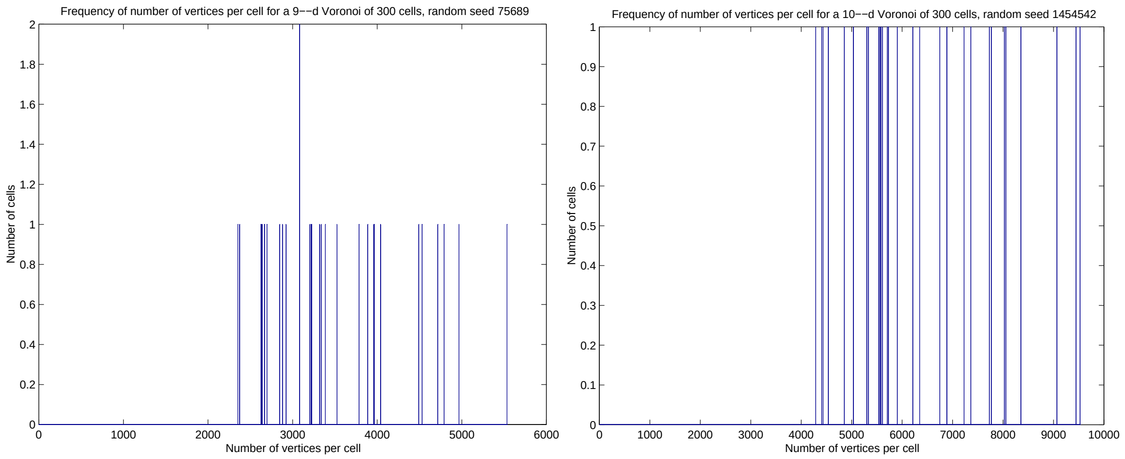
Dimension	4	5	6	8	9	10
N_c	300	300	300	30	30	30
N_v	6,577	27,150	118,534	5,465	10,467	17,442
n_c	132	104	14	2	3	0
$\min(n_c^v)$	26	244.00	1,020.0	952.002	2,353.0	4,287.0
$\max(n_c^v)$	255	1,142.0	5,852.0	2,638.0	5,534.0	9,528.0
$\bar{n}_c^v$	109.90	543.51	2,766.5	1,640.2	3,489.7	6,396.0
$(\sigma_{n_v}^2)_c$	1,155.9	22,974	6.0898×10^5	1.7964×10^5	7.0328×10^5	2.3635×10^6
$\sigma_c^{n_v}$	33.999	151.57	780.37	423.84	838.62	1.5374×10^3
$\mu_g(n_c^v)$	104.41	523.06	2,666.1	1,589.9	3,398.4	6,225.6
$\mu_h(n_c^v)$	98.389	502.94	2,571.4	1,542.3	3,313.9	6,065.2
$\mathrm{med}(n_c^v)$	107.50	527.00	2,651.0	1,568.5	3,272.5	5,816.5
$\mathrm{mad}(n_c^v)$	27.366	119.39	588.19	331.96	687.47	1,296.6
$m^2(n_c^v)$	1,152.1	22,897	6.0695×10^5	1.7365×10^5	6.7984×10^5	2.2847×10^6
$m^3(n_c^v)$	15,555	2.2871×10^6	5.1556×10^8	4.7965×10^7	3.9108×10^8	1.8988×10^9
$m^4(n_c^v)$	4.5760×10^6	1.8905×10^9	1.7964×10^{12}	8.5043×10^{10}	1.1967×10^{12}	1.1638×10^{13}
$\kappa(n_c^v)$	3.4476	3.6059	4.8764	2.8202	2.5892	2.2296

Table 3.10 *Statistics of the number of vertices in 4, 5, 6, 8, 9, and 10 dimensions.*



(a) (b)

(c) (d)



(e) (f)

Figure 3.28 *Distribution of vertices in* (a) *4,* (b) *5,* (c) *6, and* (d) *8,* (e) *9,* (f) *10 dimensions.*

To obtain the number of vertices per cell, n_c^v, of a six-dimensional Voronoi structure of 1,000 cells a batch programme was used, for instance the one listed in § A.19. The Matlab macro that this programme refers to opens and reads from a file the number of vertices and then finds the statistical values, *viz.* $\min(n_c^v) = 1{,}198$; $\max(n_c^v) = 9{,}923$; $\bar{n}_c^v = 4{,}201.1$; $(\sigma^2_{n_v})_c = 1.4069 \times 10^6$; $\sigma_c^{n_v} = 1186.1$; $(\bar{n}_g)_c^v = 4035.4$; $(\bar{n}_h)_c^v = 3866.8$; $\mathrm{med}(n_c^v) = 4122.5$; $\mathrm{mad}(n_c^v) = 931.81$; $m^2(n_c^v) = 1.4055 \times 10^6$; $m^3(n_c^v) = 1.0462 \times 10^9$; $m^4(n_c^v) = 7.7148 \times 10^{12}$; and $\kappa(n_c^v) = 3.9053$.

The following figure shows the frequency of the number of vertices. Interested readers may find the data in § B.6 which give rise to Figure 3.29 interesting. These results are not the most useful ones because of the presence of border cells.

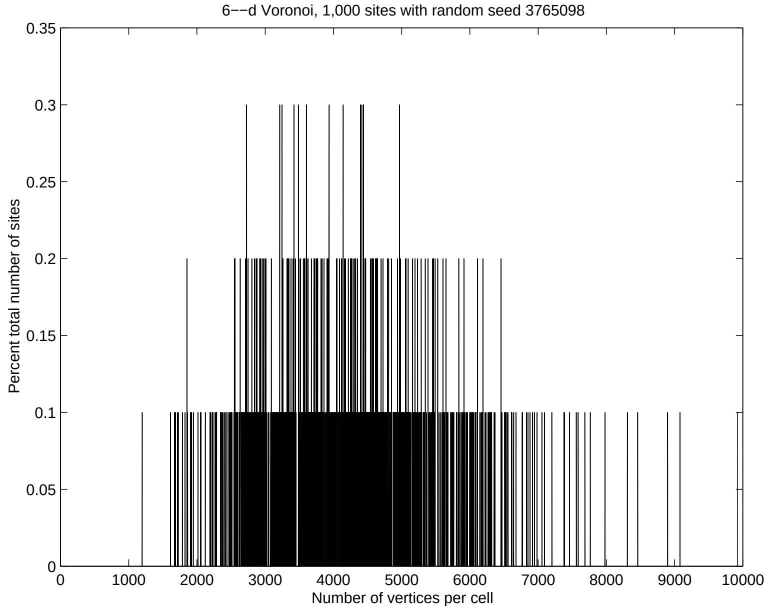


Figure 3.29 *Distribution of vertices in a 6-d Voronoi structure*

AVS

The first picture I created on AVS was a Trigonal Dipyramidal. The programme was the following .inp file.

```
5 2 0 0 0
1 0 0 1.2910
2 2 0 1.2910
3 1 1.7321 1.2910
4 1 0.5774 0
5 1 0.5774 2.5820
1 1 tet 1 2 3 4
2 1 tet 1 2 3 5
1 2
```

The connection of modules was ReadUCD to ExternalEdges to UViewer3D.

The following simulation was done on a 3–d systems with a joggled-input option in `qhull`. The number of vertices, however, also takes into account those belonging to the boundary cells. The minimum $n_{v,c}$ was 6, maximum $n_{v,c}$ 52, $\bar{n}_{v,c}$ 25.269, 2^{nd}–Moment 49.363, 3^{rd}–Moment 228.87, 4^{th}–Moment 8,704.9, $\sigma^2_{n_{v,c}}$ 49.412, $\sigma_{n_{v,c}}$ 7.0294, $\mu_{g,n_{v,c}}$ 24.317, $\mu_{h,n_{v,c}}$ 23.365, $\widetilde{n}_{v,c}$ 24, $\delta_\mu(n_{v,c})$ 5.5428, $\kappa(n_{v,c})$ 3.5725 .

Following figure is a distribution graph.

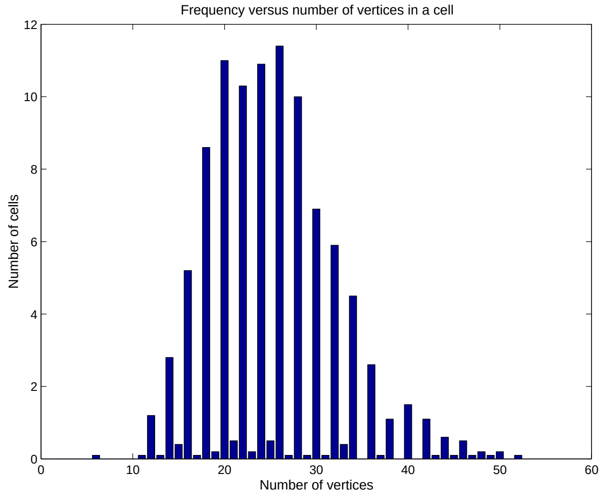


Figure 3.30 Number of cells having a particular number of vertices.

In Figure 3.30 boundary cells have not been excluded. So there are still some cells which have an odd number of vertices. But even these are few and far between, and only because of them that the lack of odd vertice-numbered cells has been noticed. Had there been nothing there it is difficult to see how this uniform characteristic, or invariance, of the Voronoi tessellation could have come to light.

Theorems 3.3 and 3.4 above are observations which are not only useful but essential when you try to understand the valence relations. Assuming that the Euler's theorem is true, then these two theorems give rise to Algorithm 3.2 which I have devised for counting all the components in these relations. Here n_e and n_v are all the edges, and respectively vertices, of the original structure originally counted, n^e_b all the additional edges, going to infinity, which are drawn to complete the valences of some of the boundary vertices, c_n cells with n edges and f_n the frequency occurences of c_n.

Algorithm 3.2 *Valence relations for planes in two dimensions.*

```
draw an addition bond for all boundary vertices which have valence 2;
$e \leftarrow \lfloor \mathbf{n_e + n^e_b/3} \rfloor$;
draw an additional $(n_e + n^e_b - e)$ on boundary vertices;
if $\lfloor n_v/2 \rfloor \bmod 2 \neq 0$ then
  draw an additional vertex at infinity;
  $v \leftarrow n_v + 1$;
else
  $v \leftarrow n_v$;
endif
$f \leftarrow 1 + e - v$;
label all the $f_n$ bound faces;
if $(n_f + n^f_b) = f$ then
  label all the $f^n_b$ unbound faces;
else
```

```
    label $(f - n_f)$ unbound faces;
    label all the remaining $(n_f + n_b^f - f)$ unbound faces together as a single face;
endif
```

□

In Algorithm 3.2 the vertex at infinity mentioned does not necessarily have to be literally at the infinity. Although Algorithm 3.2 is for planes in two dimensions, it should be possible to extend it to other topolgical objects and to higher dimensions by simply changing the Euler's equation to the appropriate one. When labelling f, if $(n_f + n_b^f) \neq f$, we know that $(n_f + n_b^f) > f$ with probability one.

Paradoxically this relation holds only for infinite networks, but its derivation, as well as its verification, can only be done on a tessellation of finite size. Counting the components of the valence relation one sees in a graph involves both the Euler's theorem and Theorem's 3.3 and 3.4.

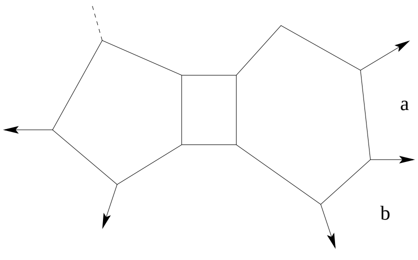


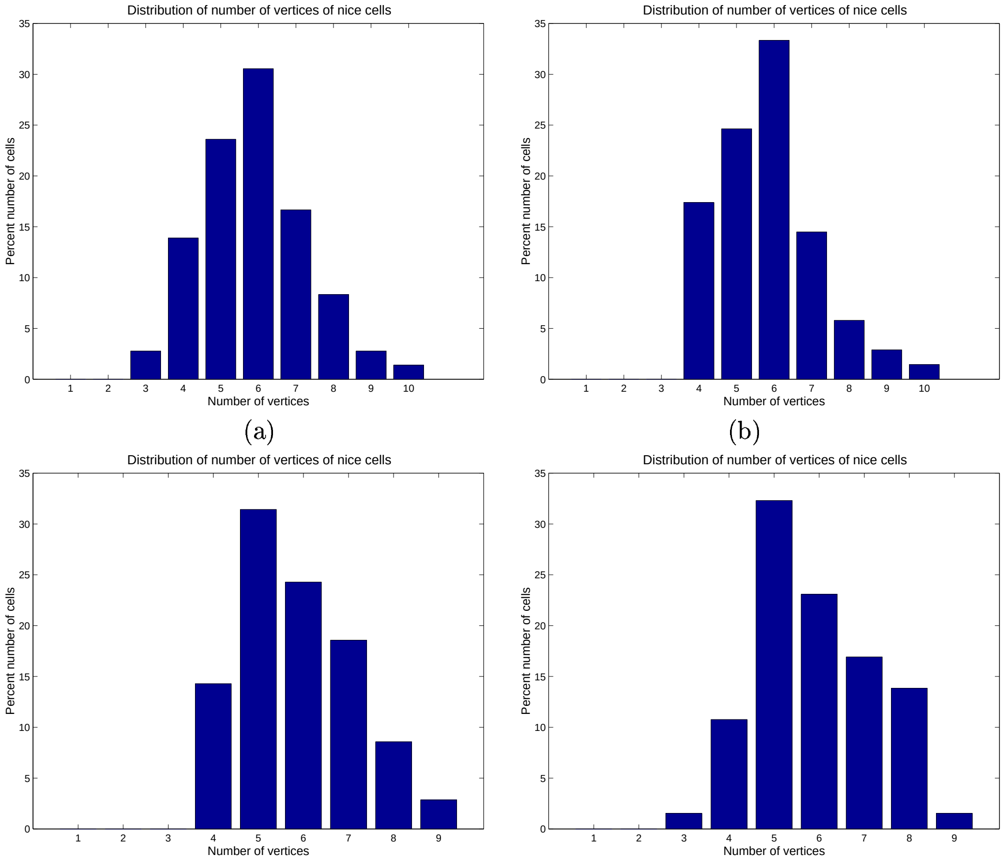
Figure 3.31 Counting the valences.

§ 3.8 Faces in different dimensions

Considering only those cells bound within the unit box, vertices and all, the following, namely Tables 3.11, 3.12 and 3.13, are the results from five simulations in two, three and four dimensions respectively, using the programme listed in § A.16.

	Random seed				
	829247	134315	67453	432243	231215
N_v	187	187	186	189	183
n_v	170	167	168	163	170
n_c	72	69	70	65	72
n_c^v	169	165	166	159	170
μ_c^v	5.875	5.8116	5.8429	5.9077	5.8194
$(\sigma_v^2)_c$	2.0264	1.8022	1.6706	1.7726	1.5022
$m^2(n_c^v)$	1.9983	1.7761	1.6467	1.7453	1.4813
$m^3(n_c^v)$	1.1263	1.6917	0.96591	0.57642	0.96103
n_{1f}	241	235	237	227	241
n_c^{1f}	240	233	235	223	241
$\mathrm{t_{CPU}}$(second)	20.36	20.39	20.09	20.6	19.76

Table 3.11 *Faces of Voronoi in two dimensions.* $\mathrm{N}_c = 100$.



(a)

(b)

(c)

(d)

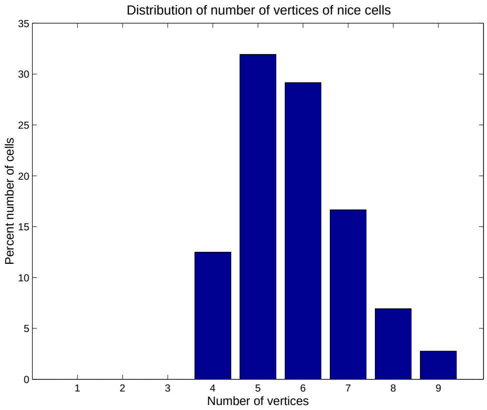


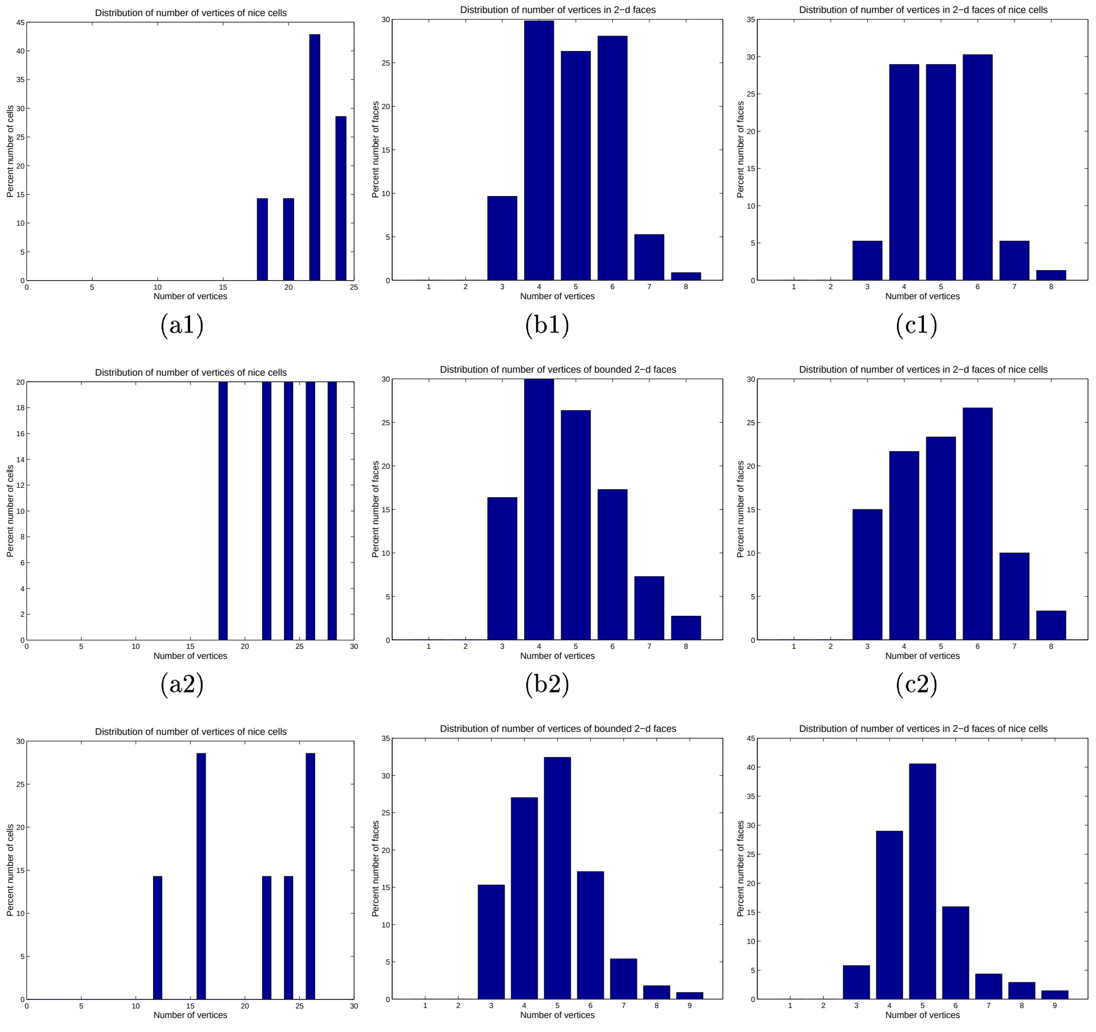
Figure 3.32 shows the distribution of v_c in each simulation on 2-d Voronoi. The size of the network is very small, which explains why the shape of the distribution varies greatly from one simulation to another. This is also the reason why all the bars in the graph have only limited number of possible heights.

(e)

Figure 3.32 Distribution of v_c in two dimensions.

	Random seed				
	42398198	83250	34959	743690	1321
N_v	204	221	224	225	214
n_v	148	147	143	155	146
n_c	7	5	7	8	7
n_c^v	99	79	85	109	97
μ_c^v	21.714	23.6	20.286	21	21.714
$(\sigma_v^2)_c$	4.5714	14.8	31.238	34.286	21.905
$m^2(n_c^v)$	3.9184	11.84	26.776	30	18.776
$m^3(n_c^v)$	-4.6181	-16.128	-42.402	171	69.831
$n_{2\mathrm{f}}$	114	110	111	123	113
$\mu_{2\mathrm{f}}^v$	4.9211	4.7727	4.7928	4.8293	4.7788
$(\sigma_v^2)_{2\mathrm{f}}$	1.2592	1.6268	1.5476	2.0772	1.656
$m^2(n_{2\mathrm{f}}^v)$	1.2482	1.612	1.5336	2.0603	1.6413
$m^3(n_{2\mathrm{f}}^v)$	0.16453	0.99264	1.1334	1.7676	1.1444
$n_c^{2\mathrm{f}}$	76	60	69	88	75
$(\mu_{2\mathrm{f}}^v)_c$	5.0526	5.05	4.9855	4.9545	5.0533
$(\sigma_v^2)_c^{2\mathrm{f}}$	1.1439	1.811	1.3674	2.0209	1.7268
$m^2((n_{2\mathrm{f}}^v)_c)$	1.1288	1.7808	1.3476	1.9979	1.7038
$m^3((n_{2\mathrm{f}}^v)_c)$	0.19004	0.28275	1.5224	1.4544	0.98057
$n_c^{1\mathrm{f}}$	167	133	146	187	163
$\mathrm{t_{CPU}}$(second)	24.28	25.67	28.49	34.23	26.46

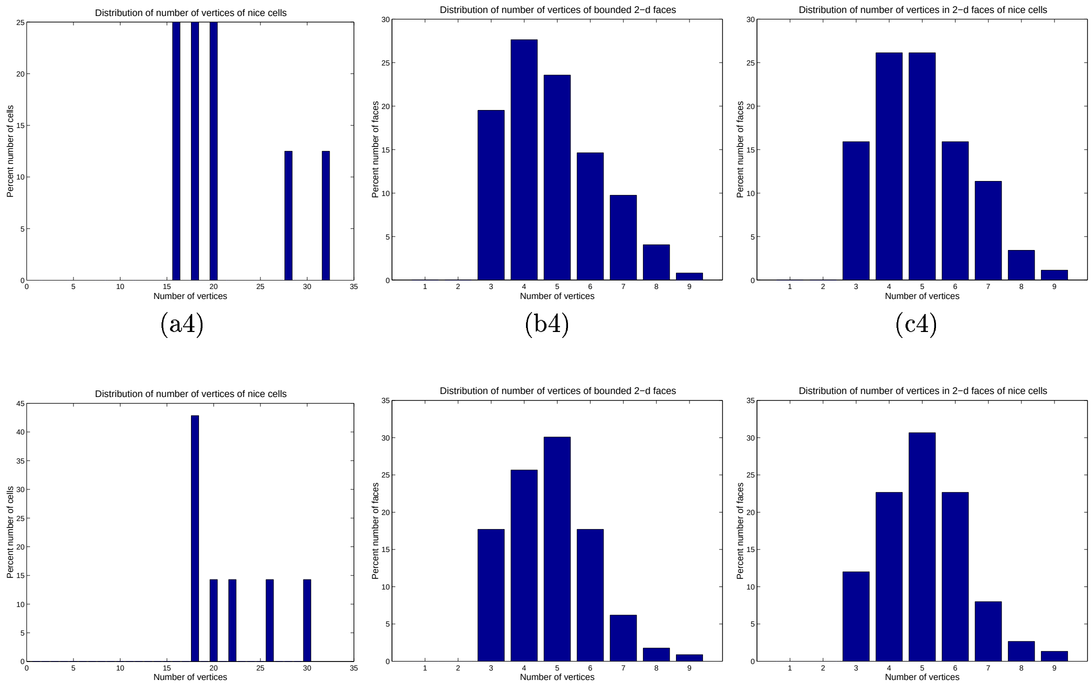
Table 3.12 *Various faces of Voronoi in three dimensions.* $\mathrm{N}_c = 50$.



(a1) (b1) (c1)

(a2) (b2) (c2)

(a3) (b3) (c3)



(a4) (b4) (c4)

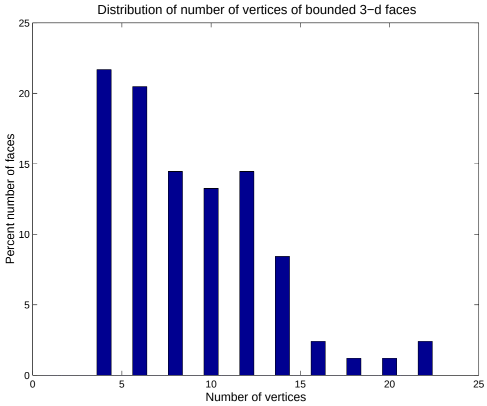
(a5) (b5) (c5)

Figure 3.33 *Distribution of* (ai) v_c, (bi) $v_{2\mathrm{f}}$, *and* (ci) $v_c^{2\mathrm{f}}$ *of the* i^{th} *simulation on 3-d Voronoi.*

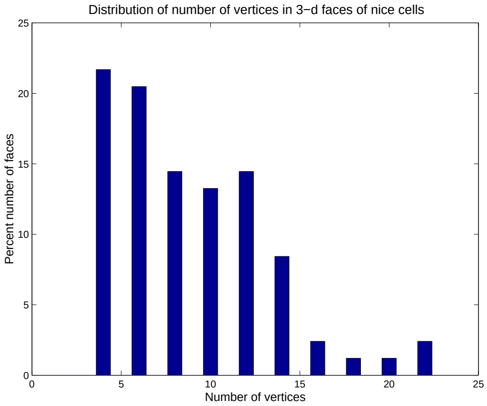
	Random seed				
	24098	802723	1453	732849	20480
N_v	1684	1719	1674	1586	1600
n_v	938	921	860	889	904
n_c	5	1	1	1	2
n_c^v	442	172	149	117	200
μ_c^v	114	172	149	117	110
$(\sigma_v^2)_c$	174.5	0	0	0	72
$m^2(n_c^v)$	139.6	0	0	0	36
$m^3(n_c^v)$	-547.2	0	0	0	0
$n_{3\mathrm{f}}$	83	102	84	83	95
$\mu_{3\mathrm{f}}^v$	8.8675	8.3137	8.5952	8.9398	8.7789
$(\sigma_v^2)_{3\mathrm{f}}$	18.848	18.198	19.449	17.496	19.77
$m^2(n_{3\mathrm{f}}^v)$	18.621	18.019	19.217	17.286	19.562
$m^3(n_{3\mathrm{f}}^v)$	75.225	102.38	95.854	62.919	118.61
$n_c^{3\mathrm{f}}$	32	8	10	7	19
$(\mu_{3\mathrm{f}}^v)_c$	8.8675	9.25	10.8	11.143	9.8947
$(\sigma_v^2)_c^{3\mathrm{f}}$	18.848	13.643	21.511	14.476	28.655
$m^2((n_{3\mathrm{f}}^v)_c)$	18.621	11.938	19.36	12.408	27.147
$m^3((n_{3\mathrm{f}}^v)_c)$	75.225	18.281	66.624	8.5364	195.1
$n_c^{2\mathrm{f}}$	11	0	5	2	7
$n_c^{1\mathrm{f}}$	2	0	0	0	1
$\mathrm{t_{CPU}}$(second)	315.07	358.94	324.56	281.85	283.82

Table 3.13 *Faces of Voronoi in four dimensions.* $\mathrm{N}_c = 100$.

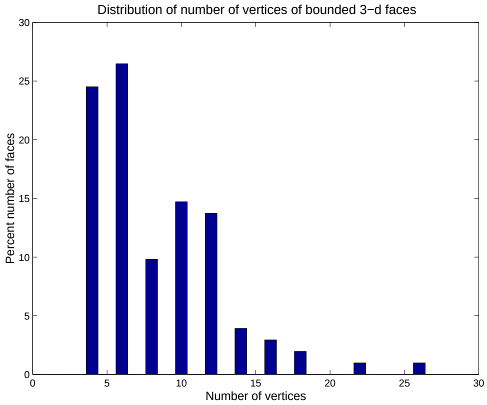
The number of vertices in each 2-d face is a constant equal to three while that of a 1-d face is two. In the first run the number of vertices in each of the cells is 95, 108, 117, 120 and 130; in the last run, this number is 104 and 116.



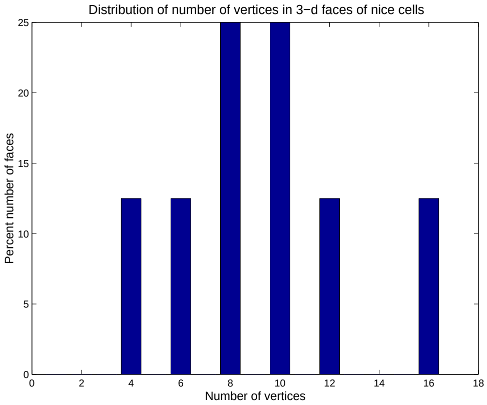
(a1)



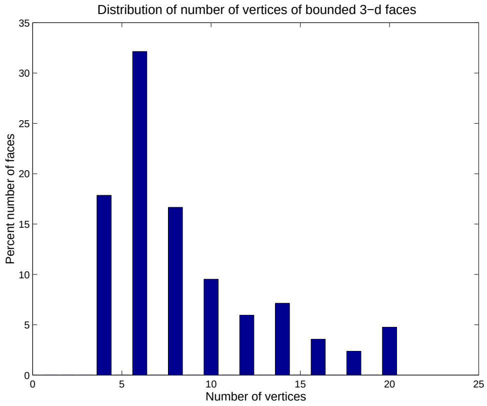
(b1)



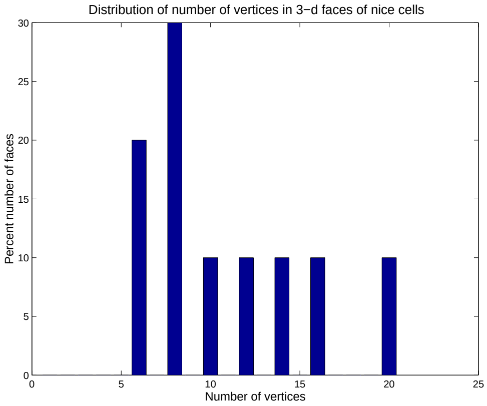
(a2)



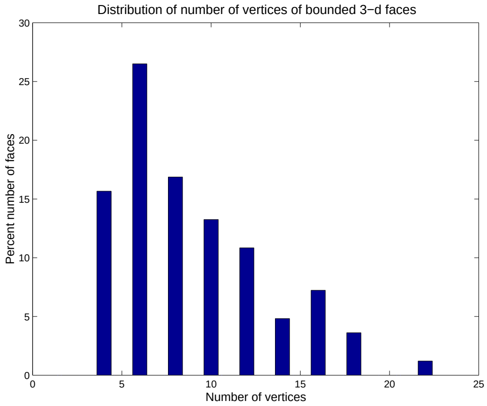
(b2)



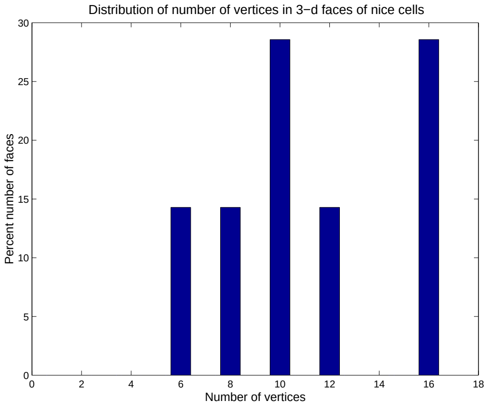
(a3)



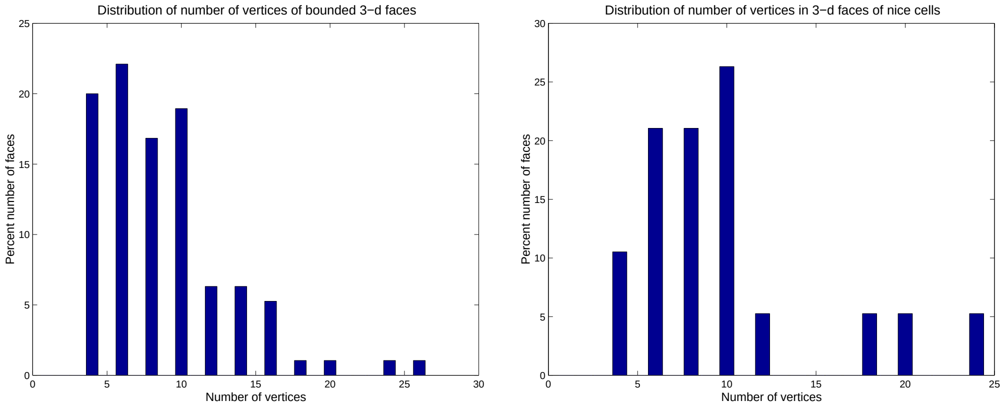
(b3)



(a4)



(b4)



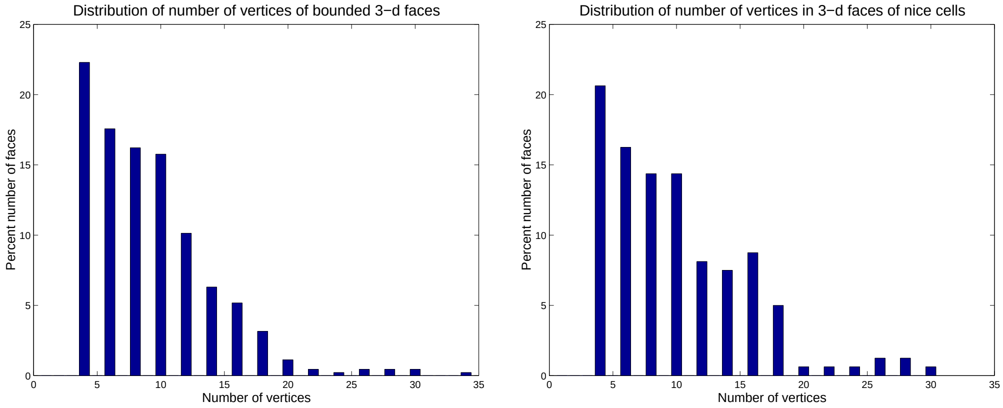
(a5) (b5)

Figure 3.34 *Distribution of* (ai) $v_{3\mathrm{f}}$ *and* (bi) $v_c^{3\mathrm{f}}$ *of the* i^{th} *simulation on 4-d Voronoi.*

Table 3.14 contains the results obtained from a 4-d Voronoi network of 300 original nuclei. The numbers of vertices of the sixteen cells are 97, 99, 112, 141, 145, 160, 170, 171, 176, 176, 184, 186, 188, 192, 216 and 235.

Random seed	91876				
N_v	6776	$m^3(n_c^v)$	-1.7507×10^4	$(\mu_{3\mathrm{f}}^v)_c$	9.9875
n_v	3848	$n_{3\mathrm{f}}$	444	$(\sigma_v^2)_c^{3\mathrm{f}}$	31.1697
n_c	16	$\mu_{3\mathrm{f}}^v$	9.1486	$m^2((n_{3\mathrm{f}}^v)_c)$	30.9748
n_c^v	1687	$(\sigma_v^2)_{3\mathrm{f}}$	24.0411	$m^3((n_{3\mathrm{f}}^v)_c)$	203.3116
μ_c^v	165.5000	$m^2(n_{3\mathrm{f}}^v)$	23.9869	$n_c^{2\mathrm{f}}$	60
$(\sigma_v^2)_c$	1.5100×10^3	$m^3(n_{3\mathrm{f}}^v)$	170.4755	$n_c^{1\mathrm{f}}$	3
$m^2(n_c^v)$	1.4156×10^3	$n_c^{3\mathrm{f}}$	160	$\mathrm{t_{CPU}}$(second)	1.6013×10^4

Table 3.14 *Face statistics of 300 nuclei Voronoi in four dimensions.*



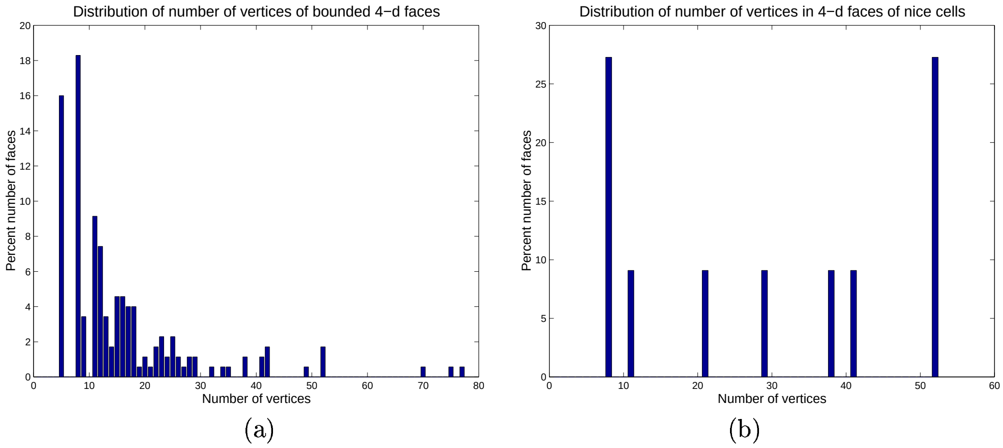
(a) (b)

Figure 3.35 *Distribution of* (a) $v_{3\mathrm{f}}$ *and* (bi) $v_c^{3\mathrm{f}}$ *of 4-d Voronoi, 300 nuclei.*

Table 3.15 is obtained from Voronoi in five dimensions.

Random seed	39378			N_v	16212
n_v	6449	$n_{4\mathrm{f}}$	175	$(\sigma_v^2)_c^{4\mathrm{f}}$	352.2909
n_c	1	$\mu_{4\mathrm{f}}^v$	15.9200	$m^2((n_{4\mathrm{f}}^v)_c)$	320.2645
n_c^v	864	$(\sigma_v^2)_{4\mathrm{f}}$	163.7752	$m^3((n_{4\mathrm{f}}^v)_c)$	351.8362
μ_c^v	864	$m^2(n_{4\mathrm{f}}^v)$	162.8393	$n_c^{3\mathrm{f}}$	1
$(\sigma_v^2)_c$	0	$m^3(n_{4\mathrm{f}}^v)$	4.8764×10^3	$(\mu_{3\mathrm{f}}^v)_c$	4
$m^2(n_c^v)$	0	$n_c^{4\mathrm{f}}$	11	$n_c^{2\mathrm{f}}$	1
$m^3(n_c^v)$	0	$(\mu_{4\mathrm{f}}^v)_c$	29.0909	$\mathrm{t_{CPU}}$(second)	1.8908×10^4

Table 3.15 *Face statistics of 200 nuclei Voronoi in five dimensions.*

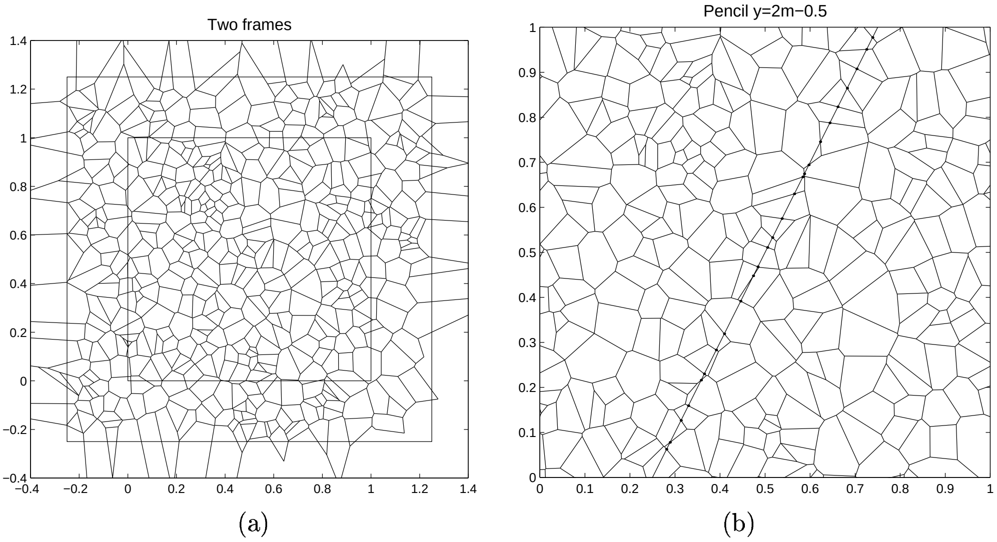


(a) (b)

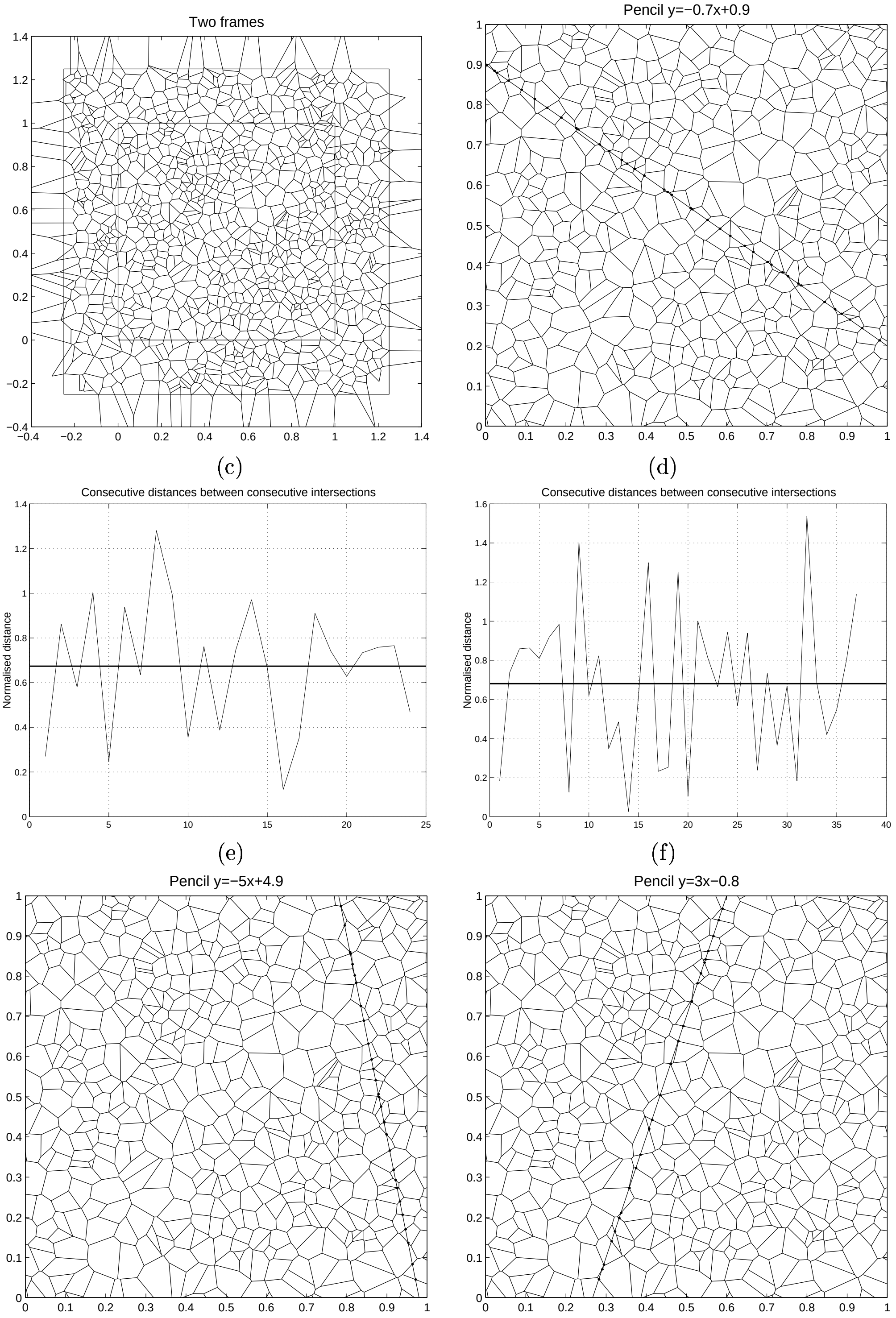
Figure 3.36 *Distribution of* (a) $v_{4\mathrm{f}}$ *and* (bi) $v_c^{4\mathrm{f}}$ *of 5-d Voronoi, 200 nuclei.*

§ 3.9 Beam intersection study

For this study of sectioning by a line the Voronoi in two dimensions, first generate on Matlab 500 points within a square box from -0.25 to 1.25 in both axes. The beam is simply a straight line $y = mx + c$ where m is the slope and c a constant. The term *pencil beam* is used in Astronomy to describe the probing into the depth of the universe with a window of very narrow width. The data collected thus therefore in effect represent data along a Euclidean straight line at various distances. It is largely a very limited technique, but there are no other choices in Astronomy as regard the vast space that separates us from anywhere.



(a) (b)



(c) (d) (e) (f) (g) (h)

Figure 3.37 *Intersection by a line.* (a) *is intersected by* (b) $y = 2x - 0.5$*;* (c) *is intersected by* (d) $y = -0.7x + 0.9$*;* (e) *and* (f) *are corresponding distances of respectively* (b) *and* (d)*;* (c) *is intersected by* (g) $y = -5x + 4.9$ *and* (h) $y = 3x - 0.8$.

A natural basis for the normalisation is the expected distance. Another possible basis is $(1/\sqrt{239}) = 0.064685$. I call *mean normalisation* the normalisation using the first-, and *homogeneity normalisation* the second basis. Graphs of the closest pair distances look the same for both types of normalisation.

Simulation	1	2	3	4
N_c	500	1,000		
N_e	1,463	2,955		
n_c	239	463		
n_e	789	1,455		
Line equation	$y = 2x - 0.5$	$y = -0.7x + 0.9$	$y = -5x + 4.9$	$y = 3x - 0.8$
$\bar{d}$	4.6867×10^{-2}	3.163×10^{-2}	3.2448×10^{-2}	3.7843×10^{-2}
σ_d^2	3.9422×10^{-4}	3.1174×10^{-4}	2.9005×10^{-4}	4.9535×10^{-4}
Normalisation				
by mean	1 ± 0.4237 0.1713 -8.1332×10^{-3}	1 ± 0.5582 0.3032 3.5783×10^{-2}	1 ± 0.5249 0.2656 -3.7762×10^{-3}	1 ± 0.5881 0.3321 2.2989×10^{-2}
by homogeneity	0.7245 ± 0.3070 8.9936×10^{-2} -3.0936×10^{-3}	0.68059 ± 0.3799 0.1404 1.128×10^{-2}	0.6982 ± 0.3665 0.1295 -1.2853×10^{-3}	0.81428 ± 0.4789 0.2202 1.2412×10^{-2}

The programme used is listed in § A.17. The space position vector of the intersection between the two vectors AB and CD is $P = A + r(B - A)$, where $AB = B - A$ and $CD = D - C$ are vectors or directed lines and A, B, C, D are space vectors, $r = \begin{vmatrix} CD' \\ CA' \end{vmatrix} \bigg/ \begin{vmatrix} AB' \\ CD' \end{vmatrix}$ and $s = \begin{vmatrix} AB' \\ CA' \end{vmatrix} \bigg/ \begin{vmatrix} AB' \\ CD' \end{vmatrix}$ Here $AB = A + r(B - A)$, and $CD = C + s(D - C), 0 \leq r, s \leq 1$ are directed lines. P exists if $0 \leq r \leq 1$ and $\quad 0 \leq s \leq 1$.

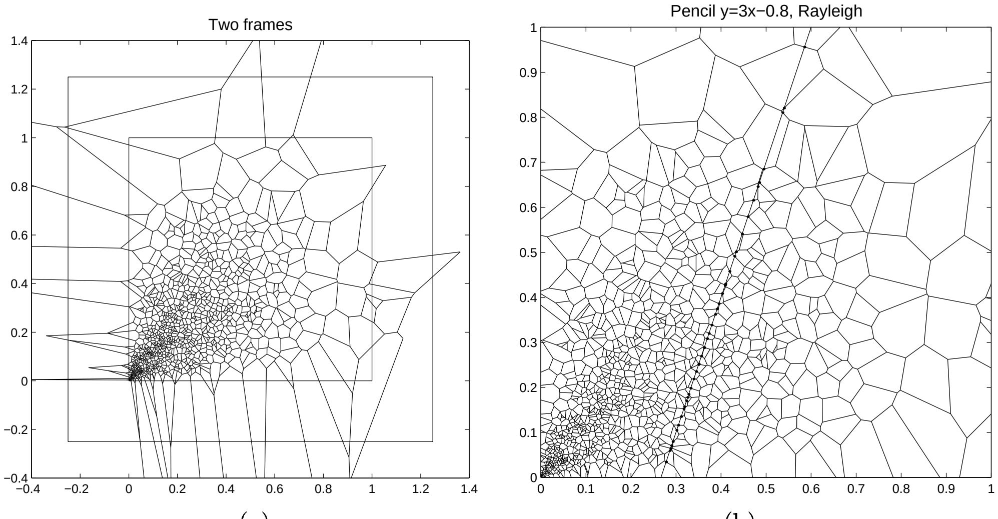
If the denominator $\begin{vmatrix} AB' \\ CD' \end{vmatrix}$ is zero, then the two lines are parallel. Also, if the numerator of r is zero, that is $\begin{vmatrix} CD' \\ CA' \end{vmatrix} = 0$, then both lines are collinear.

Consider the line section of Rayleigh distributed Voronoi where both the coordinates x and y are random numbers with Rayleigh distribution. The probability density function of the Rayleigh distribution is $y = f(x/b) = (x/b^2)\exp(-x^2/2b^2)$. The mean of this distribution is $b\sqrt{\pi/2}$ and the variance is $((4 - \pi)/2)b^2$. With $b = 1, 2, \ldots, 1000$ choose the random numbers from the Rayleigh distribution, then scale and use them as the coordinates.

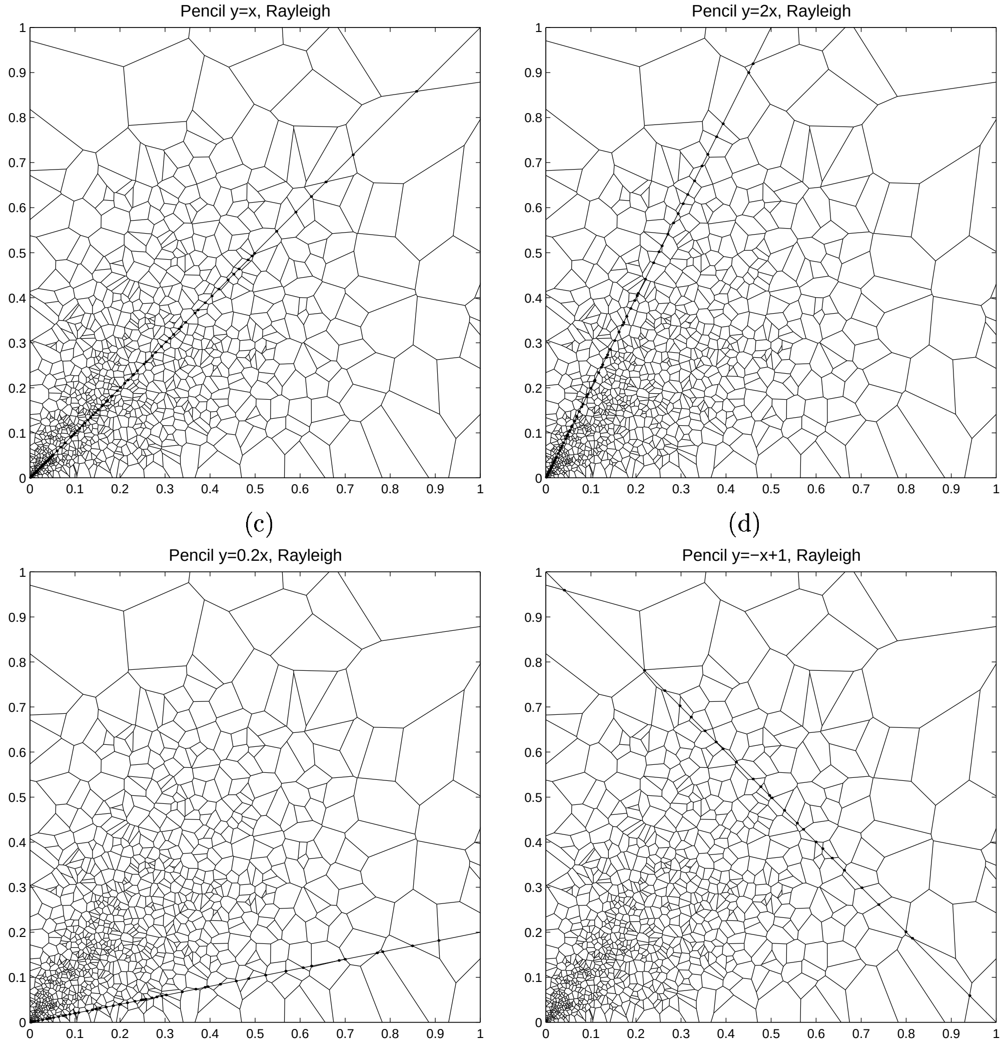
In Figure 3.38 the structure in (a) is intersected by (b) $y = 3x - 0.8$, (c) $y = x$, (d) $y = 2x$, (e) $y = 0.2x$, (f) $y = -x + 1$, (g) $y = -x + 0.3$.

The following codes generate and scale the points of Rayleigh distribution.

```
X=raylrnd([1:NumCell])';Y=raylrnd([1:NumCell])';Max=0.8*max([X;Y]);X=X/Max;Y=Y/Max;
```

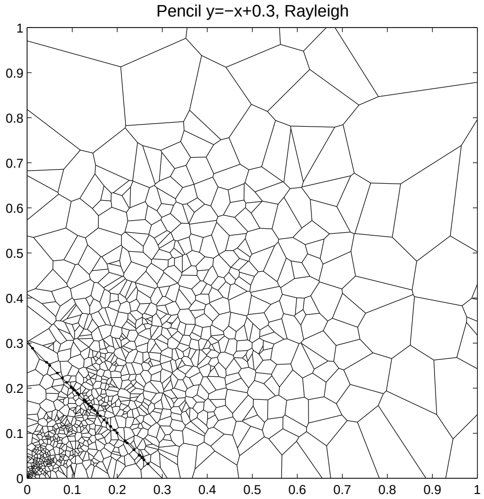


(a) (b)



(c) (d)

(e) (f)



The values of Cx and Dx will need to be adjusted manually from the pencil beam equation. It is the value of x at both points of intersection between the beam and the $[0, 1]$ square box. A random Voronoi network can either be both homogeneous and isotropic or, nonhomogeneous and nonisotropic depending on whether the probability distribution function is a constant.

(g)

Figure 3.38 Line intersection in a Voronoi with Rayleigh distribution.

Simulation	1	2	3	4	5	
N_c	1,000					
N_e	2,975					
n_c	988					
n_e	2,912					
Line eq.	$y = 3x - 0.8$	$y = x$	$y = 2x$	$y = 0.2x$	$y = -x + 1$	$y = -x + 0.3$
$\bar{d}$	2.3621×10^{-2}	1.4253×10^{-2}	1.3853×10^{-2}	1.7434×10^{-2}	4.4401×10^{-2}	1.027×10^{-2}
σ_d^2	8.1582×10^{-4}	6.2922×10^{-4}	2.9647×10^{-4}	3.0991×10^{-4}	1.1584×10^{-3}	3.5365×10^{-5}
Normalised						
by mean	1±1.2092 1.4256 5.5516	1±1.7599 3.0609 28.351	1±1.2429 1.5239 7.6499	1±1.0098 1.0004 1.6643	1±0.7665 0.56204 1.2114	1±0.5791 0.3245 0.1755
by homog.	0.7425±0.8978 0.7859 2.2721	0.4480±0.7885 0.6144 2.5493	0.4354±0.5412 0.2890 0.6316	0.548±0.5533 0.3004 0.2739	1.3956±1.0698 1.0947 3.293	0.3228±0.1869 3.3813×10^{-2} 5.9021×10^{-3}

Table 3.17 *Line intersection statistics of Rayleigh distributed Voronoi.*

§ 3.10 Voronoi of a Voronoi

A Voronoi of a Voronoi is still a Voronoi, provided that by 'Voronoi' one means Voronoi Tessellation. But what are the effects of applying the Voronoi tessellating operator on a set of points? And what are the effects of applying this operator twice?

It is an interesting question to consider whether Voronoi tessellation as an operator alters, for instance, the nature of the distribution of the original set of points. Figure 3.39 shows the first Voronoi operator applied to the original points and then the second Voronoi operator applied to the vertices obtained. Here the Voronoi operator is applied twice in succession. In Figure the Voronoi operator is twiced in succession, *i.e.* $\mathcal{V}^2(\cdot)$, starting from a set of 100 points.

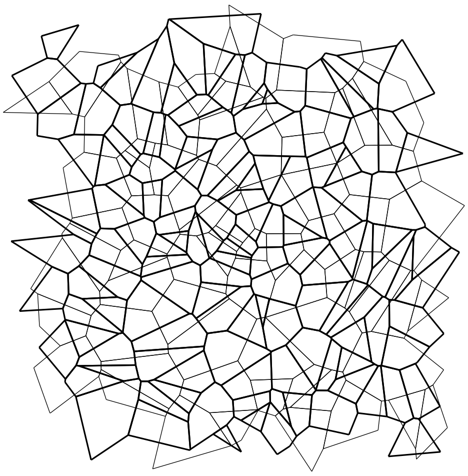
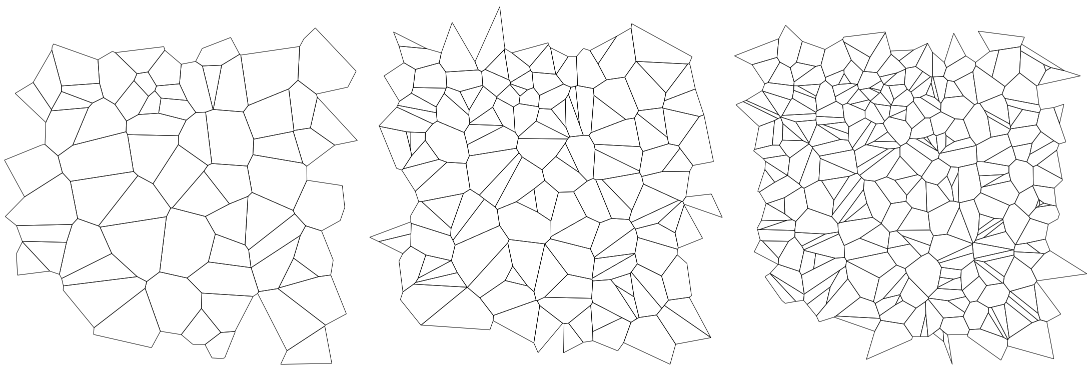
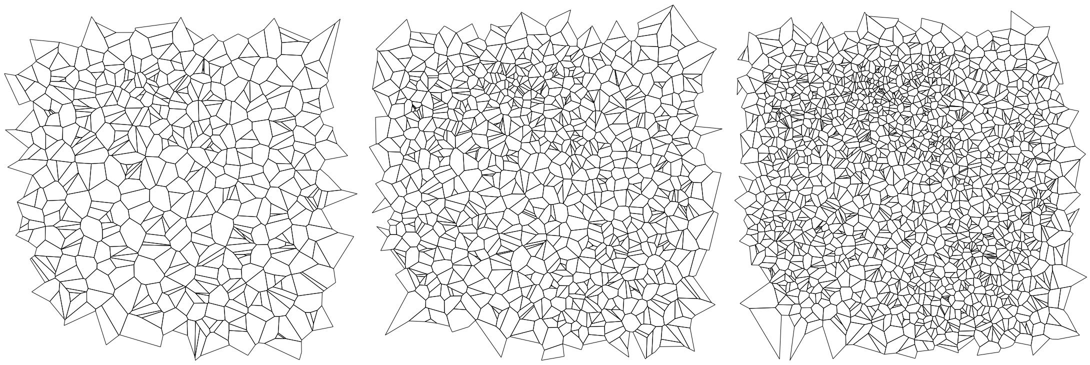


Figure 3.39 Two Voronoi operators applied in succession.

Let $\mathcal{V}^n(\cdot)$ be the n^{th}-order Voronoi operator. Figure 3.39 shows as an example $\mathcal{V}^2(x)$ where x is the original generator points. In Figure 3.40 this operator is applied six times in succession. The Voronoi operator is a mapping which maps a set of points into a set of Voronoi vertices having the former points as the Voronoi nuclei. The programme used for Figures 3.39 to 3.41 is listed in § A.20.:



(a) (b) (c)



(d) (e) (f)

Figure 3.40 *The Voronoi operations,* (a) $\mathcal{V}(x)$, (b) $\mathcal{V}^2(x)$, (c) $\mathcal{V}^3(x)$, (d) $\mathcal{V}^4(x)$, (e) $\mathcal{V}^5(x)$, (f) $\mathcal{V}^6(x)$, *where x is the original set of 100 points.*

Figure 3.41 shows the effect produced by the Voronoi operators of various degrees on the number of cells, n_c, and vertices, n_v of the network.

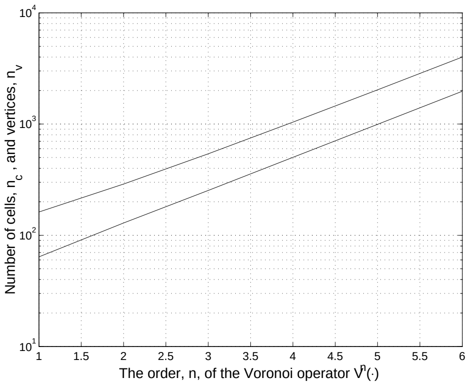
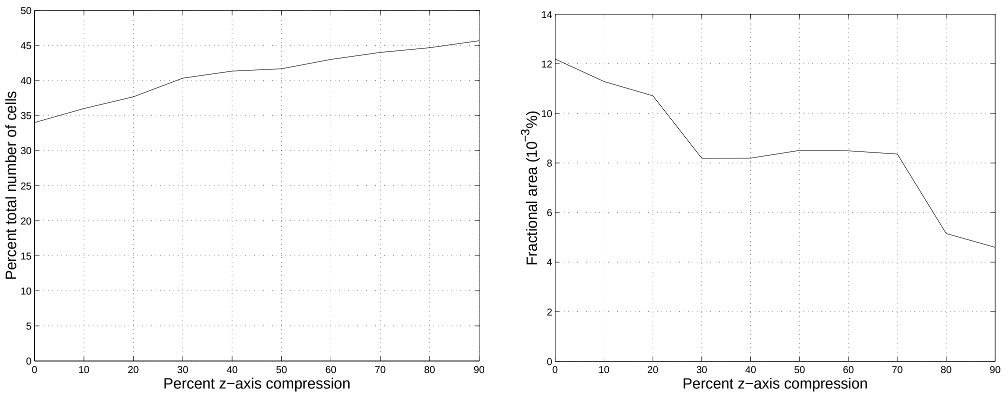


Figure 3.41 recursively apply the Voronoi operators of various orders. The number of cells and vertices when recursively applying the Voronoi operators of various orders on the set of point x is shown in Figure 3.41, that is to say, $\mathcal{V}(x)$, $\mathcal{V}^2(x)$, $\cdots$, $\mathcal{V}^6(x)$. From Figure 3.41, the increase of n_c and n_v with the order n of $\mathcal{V}^n(\cdot)$ is exponential. And from Figure 3.40 (a) to (f), apart from the unevenness affected at the boundary, the graphs shows the degree of lumpiness in the original distribution retained by the Voronoi operator. This is analogous to the increase in the entropy in physical processes.

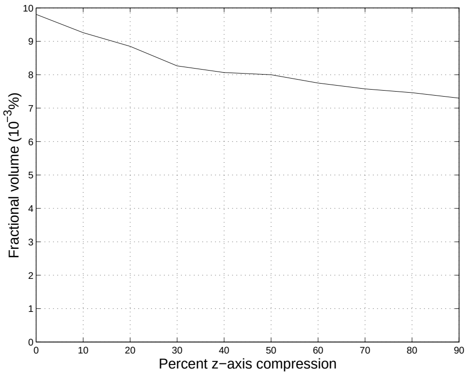
Figure 3.41 The number of cells and vertices when recursively applying the Voronoi operator.

§ 3.11 Transformations of a Voronoi



(a) (b)

The simplest of transformations is compression where the coordinates of each vertex in multiplied by a factor less than one. Figure 3.42 shows, within a fixed box, the effect the compression along the z-axis has on the number of cells, surface area and volume of cells within the box. It shows the effects of a z-axis compression on (a) the number of cells in the box, (b) the surface area of a cell and (c) the volume of a cell.

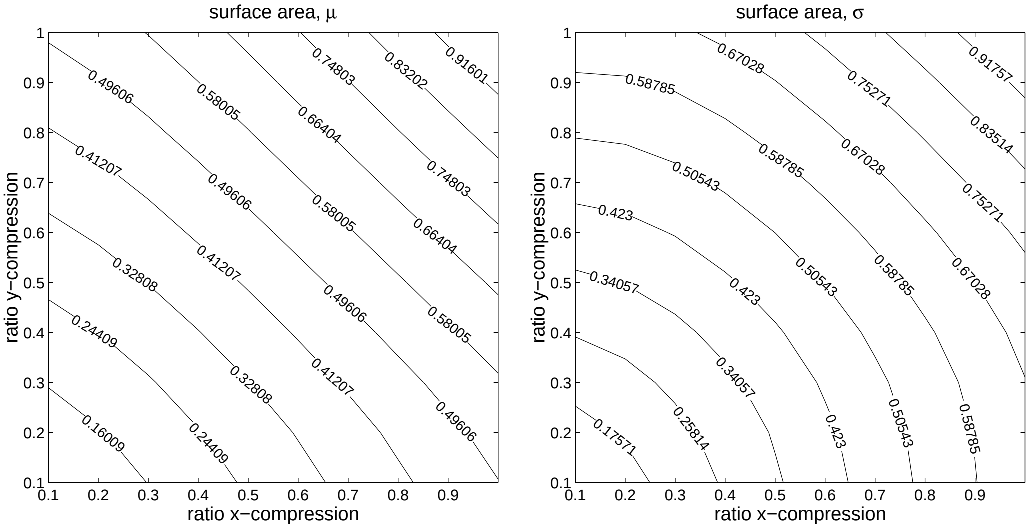


(c)

Figure 3.42 The effects of axis compression.

§ 3.12 Compressed Voronoi

The following study looks at compression of the Voronoi tessellation in three dimensions. With the compression simultaneously in the x- and y axes, the mean and standard deviation of cell's surface area are shown as contours in Figure 3.43. The numerical values are shown in Figure's 3.18 and 3.19. Here x_i and y_i are respectively transformed to $r_x x_i$ and $r_y y_i$. All the values here are normalised by their corresponding values in the case without compression. In this case, the network was created from 100 generators within one unit cube, only 21 inner cells are considered, the mean and the standard deviation of the cell surface area are respectively 0.2330 and 0.0544.



(a) (b)

Figure 3.43 (a) *Mean, and* (b) *standard deviation of surface area of cells under compression, normalised against the corresponding values of the uncompressed case. In other words,* (a) $\mu(a_{ij})/\mu(a_{11})$, *and* (b) $\sigma(a_{ij})/\sigma(a_{11})$.

$\boxed{\mu(a_{ij})}$		r_x									
		1.0	*0.9*	*0.8*	*0.7*	*0.6*	*0.5*	*0.4*	*0.3*	*0.2*	*0.1*
	1.0	1.0000	0.9337	0.8688	0.8058	0.7450	0.6870	0.6326	0.5828	0.5396	0.5060
	0.9	0.9318	0.8683	0.8061	0.7457	0.6873	0.6315	0.5790	0.5310	0.4892	0.4566
	0.8	0.8650	0.8042	0.7447	0.6866	0.6305	0.5768	0.5261	0.4797	0.4391	0.4073
	0.7	0.8001	0.7419	0.6847	0.6290	0.5749	0.5230	0.4740	0.4289	0.3893	0.3581
	0.6	0.7375	0.6816	0.6267	0.5730	0.5208	0.4706	0.4230	0.3789	0.3400	0.3092
r_y	*0.5*	0.6777	0.6240	0.5711	0.5193	0.4687	0.4199	0.3734	0.3300	0.2914	0.2605
	0.4	0.6217	0.5699	0.5188	0.4685	0.4193	0.3715	0.3257	0.2826	0.2439	0.2123
	0.3	0.5707	0.5205	0.4708	0.4217	0.3735	0.3264	0.2808	0.2375	0.1979	0.1648
	0.2	0.5269	0.4779	0.4293	0.3810	0.3333	0.2863	0.2404	0.1962	0.1547	0.1188
	0.1	0.4939	0.4457	0.3976	0.3497	0.3020	0.2546	0.2078	0.1617	0.1171	0.0761

Table 3.18 *Mean of the cell surface area of compressed Voronoi*

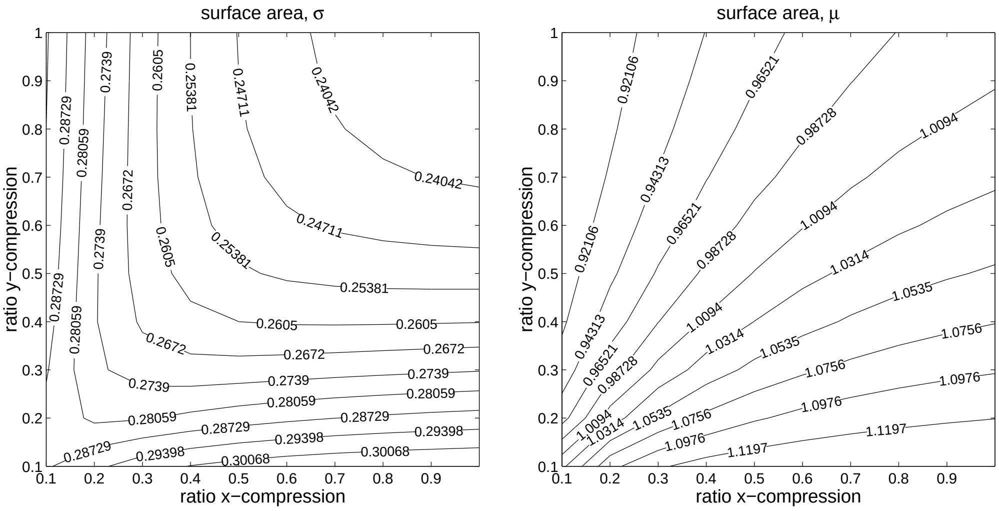
$\boxed{\sigma(a_{ij})}$		r_x									
		1.0	*0.9*	*0.8*	*0.7*	*0.6*	*0.5*	*0.4*	*0.3*	*0.2*	*0.1*
	1.0	1.0000	0.9381	0.8789	0.8231	0.7715	0.7255	0.6868	0.6577	0.6407	0.6382
	0.9	0.9360	0.8761	0.8186	0.7641	0.7135	0.6678	0.6288	0.5986	0.5799	0.5751
	0.8	0.8756	0.8173	0.7611	0.7075	0.6573	0.6116	0.5720	0.5403	0.5195	0.5122
	0.7	0.8199	0.7626	0.7071	0.6539	0.6037	0.5574	0.5165	0.4830	0.4596	0.4495
	0.6	0.7699	0.7130	0.6576	0.6041	0.5532	0.5058	0.4631	0.4271	0.4005	0.3870
r_y	*0.5*	0.7269	0.6696	0.6136	0.5592	0.5069	0.4576	0.4123	0.3731	0.3426	0.3249
	0.4	0.6924	0.6340	0.5766	0.5205	0.4661	0.4141	0.3654	0.3219	0.2864	0.2634
	0.3	0.6676	0.6076	0.5482	0.4897	0.4324	0.3769	0.3240	0.2751	0.2331	0.2030
	0.2	0.6536	0.5914	0.5296	0.4682	0.4076	0.3481	0.2902	0.2350	0.1848	0.1451
	0.1	0.6495	0.5853	0.5212	0.4573	0.3936	0.3302	0.2675	0.2059	0.1466	0.0933

Table 3.19 *Standard deviation of the cell surface area of compressed Voronoi*

Mistakes reveal some interesting characteristics worth investigating further, namely those shown in Figure 3.44.

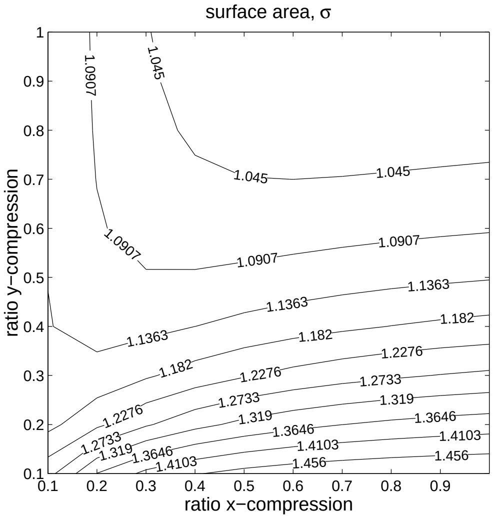
Let a_{ij} be the surface area of cells when subjected to compressions i and j respectively along the x- and y axes, that is to say, $r_x = i$ and $r_y = j$. And let $a_{ij}(k)$ be that surface area of the k^{th} cell among those undergoing these compressions ij. Then Figure 3.44 (b) and (c) are possible when we pick one cell as the basis for our normalisation, while Figure 3.44 (a) is when we normalise each compressed case by its own mean.

For cell volume of the same network the results are shown in Figure 3.45. The mean value and the standard deviation of cell volumes are respectively 0.0075 and 0.0032.



(a)

(b)

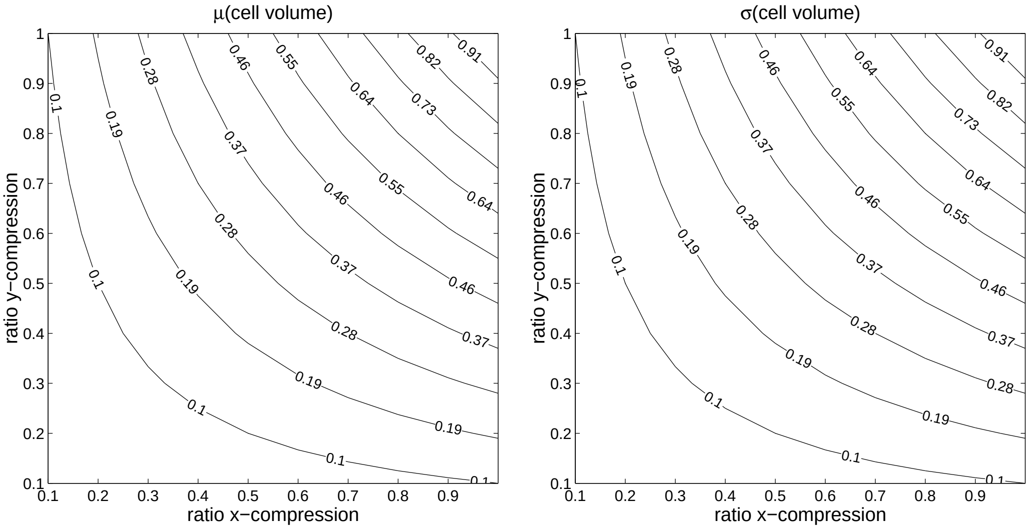


In Figure 3.44 some hidden characteristics are revealed from our choice of the normalisation basis; respectively for (a), (b) and (c) the following,

$$\sigma\left(\frac{a_{ij}}{\mu(a_{ij})}\right), \quad \frac{\mu(a_{ij}/a_{ij}(1))}{\mu(a_{11}/a_{11}(1))}, \quad \frac{\sigma(a_{ij}/a_{ij}(1))}{\sigma(a_{11}/a_{11}(1))}.$$

(c)

Figure 3.44 Some hidden characteristics revealed.



(a) (b)

Figure 3.45 *Volume of compressed Voronoi cells;* (a) $\mu(V_{ij}))and$(b) $\sigma(V_{ij})$.

Table 3.20 gives the numerical values of the mean, while Table 3.21 those of the standard deviation plotted in Figure 3.45.

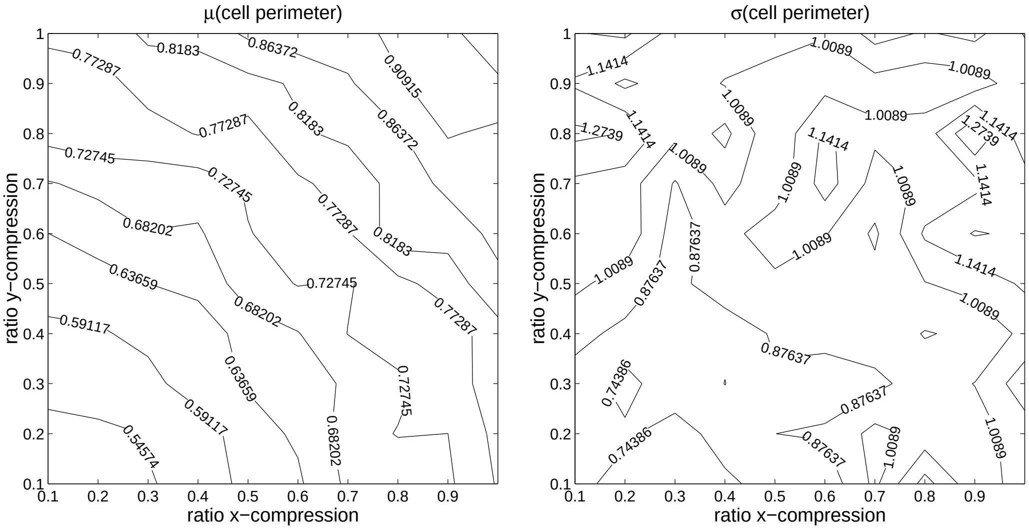
$\mu(V_{ij})$						r_x					
		1.0	*0.9*	*0.8*	*0.7*	*0.6*	*0.5*	*0.4*	*0.3*	*0.2*	*0.1*
	1.0	1.0000	0.9000	0.8000	0.7000	0.6000	0.5000	0.4000	0.3000	0.2000	0.1000
	0.9	0.9000	0.8100	0.7200	0.6300	0.5400	0.4500	0.3600	0.2700	0.1800	0.0900
	0.8	0.8000	0.7200	0.6400	0.5600	0.4800	0.4000	0.3200	0.2400	0.1600	0.0800
	0.7	0.7000	0.6300	0.5600	0.4900	0.4200	0.3500	0.2800	0.2100	0.1400	0.0700
	0.6	0.6000	0.5400	0.4800	0.4200	0.3600	0.3000	0.2400	0.1800	0.1200	0.0600
r_y	*0.5*	0.5000	0.4500	0.4000	0.3500	0.3000	0.2500	0.2000	0.1500	0.1000	0.0500
	0.4	0.4000	0.3600	0.3200	0.2800	0.2400	0.2000	0.1600	0.1200	0.0800	0.0400
	0.3	0.3000	0.2700	0.2400	0.2100	0.1800	0.1500	0.1200	0.0900	0.0600	0.0300
	0.2	0.2000	0.1800	0.1600	0.1400	0.1200	0.1000	0.0800	0.0600	0.0400	0.0200
	0.1	0.1000	0.0900	0.0800	0.0700	0.0600	0.0500	0.0400	0.0300	0.0200	0.0100

Table 3.20 *Numerical values of the mean and standard deviation of compressed cell volume.*

$\sigma(V_{ij})$						r_x					
		1.0	*0.9*	*0.8*	*0.7*	*0.6*	*0.5*	*0.4*	*0.3*	*0.2*	*0.1*
	1.0	1.0000	0.9000	0.8000	0.7000	0.6000	0.5000	0.4000	0.3000	0.2000	0.1000
	0.9	0.9000	0.8100	0.7200	0.6300	0.5400	0.4500	0.3600	0.2700	0.1800	0.0900
	0.8	0.8000	0.7200	0.6400	0.5600	0.4800	0.4000	0.3200	0.2400	0.1600	0.0800
	0.7	0.7000	0.6300	0.5600	0.4900	0.4200	0.3500	0.2800	0.2100	0.1400	0.0700
	0.6	0.6000	0.5400	0.4800	0.4200	0.3600	0.3000	0.2400	0.1800	0.1200	0.0600
r_y	*0.5*	0.5000	0.4500	0.4000	0.3500	0.3000	0.2500	0.2000	0.1500	0.1000	0.0500
	0.4	0.4000	0.3600	0.3200	0.2800	0.2400	0.2000	0.1600	0.1200	0.0800	0.0400
	0.3	0.3000	0.2700	0.2400	0.2100	0.1800	0.1500	0.1200	0.0900	0.0600	0.0300
	0.2	0.2000	0.1800	0.1600	0.1400	0.1200	0.1000	0.0800	0.0600	0.0400	0.0200
	0.1	0.1000	0.0900	0.0800	0.0700	0.0600	0.0500	0.0400	0.0300	0.0200	0.0100

Table 3.20 *Numerical values of the mean and standard deviation of compressed cell volume.*

Perimeters prove to be the most difficult to find. Figure 3.46 (a) and (b) show the mean and standard deviation of the cell perimeter while Table 3.22 lists the $\mu(s_c)$ matrix that makes up one of these graphs. The values before normalisation are $\mu(s_{ij}) = 3.0158$ and $\sigma(s_{ij}) = 0.4291$.



(a)

(b)

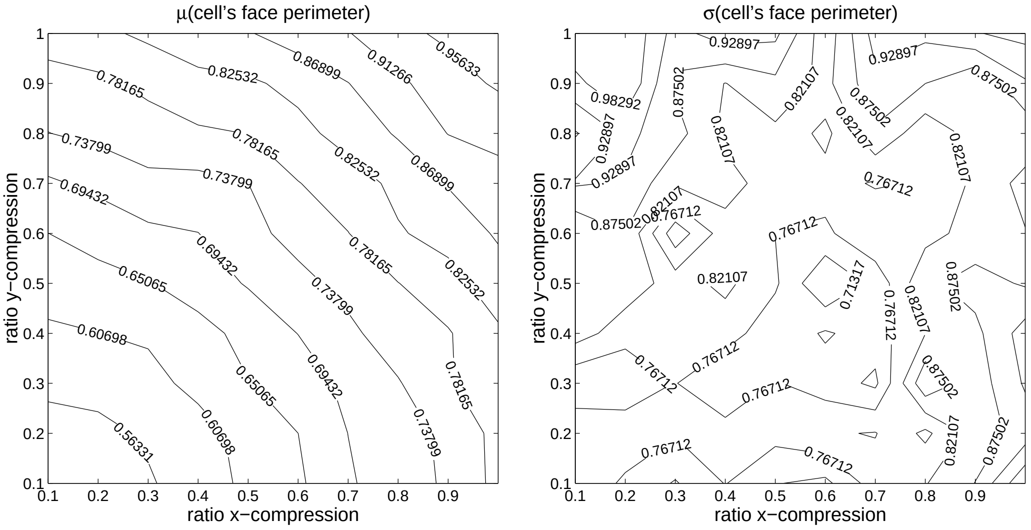
Figure 3.46 *Cell perimeter;* (a) *the normalised mean, i.e.* $\mu(s_{ij})/\mu(s_{11})$*, and* (b) *the standard deviation,* $\sigma(s_{ij})/\sigma(s_{11})$*.*

$\mu(s_{ij})$						r_x					
		1.0	*0.9*	*0.8*	*0.7*	*0.6*	*0.5*	*0.4*	*0.3*	*0.2*	*0.1*
	1.0	1.0000	0.9376	0.9227	0.8878	0.8918	0.8717	0.8324	0.8271	0.7906	0.7835
	0.9	0.9437	0.9333	0.9039	0.8577	0.8241	0.8046	0.7922	0.7906	0.7556	0.7472
	0.8	0.8996	0.9127	0.8580	0.8266	0.8131	0.7558	0.7752	0.7560	0.7580	0.7441
	0.7	0.8840	0.8766	0.8334	0.7928	0.7644	0.7313	0.7054	0.7042	0.6937	0.6788
	0.6	0.8780	0.8339	0.8473	0.7684	0.7392	0.7261	0.6757	0.6806	0.6575	0.6363
r_y	*0.5*	0.8509	0.7948	0.7592	0.7228	0.7301	0.7017	0.6550	0.6364	0.6166	0.6036
	0.4	0.8019	0.7457	0.7649	0.7284	0.6798	0.6613	0.6011	0.6036	0.5858	0.5847
	0.3	0.7852	0.7609	0.7129	0.6883	0.6618	0.6408	0.6161	0.5767	0.5755	0.5662
	0.2	0.7860	0.7273	0.7305	0.7008	0.6431	0.6189	0.5638	0.5476	0.5334	0.5264
	0.1	0.7771	0.7198	0.6895	0.6911	0.6296	0.6090	0.5538	0.5440	0.5124	0.5003

Table 3.22 *Numerical means of the cell perimeter of compressed Voronoi.*

Figure 3.47 shows the plot of the change in the perimeter of faces when the Voronoi structure is compressed. The normalised mean and standard deviation are shown in the form of contours. The values before normalisation are $\mu(\mu(z_{ij})) = 0.3766$ and $\sigma(\mu(z_{ij})) = 0.0532$.

Both the surface and the perimeter are embedded in three dimensions, the former as 2-d facets while the latter as 1-d facets. But, comparing Figure 3.43 (a) with Figure 3.46 (a), the trend of changes in the mean is smoother for surfaces than for perimeters. The difference between the thrend of the standard deviation is even more pronounced, as can be readily seen by comparing Figure 3.43 (b) with Figure 3.46 (b). This may imply the reduction in the correlation between different statistical properties as the structural differences increase, as surface is directly related to the convex hull of the volume while perimeter is a convex hull embedded in another convex hull. Also, this means that convex hull as a mapping or function is not smooth, or it could mean that it is a smooth function only up to the order one, that is to say, when we apply it only once not twice or more.



(a) (b)

Figure 3.47 *Face perimeter;* (a) *its normalised mean, i.e.* $\mu(\mu(z_{ij}))/\mu(\mu(z_{11}))$*, and* (b) *the standard deviation,* $\sigma(\mu(s_{ij}))/\sigma(\mu(s_{11}))$.

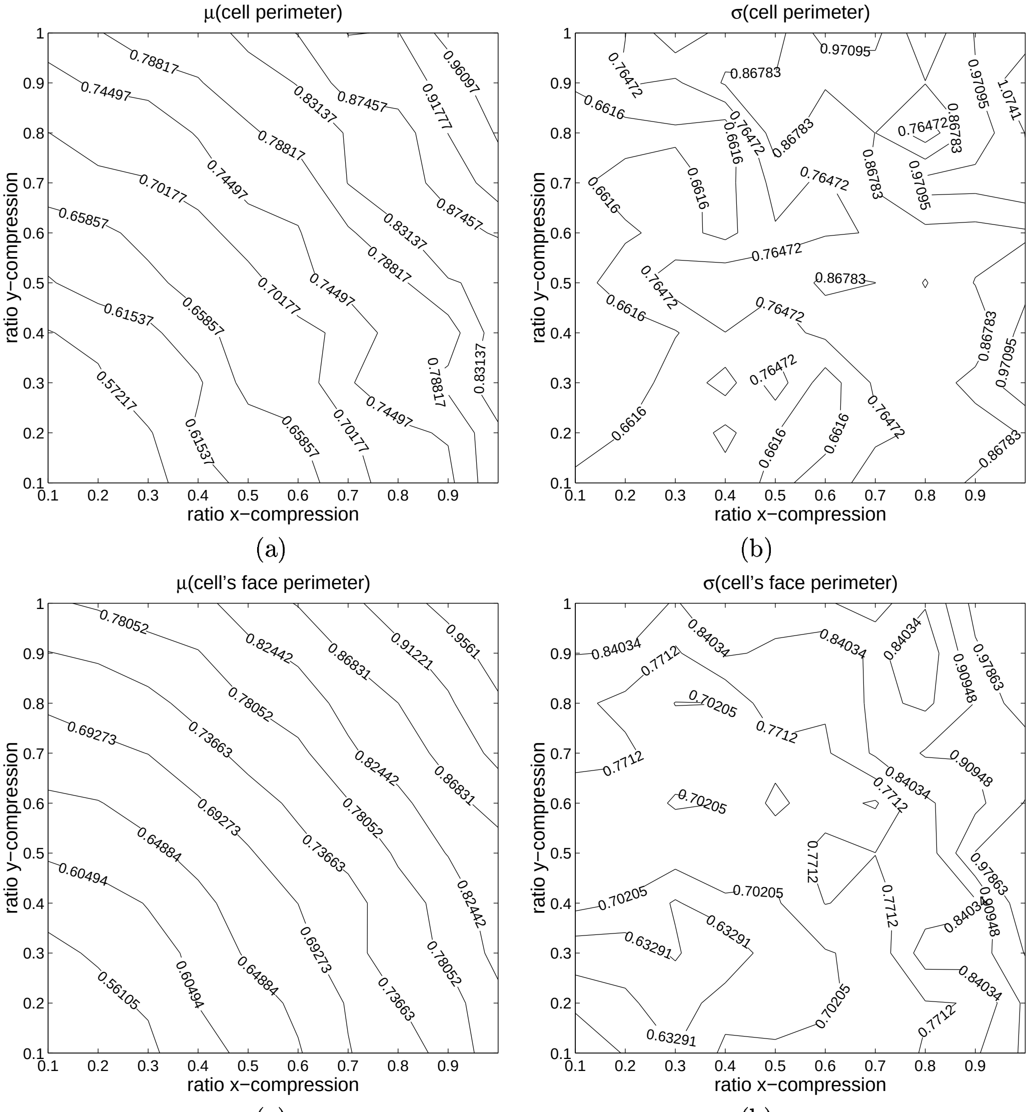
$\mu(\mu(z_{ij}))$		r_x *1.0*	*0.9*	*0.8*	*0.7*	*0.6*	*0.5*	*0.4*	*0.3*	*0.2*	*0.1*
	1.0	1.0000	0.9640	0.9460	0.9103	0.8907	0.8659	0.8523	0.8339	0.8155	0.8056
	0.9	0.9620	0.9384	0.9040	0.8681	0.8371	0.8186	0.8124	0.7927	0.7717	0.7606
	0.8	0.9262	0.9137	0.8738	0.8409	0.8129	0.7823	0.7755	0.7608	0.7489	0.7372
	0.7	0.8956	0.8722	0.8355	0.8058	0.7801	0.7378	0.7246	0.7276	0.7053	0.6874
	0.6	0.8745	0.8404	0.8214	0.7807	0.7527	0.7252	0.6936	0.6849	0.6661	0.6504
r_y	*0.5*	0.8485	0.8093	0.7808	0.7514	0.7255	0.6977	0.6734	0.6479	0.6368	0.6283
	0.4	0.8186	0.7780	0.7614	0.7284	0.6951	0.6659	0.6335	0.6139	0.6065	0.5987
	0.3	0.8125	0.7771	0.7347	0.7075	0.6670	0.6490	0.6210	0.5916	0.5772	0.5731
	0.2	0.7958	0.7459	0.7225	0.6946	0.6506	0.6254	0.5870	0.5685	0.5528	0.5466
	0.1	0.7938	0.7451	0.7154	0.6897	0.6433	0.6181	0.5815	0.5594	0.5350	0.5196

Table 3.23 *Numerical mean values of the face perimeter in a compressed Voronoi.*

$\sigma(\mu(z_{ij}))$		r_x *1.0*	*0.9*	*0.8*	*0.7*	*0.6*	*0.5*	*0.4*	*0.3*	*0.2*	*0.1*
	1.0	1.0000	0.9795	0.9413	0.9524	0.7858	0.9426	0.9614	0.9060	1.0364	1.0311
	0.9	0.9221	0.8227	0.8754	0.9145	0.8047	0.8613	0.8197	0.8653	1.0369	0.9671
	0.8	0.8363	0.8407	0.7857	0.8670	0.7517	0.8090	0.8055	0.8980	0.9424	0.8702
	0.7	0.8939	0.8322	0.7718	0.7612	0.7903	0.7904	0.8453	0.8296	0.9233	0.9350
	0.6	0.8724	0.8413	0.8031	0.8155	0.7560	0.7662	0.7972	0.6784	0.8740	0.8279
r_y	*0.5*	0.8689	0.8957	0.8640	0.7293	0.6592	0.7756	0.8340	0.7986	0.8512	0.8750
	0.4	0.9538	0.8608	0.8199	0.7414	0.7733	0.7324	0.7916	0.7954	0.7916	0.8463
	0.3	0.9359	0.8457	0.9174	0.7016	0.7422	0.7711	0.7139	0.7697	0.7137	0.7206
	0.2	0.8985	0.8447	0.7545	0.8238	0.8155	0.7855	0.7918	0.7724	0.8133	0.8138
	0.1	1.0322	0.8763	0.8181	0.7953	0.6991	0.7159	0.7692	0.7085	0.7546	0.8212

Table 3.24 *Numerical standard deviations of the face perimeter in Voronoi network under various degrees of compression.*

The simulation above is then repeated again for a second time, where now $\mu(s_{11}) = 2.1206$, $\sigma(s_{11}) = 0.8154$, $\mu(\mu(z_{11})) = 0.3840$ and $\sigma(\mu(z_{11})) = 0.0518$. Figure 3.48 shows the results for both the cell perimeter s and the face perimeter z. Also, compare these pictures with Table 3.25.



(a) (b)

(a) (b)

Figure 3.48 *The cell perimeter and the face perimeter;* (a) $\mu(s_{ij})/\mu(s_{11})$, (b) $\sigma(s_{ij})/\sigma(s_{11})$, (c) $\mu(\mu(z_{ij}))/\mu(\mu(z_{11}))$ *and* (d) $\sigma(\mu(z_{ij}))/\sigma(\mu(z_{11}))$

$\boxed{\mu(s_{ij})}$

	r_x	*1.0*	*0.9*	*0.8*	*0.7*	*0.6*	*0.5*	*0.4*	*0.3*	*0.2*	*0.1*
r_y	*1.0*	1.0000	0.9781	0.9164	0.9188	0.8766	0.8417	0.8126	0.8282	0.7834	0.7633
	0.9	1.0042	0.9292	0.9004	0.8959	0.8330	0.8144	0.7851	0.7555	0.7511	0.7323
	0.8	0.9654	0.9185	0.8512	0.8339	0.8090	0.7808	0.7475	0.7256	0.7120	0.7015
	0.7	0.9422	0.8868	0.8596	0.8327	0.7596	0.7573	0.7262	0.6988	0.6964	0.6763
	0.6	0.8780	0.8621	0.8241	0.7817	0.7425	0.7278	0.6810	0.6723	0.6474	0.6397
	0.5	0.8411	0.8281	0.7785	0.7604	0.7350	0.6821	0.6752	0.6475	0.6352	0.6116
	0.4	0.8568	0.7661	0.7653	0.7156	0.6856	0.6833	0.6423	0.6052	0.5874	0.5698
	0.3	0.8605	0.7999	0.7731	0.7409	0.6744	0.6747	0.6093	0.6080	0.5627	0.5641
	0.2	0.8234	0.7491	0.7368	0.6950	0.6551	0.6370	0.6231	0.5672	0.5677	0.5361
	0.1	0.8255	0.7332	0.7161	0.6898	0.6345	0.6265	0.5977	0.5548	0.5552	0.5290

(a)

$\boxed{\sigma(s_{ij})}$						r_x					
		1.0	*0.9*	*0.8*	*0.7*	*0.6*	*0.5*	*0.4*	*0.3*	*0.2*	*0.1*
	1.0	1.0000	0.9773	0.7894	0.9984	1.0195	0.8271	0.8252	0.9487	0.7626	0.7128
	0.9	1.1772	0.9638	0.8714	0.9202	0.8793	0.8753	0.8798	0.7462	0.7780	0.6670
	0.8	1.0766	0.9053	0.7051	0.8686	0.7943	0.9349	0.5831	0.6456	0.6093	0.6356
	0.7	1.0506	1.0094	1.0137	0.8884	0.7487	0.7961	0.6260	0.7008	0.7110	0.6383
	0.6	0.8534	0.8277	0.8383	0.7692	0.7554	0.7554	0.6306	0.6991	0.6439	0.5835
r_y	*0.5*	0.9281	0.8727	0.7584	0.8685	0.8945	0.7790	0.8540	0.8201	0.7273	0.6109
	0.4	1.0266	0.7761	0.8214	0.7686	0.7893	0.7270	0.7635	0.6527	0.6022	0.5956
	0.3	1.0474	0.9116	0.7985	0.7823	0.6053	0.8193	0.6155	0.7399	0.5585	0.5668
	0.2	0.8807	0.8048	0.7829	0.7438	0.6276	0.6633	0.7936	0.6742	0.6638	0.6000
	0.1	0.8996	0.8820	0.8161	0.8172	0.7064	0.6345	0.7206	0.6647	0.7240	0.6899

(b)

$\boxed{\mu(\mu(z_{ij}))}$						r_x					
		1.0	*0.9*	*0.8*	*0.7*	*0.6*	*0.5*	*0.4*	*0.3*	*0.2*	*0.1*
	1.0	1.0000	0.9740	0.9327	0.9028	0.8710	0.8447	0.8119	0.8070	0.7864	0.7749
	0.9	0.9817	0.9287	0.9023	0.8735	0.8320	0.8126	0.7782	0.7608	0.7442	0.7353
	0.8	0.9482	0.9068	0.8681	0.8373	0.8039	0.7809	0.7508	0.7248	0.7079	0.7004
	0.7	0.9246	0.8814	0.8465	0.8174	0.7699	0.7500	0.7208	0.6933	0.6843	0.6670
	0.6	0.8846	0.8544	0.8126	0.7773	0.7453	0.7189	0.6878	0.6667	0.6467	0.6424
r_y	*0.5*	0.8510	0.8268	0.7861	0.7496	0.7240	0.6879	0.6644	0.6370	0.6243	0.6118
	0.4	0.8542	0.7924	0.7657	0.7184	0.6930	0.6720	0.6354	0.6071	0.5865	0.5697
	0.3	0.8378	0.7876	0.7612	0.7213	0.6755	0.6554	0.6135	0.5900	0.5643	0.5549
	0.2	0.8126	0.7632	0.7354	0.6962	0.6575	0.6289	0.6062	0.5653	0.5530	0.5329
	0.1	0.8088	0.7494	0.7170	0.6906	0.6449	0.6155	0.5873	0.5531	0.5413	0.5172

(c)

$\boxed{\sigma(\mu(z_{ij}))}$						r_x					
		1.0	*0.9*	*0.8*	*0.7*	*0.6*	*0.5*	*0.4*	*0.3*	*0.2*	*0.1*
	1.0	1.0000	0.9998	0.8469	0.9409	0.9015	0.8691	0.9077	0.8331	0.8906	0.8925
	0.9	1.0290	0.9684	0.7922	0.8566	0.7914	0.8284	0.8495	0.7599	0.8379	0.8420
	0.8	1.0285	0.9118	0.8258	0.8582	0.7783	0.8267	0.7019	0.7007	0.7504	0.7876
	0.7	0.9241	0.9131	0.9166	0.8512	0.7612	0.7458	0.7123	0.7265	0.7861	0.7844
	0.6	0.9830	0.8897	0.8278	0.6931	0.7288	0.7882	0.7028	0.6968	0.7300	0.7506
r_y	*0.5*	0.9891	0.9817	0.7976	0.7720	0.8001	0.7218	0.7600	0.7386	0.7329	0.7131
	0.4	1.0478	0.8275	0.8022	0.7546	0.7731	0.6930	0.6875	0.6249	0.6975	0.7238
	0.3	0.9820	0.8738	0.8755	0.7251	0.6961	0.6874	0.5638	0.6426	0.5881	0.5863
	0.2	0.9251	0.7740	0.7667	0.7174	0.6943	0.6578	0.6800	0.5811	0.6519	0.6929
	0.1	0.9230	0.8312	0.7762	0.7384	0.7160	0.7186	0.7152	0.6357	0.6987	0.7378

(d)

Table 3.25 *Numerical values of cell and face perimeters.* (a) $\mu(s_{ij})/\mu(s_{11})$, (b) $\sigma(s_{ij})/\sigma(s_{11})$, (c) $\mu(\mu(z_{ij}))/\mu(\mu(z_{11}))$ *and* (d) $\sigma(\mu(s_{ij}))/\sigma(\mu(s_{11}))$.

Codes for finding volume, surface area, cell- and face perimeters can be found in § A.28. There only some of the printing commands have been left out.

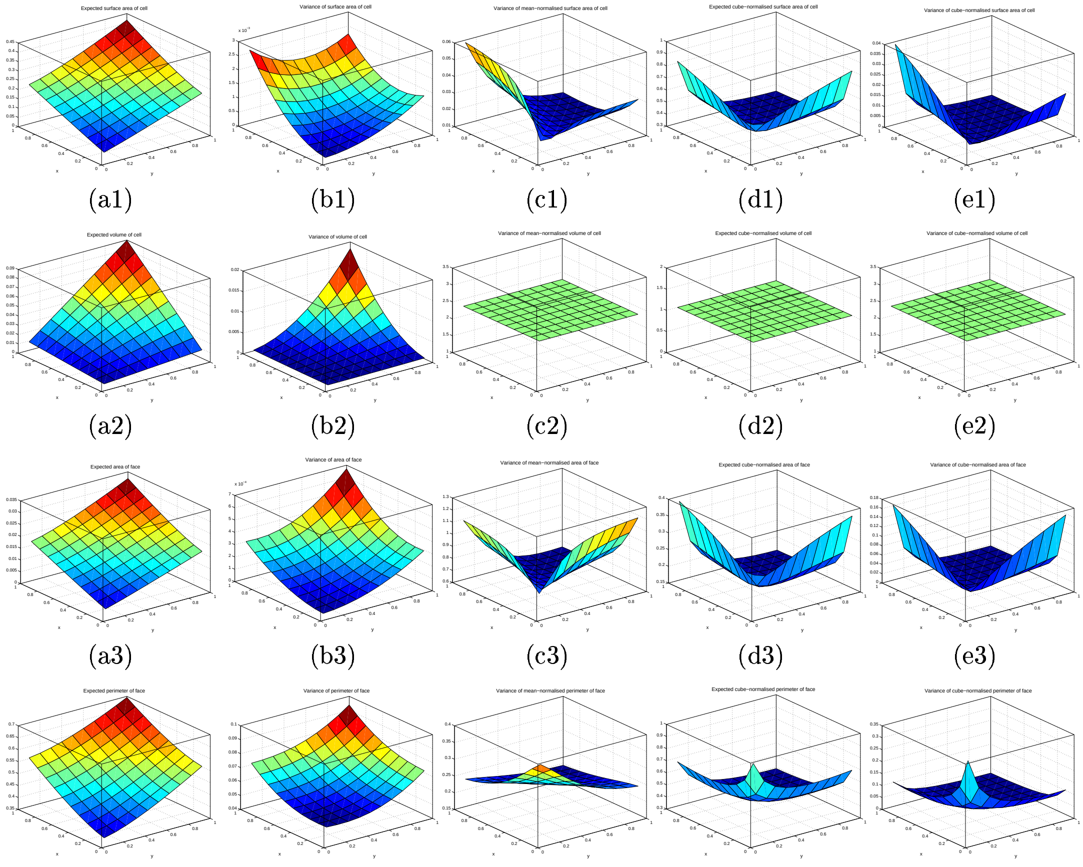
The procedure in general follows Algorithm 3.3. The volume is the summation of all tetrahedral volumes obtained from the triangulation. The surface area is summed over all triangular faces of a convex hull. Plane parameters are then calculated for all of these faces, *viz.* $a = |1, y_i, z_i|$, $b = |x_i, 1, z_i|$, $c = |x_i, y_i, 1|$ and $d = |x_i, y_i, z_i|$, in order to group them together into polyhedral faces.

Algorithm 3.3 *Volume, area and perimeter algorithms.*

```
$(v, v_c) \leftarrow$ Voronoi tessellation;
exclude boundary vertices;
exclude boundary cells;
for every cell do
    find its Delaunay triangulation;
    $V \leftarrow \sum |x_i, y_i, z_i, 1|/6$;
    find its convex hull;
    $A \leftarrow \sum (s(s-a)(s-b)(s-c))^{1/2}$;
    for all facets of the convex hull do
        $(a, b, c, d)$ ← plane parameters;
        group hull facets into polyhedral faces;
        for all hull facets in every face do
            $\{v_f\}$ ← vertices of all coplanar facets;
            find convex hull of $v_f$;
            $\beta$ ← sum area of these hull segments;
            count number of occurrences of their edges;
            $\{e_f\}$ ← edges counted only once;
            $\{e_c\}$ ← $e_f$;
            $z \leftarrow \sum(\sum(\Delta x_i)^2)^{1/2}$;
        endfor
    endfor
    $s \leftarrow (\sum z_i)/2$;
endfor
```
□

The programme `varea.m` was written long before `vareac.m`. It is listed after the latter in § A.28, although most of the variable names have been changed. The old names were long, for example `CubeNormalVolumePerInnerCell` which is now changed into `cbnafn`. The original programme has not been published (Tiyapan, 2001, KNT8(i)).

Some of the past results are shown in Figure 3.49.



(a1) (b1) (c1) (d1) (e1)

(a2) (b2) (c2) (d2) (e2)

(a3) (b3) (c3) (d3) (e3)

(a4) (b4) (c4) (d4) (e4)

Figure 3.49 *Past results of area, volume and perimeter.* (a), (b), (c), (d) *and* (e) *are respectively expected value, variance, variance of mean-normalised values, expected cube-normalised value and variance of the cube-normalised values, whereas* (1) *are values for surface area,* (2) *the volume of cell,* (3) *the area of face and* (4) *the perimeter of face.*

In addition, there is Table 3.26 lists the numerical statistics of the area and perimeter of face. Both this table and Figure 3.49 are from an unpublished work (Tiyapan, 2001, KNT8(i), *ibid.*).

$\boxed{\beta_{ij}}$		r_x									
		1.0	*0.9*	*0.8*	*0.7*	*0.6*	*0.5*	*0.4*	*0.3*	*0.2*	*0.1*
	1.0	0.0322	0.0300	0.0278	0.0257	0.0237	0.0218	0.0200	0.0183	0.0169	0.0158
	0.9	0.0301	0.0280	0.0259	0.0239	0.0219	0.0201	0.0183	0.0167	0.0153	0.0142
	0.8	0.0280	0.0260	0.0240	0.0221	0.0202	0.0184	0.0167	0.0151	0.0138	0.0127
	0.7	0.0261	0.0241	0.0222	0.0203	0.0185	0.0168	0.0151	0.0136	0.0122	0.0112
	0.6	0.0242	0.0223	0.0204	0.0186	0.0169	0.0152	0.0136	0.0121	0.0107	0.0097
r_y	*0.5*	0.0224	0.0205	0.0188	0.0170	0.0153	0.0136	0.0120	0.0106	0.0092	0.0082
	0.4	0.0206	0.0189	0.0172	0.0155	0.0138	0.0122	0.0106	0.0091	0.0078	0.0067
	0.3	0.0191	0.0174	0.0157	0.0140	0.0124	0.0108	0.0092	0.0077	0.0064	0.0052
	0.2	0.0177	0.0161	0.0144	0.0128	0.0112	0.0096	0.0080	0.0065	0.0051	0.0038
	0.1	0.0166	0.0150	0.0134	0.0118	0.0102	0.0086	0.0070	0.0054	0.0039	0.0025

(a)

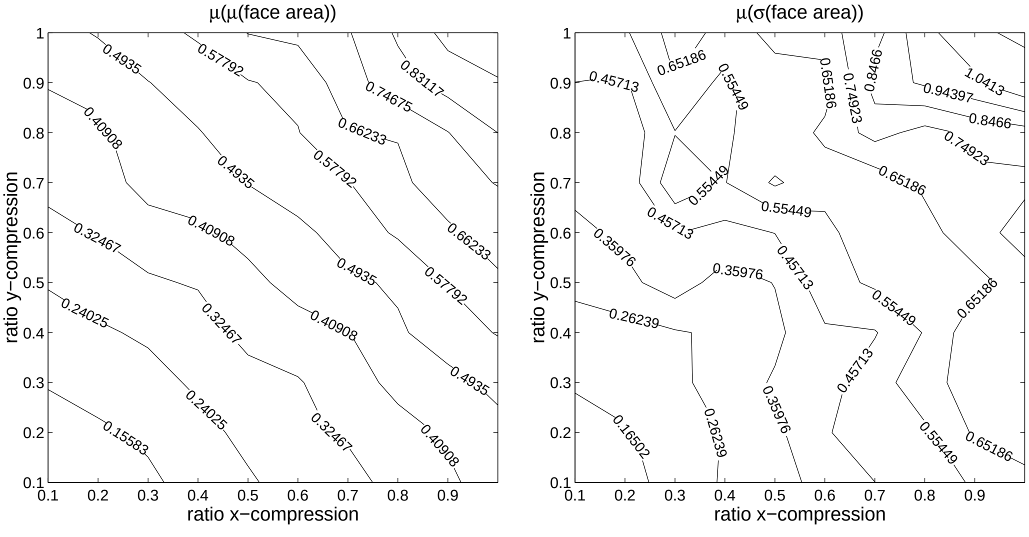
$\boxed{\sigma(\mu(z_{ij}))}$		r_x									
		1.0	*0.9*	*0.8*	*0.7*	*0.6*	*0.5*	*0.4*	*0.3*	*0.2*	*0.1*
	1.0	0.6952	0.6728	0.6514	0.6313	0.6126	0.5956	0.5806	0.5679	0.5581	0.5516
	0.9	0.6733	0.6504	0.6285	0.6079	0.5886	0.5711	0.5555	0.5423	0.5320	0.5251
	0.8	0.6526	0.6291	0.6067	0.5855	0.5657	0.5475	0.5313	0.5175	0.5067	0.4995
	0.7	0.6330	0.6090	0.5861	0.5643	0.5438	0.5251	0.5082	0.4938	0.4824	0.4747
$\delta(y]$	*0.6*	0.6147	0.5903	0.5668	0.5444	0.5234	0.5039	0.4865	0.4714	0.4593	0.4511
	0.5	0.5981	0.5731	0.5491	0.5261	0.5045	0.4844	0.4662	0.4504	0.4377	0.4289
	0.4	0.5832	0.5578	0.5332	0.5097	0.4875	0.4667	0.4478	0.4313	0.4178	0.4084
	0.3	0.5706	0.5447	0.5196	0.4956	0.4727	0.4514	0.4318	0.4145	0.4002	0.3900
	0.2	0.5605	0.5342	0.5087	0.4842	0.4608	0.4389	0.4186	0.4006	0.3856	0.3746
	0.1	0.5537	0.5270	0.5012	0.4763	0.4525	0.4301	0.4094	0.3908	0.3750	0.3633

(a)

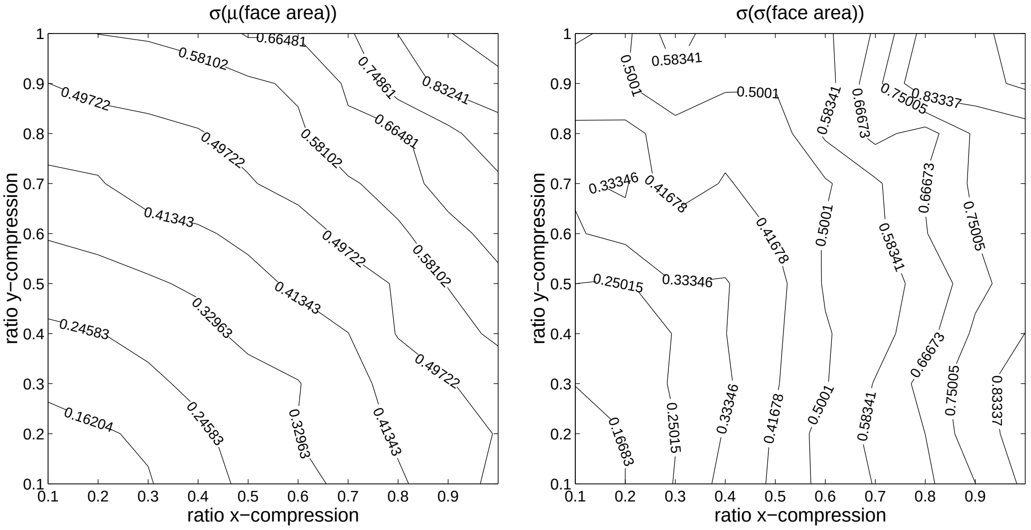
Table 3.26 (a) *The area of face and* (b) *the perimeter of face.*

These much earlier results show in addition the face area statistics. Another set of codes to do the same job has been developed in the style presently used. It used to be `qhull` then, but now `voronoin` on Matlab is used instead. This is the reason why there are many programmes listed in the appendix, from page 221 to page 312. It is a good practice to use more than one programme for doing the same job, in order to cross check between the programmes. As a programme develops, the style and approach of the programmes he writes also change. This is why it is difficult to leave out any certain piece of codes. What you see here are in fact merely snapshots that have survived from the continually changing virtual working ground.

The results for the face area of a VT under compressions are made up of $\mu(\mu(\beta_{ij}))$, $\sigma(\mu(\beta_{ij}))$, $\mu(\sigma(\beta_{ij}))$ and $\sigma(\sigma(\beta_{ij}))$. These are normalised respectively by $\mu(\mu(\beta_{11})) = 0.0065$, $\sigma(\mu(\beta_{11})) = 0.0018$, $\mu(\sigma(\beta_{11})) = 0.0065$ and $\sigma(\sigma(\beta_{11})) = 0.0029$, where β_{ij} is the face area under a compression such that $r_x = i$ and $r_y = j$. These resusts are shown together as Figure 3.50.



(a) (b)



(c) (d)

Figure 3.50 *Statistics of face areas under compression;* (a) *the normalised mean of mean* $\mu(\mu(\beta_{ij}))/\mu(\mu(\beta_{11}))$*,* (b) *standard deviation of mean* $\sigma(\mu(\beta_{ij}))/\sigma(\mu(\beta_{11}))$*,* (c) *mean of standard deviation* $\mu(\sigma(\beta_{ij}))/\mu(\sigma(\beta_{11}))$ *and* (d) $\sigma(\sigma(\beta_{ij}))/\sigma(\sigma(\beta_{11}))$ *standard deviation of standard deviation.*

Table 3.27 has the normalised numerical values used for plotting these contours.

$\boxed{\beta_{ij}}$					r_x					
	1.0	*0.9*	*0.8*	*0.7*	*0.6*	*0.5*	*0.4*	*0.3*	*0.2*	*0.1*
1.0	1.0000	0.9427	0.8435	0.7402	0.6837	0.6645	0.5893	0.5488	0.5002	0.4608
0.9	0.9052	0.8674	0.7970	0.7112	0.5991	0.5730	0.5306	0.4914	0.4389	0.4155
0.8	0.8316	0.7443	0.6698	0.6590	0.5745	0.5346	0.4893	0.4592	0.3924	0.3679
0.7	0.7487	0.7311	0.6343	0.5752	0.5438	0.4958	0.4651	0.4474	0.3593	0.3512
0.6	0.7221	0.6408	0.5886	0.5331	0.4699	0.4371	0.3904	0.3614	0.3302	0.2965
0.5	0.6392	0.5815	0.5073	0.4759	0.4408	0.3837	0.3311	0.3157	0.2762	0.2468
0.4	0.5825	0.5418	0.4803	0.4062	0.3724	0.3489	0.2880	0.2522	0.2293	0.2007
0.3	0.5183	0.4655	0.4324	0.3713	0.3183	0.2948	0.2527	0.2132	0.1886	0.1624
0.2	0.4633	0.4208	0.3781	0.3317	0.3141	0.2619	0.2139	0.1702	0.1422	0.1151
0.1	0.4478	0.3951	0.3437	0.3059	0.2757	0.2296	0.1870	0.1410	0.1030	0.0714

(a)

$\boxed{\beta_{ij}}$					r_x					
	1.0	*0.9*	*0.8*	*0.7*	*0.6*	*0.5*	*0.4*	*0.3*	*0.2*	*0.1*
1.0	1.0000	1.0757	1.0285	0.8041	0.7212	0.6669	0.6257	0.6937	0.5409	0.5482
0.9	1.1387	1.0314	0.9593	0.8929	0.5949	0.6301	0.5234	0.6477	0.4421	0.4561
0.8	0.8035	0.7713	0.7159	0.7835	0.6792	0.5606	0.5531	0.5506	0.3955	0.4186
0.7	0.7238	0.7318	0.6668	0.5928	0.5840	0.6665	0.5509	0.6229	0.3910	0.3941
0.6	0.7988	0.6993	0.6248	0.6091	0.5329	0.4589	0.4261	0.4609	0.3835	0.3318
0.5	0.6963	0.6241	0.5705	0.5704	0.5172	0.3629	0.3178	0.4090	0.3335	0.2881
0.4	0.6974	0.7181	0.5614	0.4507	0.4437	0.3370	0.2802	0.2536	0.2256	0.2190
0.3	0.7246	0.6884	0.6228	0.5051	0.4242	0.3712	0.3013	0.2415	0.2232	0.1792
0.2	0.6779	0.6656	0.5349	0.4740	0.4543	0.3349	0.2668	0.2204	0.1519	0.1118
0.1	0.6377	0.5708	0.4823	0.4569	0.3863	0.3285	0.2744	0.1997	0.1332	0.0677

(a)

$\boxed{\beta_{ij}}$	r_x									
	1.0	*0.9*	*0.8*	*0.7*	*0.6*	*0.5*	*0.4*	*0.3*	*0.2*	*0.1*
1.0	1.0000	0.9084	0.8333	0.7368	0.6714	0.6736	0.6061	0.5907	0.5819	0.5499
0.9	0.8728	0.8531	0.7962	0.6763	0.5947	0.5654	0.5499	0.5301	0.5263	0.4965
0.8	0.8034	0.7278	0.6479	0.6501	0.5653	0.5264	0.4911	0.4755	0.4539	0.4318
0.7	0.7317	0.7075	0.6189	0.5687	0.5301	0.4897	0.4648	0.4613	0.4055	0.4026
0.6	0.6995	0.6318	0.5663	0.5195	0.4533	0.4326	0.4019	0.3749	0.3567	0.3367
0.5	0.6402	0.5842	0.5038	0.4658	0.4286	0.3875	0.3421	0.3191	0.2929	0.2841
0.4	0.5954	0.5522	0.5028	0.4123	0.3740	0.3565	0.2989	0.2651	0.2437	0.2298
0.3	0.5370	0.4913	0.4437	0.3856	0.3260	0.2916	0.2620	0.2316	0.2010	0.1837
0.2	0.5022	0.4578	0.4273	0.3598	0.3321	0.2795	0.2300	0.1747	0.1520	0.1252
0.1	0.5170	0.4616	0.4006	0.3424	0.3132	0.2611	0.2164	0.1553	0.1106	0.0782

(a)

$\boxed{\beta_{ij}}$	r_x									
	1.0	*0.9*	*0.8*	*0.7*	*0.6*	*0.5*	*0.4*	*0.3*	*0.2*	*0.1*
1.0	1.0000	0.8684	0.8660	0.6771	0.5656	0.5652	0.5111	0.6327	0.4786	0.5118
0.9	0.9335	0.8894	0.9131	0.7232	0.5467	0.5089	0.5093	0.5697	0.4969	0.4564
0.8	0.7918	0.7649	0.6294	0.6942	0.5991	0.4490	0.4575	0.4608	0.3869	0.4027
0.7	0.7996	0.7652	0.6721	0.5689	0.4886	0.4718	0.4056	0.4858	0.3228	0.3420
0.6	0.8140	0.7501	0.6625	0.5617	0.5076	0.4206	0.3724	0.3551	0.3605	0.3261
0.5	0.7978	0.7256	0.5948	0.5663	0.5091	0.3878	0.3282	0.3337	0.2370	0.2504
0.4	0.8344	0.7665	0.6332	0.5491	0.4925	0.4023	0.3317	0.2549	0.1905	0.2046
0.3	0.8675	0.8142	0.6973	0.5885	0.4942	0.4097	0.3201	0.2617	0.1907	0.1702
0.2	0.8592	0.8078	0.6670	0.6029	0.5355	0.4258	0.3402	0.2465	0.1663	0.1066
0.1	0.8505	0.7491	0.6472	0.5872	0.5286	0.4294	0.3623	0.2548	0.1670	0.0835

Table 3.27 *The numerical values of face area.*

§ 3.13 Voronoi tessellation in higher dimensions

In § 1.13 it is mentioned that the volume of a tetrahedron is $|x_i, y_i, z_i, 1|/6$, for which the absolute value must be taken before adding two or more together. This is what to be found in the existing literature. But I think that this definition is flawed because, for one thing, it does not work for the case of two dimensions. Imagine what happens if we let a unit cube to always have a unit volume. It works as shown in the following. What I discovered here could well be new.

Assumption 3.1. *A unit cube has a unit volume in all dimensions d, $d \geq 2$.*

But, apart from the fact that it does not work for the case of two dimensions, the volume equation above still appeals to our commonsense, because it looks symmetrical, so we assume further.

Assumption 3.2. *The equation for volume of a d-simplex is $V = \text{abs}(|x_{ij}, 1|)/k$, where x_{ij}, $1 \leq j \leq d$, is the j:th coordinate of the i^{th} point, $1 \leq i \leq d+1$, and 1 is a unit vector of $(d+1)$ dimensions.*

Under Assumption's 3.1 and 3.2, and by trial and error, a conclusion is found that the volume of a general d-simplex must be such that the k Assumption 3.2 is $d!$. Therefore we arrive at Definition 3.1. I actually tried $d(d-1)$ and $2^d - 2$ before looking at $d!$ and realised that this is the solution for k.

Definition 3.1. *The volume of a d-simplex is $\text{abs}(|x_{ij}, 1|)/d!$, where x_{ij} is the j^{th} coordinate of the i^{th} vertex of the simplex.*

We can only test the appropriateness of our definition, *i.e.* Definition 3.1, against the cases of two and three dimensions, since four dimensions and above are unfamiliar grounds. But for cubes or hypercubes of two dimensions and above, the fit looks encouraging, as can be seen in Table 3.28.

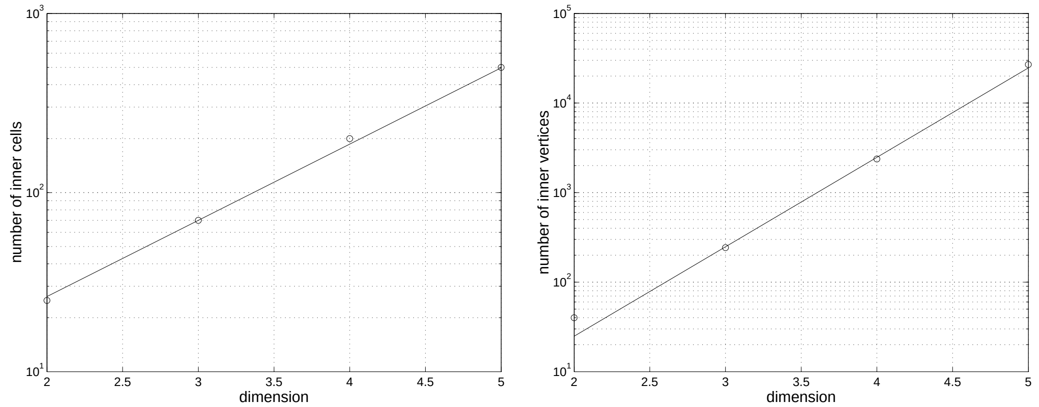
		d								
		2	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
	1	1	1	1	1	1	1	(1)	(1)	(1)
ℓ	*2*	4	8	16	32	64	128	($2^8 = 256$)	($2^9 = 512$)	($2^{10} = 1{,}024$)
	3	9	27	81	243	729	2,187	($3^8 = 6{,}561$)	($3^9 = 19{,}683$)	($3^{10} = 59{,}049$)

Table 3.28 *Volume of d-hypercubes whose dimension is ℓ. Values in brackets are implied not calculated.*

The above is, to put it in other words, our attempt at defining the equation for the volume of d-simplices and the test of this equation against squares, cubes and d-hypercubes. The equation of Definition 3.1 is also rather appealing for the reason that it allows us to calculate the volume of the cube as $V = \text{abs}(|x_{ij}, 1|)/d! = \ell^d = \prod(\Delta x_i)$, where ℓ is the length of the side of the hypercube and the product is over all coordinates x_i.

Table 3.28 justifies our intuitive equation regarding the volume of hypercubes which says that such volume is $V = \ell^d$.

The programme used in doing the simulations is shown in § A.29. There is also the code for testing the formula, which automates the generation of vertices of hypercubes before calculating their volume.

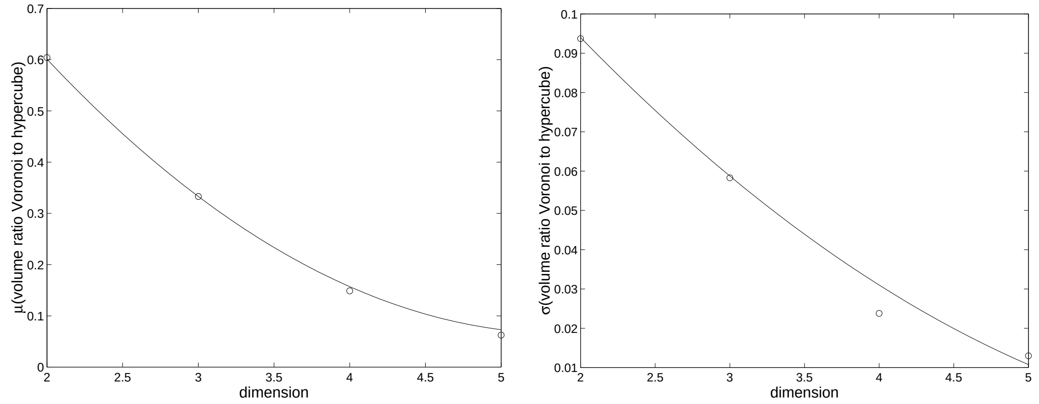


(a) (b)

Figure 3.51 *Number of generators required for ten inner cells. The dotted line in* (a) *has the equation* $y = 3.7 \exp(0.98x)$ *while for* (b) $y = 0.25 \exp(2.3x)$.

The general procedure is to generate the Voronoi cells, exclude boundary vertices, and then boundary cells (compare Algorithm 3.3, page 105). This means that producing ten usable cells, that is to say, cells in the inner reach, requires for each dimension approximately the number of generators plotted in Figure 3.51 (a). Figure 3.51 (b) is the corresponding number of vertices.

For a Voronoi cell in d dimensions we may draw a box around them such that the walls of the box are all parallel to the coordinate planes. For the plane, a Voronoi graph consumes some 60 per cent such binding rectangle. For higher dimensions, this ratio goes down from 0.6 to 0.33, 0.15 and 0.6 respectively for 3, 4 and 5 dimensions. The trend for the standard deviations is also similar to this. Both the mean and the standard deviation decrease with dimension, but their curves seemed difficult to define if we try to plot it on a semilog graph. The curve turns out to be parabola, as shown in Figure 3.52.



(a) (b)

Figure 3.52 *Mean and standard deviation of the ratio Voronoi per defining cube. Both graphs fit well with the parabolic equation* $y = ax^2 + bx + c$; (a) *has* $a = 0.046$, $b = -0.498$ *and* $c = 1.413$, *while* (b) *has* $a = 0.00375$, $b = -.054$ *and* $c = 0.187$.

Additional results on a 6-dimensional VT generated from 1,200 generators are $\mu(V_p/V_h) = 0.0252$ and $\sigma(V_p/V_h) = 8.5 \times 10^{-4}$. The results are shown in Table 3.29.

n_g	c_{in}	d	$\mu(V_p/V_h)$	$\sigma(V_p/V_h)$	V_{in}
25	10	2	0.6041	0.0937	33
30	16	2	0.6034	0.0769	44
70	13	3	0.3331	0.0583	241
200	11	4	0.1488	0.0238	2,374
500	9	5	0.0638	0.0098	
500	11	5	0.0625	0.0130	26,893
1,200	4	6	0.0252	8.5×10^{-4}	306,710

Table 3.29 *Voronoi volume in higher dimensions; V_p is the Volume of the polytope, V_h that of the hypercube and n_g number of the generators.*

The accompanying curve plotted in Figure 3.53 is a parabola $y = 0.0354x^2 - 0.428x + 1.32$. On the other hand, neither the mean value nor the standard deviation of the volume ratio can possibly follow a parabolic curve, for the obvious reason that both decrease monotonically towards zero.

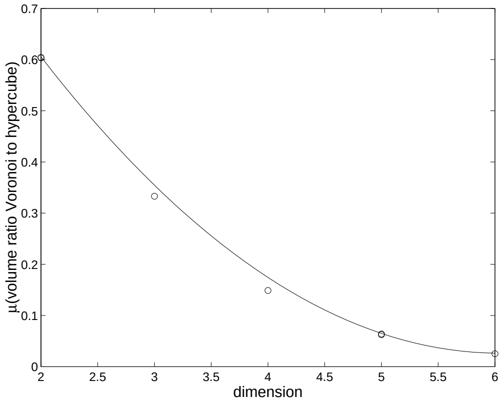


Figure 3.53 The ratio V_p/V_h, dimension from 1 to 6.

4

§ 4. Percolation

The 1970's and the 1980's saw a proliferation of variations on the theme of percolation. Every year there seemed to be a new percolation problem or two. For a *Bethe lattice* Chalupa *et al* (1979) reported a *bootstrap percolation* where those randomly occupied sites with less than m occupied neighbours are recursively emptied one by one until a stable configuration is reached. The problem they are interested in is that where the impurity concentration, dilution and crystal-field interaction compete in magnetic materials compete against the exchange interaction, resulting in the magnetic moments and consequently the magnetic order being destroyed.

Wilkinson and Willemsen (1983) introduced *invasion percolation.* Working with Schlumberger they were interested in the real problem in the oil industry where water displaces oil via capillary action. Their approach was that of a constant flow rate, not one of a constant pressure as usual. Here water displaces oil in the smallest available pores. But when water completely surrounds any region of oil, no further advance into that region will be possible, oil being incompressible. Such regions are called *trappings* and cause a problem generally known by the name of *residual oil*, an economic bane for oil industry.

In the above example the hydrophobic versus hydrophilic property plays an important role in the replacement of oil in pores with water. And water is prevented from penetrating trapped oil regions by much the same principle as that which prevents the water in the contents of a sandwich from crossing the spread layer of butter to wet the hydrophilic bread. For this latter case the pores in question are those within the bread texture, and the soaking of the bread is best prevented provided that the trapped regions of butter or margarine percolates in two dimensions to form a single layer or film which entirely covers the sectioning surface of the bread. Moreover, there is a similarity also in the internal structure of both the rock and the bread. The density of pores in bread is not homogeneous as a result of the tension on the surface of the dough caused by internal air pressure originated from the yeasts inside, as well as because of the heat applied to it when inside the oven, which dehydrolises the surface and makes it dry and hardened. The same inhomogeneity can be found inside the rocks which form the oil reservoirs where the regions of oil are surrounded by rocks which are less porous and have less permeability.

For Adler and Aharony (1988) a random walker, aka ant, treads on clusters. The ant enlarges a cluster by stepping on to an empty site next to it which meets certain conditions. They called this problem *diffusion percolation.* An example of a condition met is where the empty site has two or more occupied neighbours on a square lattice.

Most percolation in studies happens by randomly toggling the phases of sites or bonds in regular lattices. Kerstein (1983) considered randomly located spheres, take the complementary region of their union, and then perform percolation on the former. He showed that such percolation problem is equivalent to a percolation on a Voronoi lattice whose sites are the sphere centres.

In the same way as an infinite loop in computer science means that one can not come to the termination of a programme going along the time dimension, an infinite cluster in percolation means that one can not come to the end of a cluster shifting along either one of the dimensions. The former case is along the one dimension of time and is only possible because of the time flows in one direction. Therefore half the line spanned is considered as being infinite. In one nondirectional dimension, except for the trivial case where one can consider the entire space as being one single cluster, no infinite cluster is possible. In simulation, when a cluster spans the whole of the space considered along any dimension we say that it is infinite since as one moves a cross section along that dimension, it always contains a section of the cluster.

The bond percolation programme of two-dimensional Voronoi network by Tiyapan (1995, KNT3(iii), p. 78) takes $O(n^2)$ time provided we assume that the contribution made by the number of clusters together with that by each cluster comparison amount to a linear time complexity term. In order to justify this assumption, consider first the number of clusters. The maximum number of clusters possible depends on the size of the system, in other words it must vary as n^{k_1}, where $0 < k_1 < 1$. This maximum value should be approximately 0.5 because of symmetry between occupancy and void clusters. Next consider the time involved in comparing two clusters. On Matlab this is a sparse vector comparison which is likely to involve some linked lists, and similarly should take time as a function of n_2^k, $0 < k_2 < 1$. Because on average the size of clusters is always small before Pc, k_2 will be less than 0.5. Therefore $n^{k_1} \cdot n^{k_2} < n$ and it is safe to assume $O(n)$ time from both of them combined. *q.e.d.*. A C translation (Tiyapan 1995, *ibid.*, p. 80) of this programme, though not as bad as it may seem because constant coefficients are small, gives $O(n^5)$ time in comparison.

In the field of geology Miller and Nur (2000) studies the analogy between the dilatant slip in earthquakes and the hydrofraction occurred in melting and dehydration, the percolation of the latter in the permeability network internal to the fault being the cause of the former. According to them, the existence of toggle switches in the permeability rules out the assumption that the permeability throughout the whole system is homogeneous. Instead, the system reorganises itself into sytems of different scales of interaction according to the degree and nature of its inhomogeneity. At the critical state the scale of interaction is equal to the scale of the model.

The percolation probabilities given by Stauffer and Aharony (1994) are, the first number being for sites and the second for bonds, for the honey comb lattice 0.6962, 0.65271; square 0.592746, 0.50000; triangular 0.50000, 0.34729; diamond 0.43, 0.388; simple cubic 0.3116, 0.2488; body-centred cubic 0.246, 0.1803; face-centred cubic 0.198, 0.119; 4-d hypercubic 0.197, 0.1601; 5-d hypercubic 0.141, 0.1182; 6-d hypercubic 0.107, 0.0942; 7-d hypercubic 0.089, 0.0787.

De Gennes *et al* (1959) investigated disordered binary solid solution AB where active atoms A randomly replace the nodes of the periodic matrix B. There exists a critical concentration A in B below which all clusters are finite, and above which both finite and non-finite, *i.e.* infinite, clusters exist. Such solid solution in networks can represent the spin waves in alloys with one ferromagnetic component or the impurity bands in semiconductors. They cited seminal work on percolation by Broadbent *et al* (1957), but no mention was made about the Ising model.

In a way, percolation is similar to diffusion. In diffusion the particles considered move about randomly, whereas in percolation they can only crop up randomly at predetermined locations on a network which is fixed. We could imagine, for instance, cars running along the roads within a traffic network as diffusing through them. Then the percolation could occur on a larger scale, that is the scale of a road. The cars move along, that means they diffuse; but the roads remain fixed, and so their phases could percolate. In other words, in diffusion the particles move while in percolation, whether there are moving particles or not, it is the phases that percolate. Since historically percolation began as the study of diffusion of particles in a network of tubes in which the phases are naturally defined as the tubes being blocked or unblocked, these definitions have become most frequently used in other areas of application, for example in filtering membranes and traffic networks.

But this is not necessarily the case. Instead of dealing with a fixed network, one may consider a model of percolation in a continuum, for example by randomly patching an area until all the patches connect with one another somehow and percolate. The patch could be of any shape, as well as polygonal and circular. We can consider the percolation in a certain area as having occurred when there appears a cluster of patches which traverses any two opposite sides. One application of this is in the study of occurrences of epidemics. Hoyle and Wickramasinghe (1979), having given a convincing argument in favour of viruses and various forms of diseases being carried to Earth from space by comets, talk about patchiness of pathogenic clouds. According to them, simultaneous attacks across vast region rules out person-to-person theory. Moreover, influenza epidemics are generally characterised by sudden onsets and equally sudden ends. These epidemics and plagues may be thought of as the percolation of these patches in a sufficiently large, predefined area. The bacteria and viruses coming from space adding to gene give the possibility of jump patterns in evolution (*cf* Smith and Szathmáry, 1995). The cells deliberately refuse to block viruses because they could prove to be useful in the long run, generally not by individuals but by the species. Historical examples are the disease described by Thucydides between 431 and 404 BC, five epidemics of 'English Sweats' between 1485 and 1552, and more than ten influenza pandemic from 1700 to 1900. Example of diseases which are caused by bacteria and viruses are bubonic plague, chicken pox (varicella), cholera, common cold, Legionnaires' disease, leprosy, measles, mumps, poliomyelitis, small pox, tuberculosis, and trachoma. Examples of major evolutionary transitions are those going from RNA to DNA, from prokaryotes to eukaryotes, and from asexual cloning to sexual propagation. Another example is the transition from primate to human both of whom differ from each other neurologically in the ability to use language and the power to conceptualise. One description of the Great Plague in London (Dickens, 1851). There was a rumour that a few people died in the winter of 1664. In May 1665 the disease burst out in St. Giles's which raged through July and September in every part of England; approximately 10,000 people died in London alone. But then the equinox winds virtually blew the disease away, and the Plague quickly disappeared. The existence of interstellar organic matters is supported by strong evidences with more and more complex substances constantly found (*cf* Hoyle and Wickramasinghe, 1978).

Gas turns into liquid by the growth of larger and larger clusters. But unlike percolation in networks, clusters in condensation process are not well defined. Crystalisation is also characterised by the growth of one phase within another. But here the orientations are an inherent part of the clusters themselves, and there is a long range coordination among clusters. One important thing in percolation is for the system size to be infinite. Random discs overlapping one another in a continuum helps correct counts of bacteria cultures (*cf* Essam, 1980). Let p_c be the critical probability and $P(p)$ the percolation probability. Then as p approaches p_c from above, $P(p) \approx (p - p_c)^{\beta}$ where $0.4 < \beta < 0.5$ is the critical exponent. If p approaches p_c from below, $S(p) \approx (p_c - p)^{-\gamma}$ and $\xi(p) \approx (p_c - p)^{-\nu}$ where $S(p)$ is the conditional average $\langle s|F\rangle$, s the number of particles in a cluster, and F the event that the origin O is occupied and belongs to a finite cluster. In other words, $S(p)$ is the mean size of cluster at the origin given that F occurs. This last event occurs with probability $p_F = p(1 - P(p))$ and $p_F = p$ for $p < p_c$. Furthermore, $S(p) = p_F^{-1} \sum_r C(r,p)$ and $\xi^2(p) = [p_F S(p)]^{-1} \sum_r r^2 C(r,p)$ where the pair-connectedness function is $C(r,p) = \langle \eta(r)|F\rangle p_F$, and $\eta(r)$ is the indicator defined to be one if r is connected to O and zero otherwise. When F occurs, $s = \sum_r \eta(r)$. There are two different definitions of the critical probability, $p_c = \sup\{p|P(p) = 0\}$ and $\pi_c = \sup\{p|P(p) = 0 \text{ and } S(p) < \infty\}$, sometimes denoted by p_H and p_T for Hammersley and Temperley respectively. By definition, $\pi_c \leq p_c$. When p approaches p_c from above, γ' and ν' are similarly defined respectively by $S(p) \approx (p - p_c)^{\gamma'}$ and $\xi(p) \approx (p - p_c)^{\nu'}$ It is generally assumed as $\gamma' = \gamma$ and $\nu' = \nu$. No proofs for these assumptions exist, even though they are consistent with the series expansions. Estimates of γ and ν are $1.6 < \gamma < 1.7$ and $0.8 < \nu < 0.9$. In a dilute ferromagnet a cluster containing s spins each of which has a unit magnetic moment has a probability $p = \exp(\frac{1}{2}sh) / \left[\exp(-\frac{1}{2}sh) + \exp(\frac{1}{2}sh)\right]$ of being parallel to the magnetic field $H > 0$. Here $h = 2H/k_BT$ where k_BT is the thermal energy. For an infinite cluster, $p = 1$. The zero-field magnetic moment is $\mu_0(p) \propto P(p)$. The zero-field magnetic moment is $\mu_0(p) \propto P(p) \sim (p - p_c)^{\beta}$. The field dependent magnetic moment at p_c is $\langle(1 - \exp(-sh))/(1 + \exp(-sh))\rangle_F$ and $\frac{1}{2}(1 - G(p_c, h)) \leq \mu_c(h) \leq 1 - G(p_c, h)$ where $G(p, h) = \langle\exp(-sh)\rangle_F$. Therefore the critical probability δ is defined by $\mu_c(h) \sim 1 - G(p_c, h)h^{1/\delta}$. The correlation function between σ on site i and j is defined by $\Gamma_{ij} = \langle\sigma_i\sigma_j\rangle_T - \langle\sigma_i\rangle_T\langle\sigma_j\rangle_T$ where $\langle\cdot\rangle_T$ is an average over spin states. Then $\Gamma_{ij} = 1$ when i and j belong to the same cluster and zero otherwise. The fluctuation formula is $k_BT\langle\chi\rangle = \sum_j \langle\Gamma_{ij}\rangle$ where $\langle\Gamma_{ij}\rangle = C(r_j - r_i, p)$, the mean susceptibility is $\langle\chi\rangle = p_F S(p)/k_BT \sim (p_c - p)^{-\gamma}$ and the mean free energy is $\langle F\rangle = (k_BT \ln 2)K(p)$ where $K(p)$ is the mean number of clusters and $K(p) \sim (p_c - p)^{2-\alpha}$, where the index assigned to the third derivative divergence is $1 + \alpha$, $\alpha \approx -0.5$.

The growth mechanism of clusters in percolation is also found in biology. Williams and Bjerknes (1972) simulate a tumour in the basal layer of an epithelium. The basal cells become less sensitive to Charlone which controls cellular division, and thus they divide κ times faster than the normal cells where $\kappa > 1$ is the carcinogenic advantage. Abnormal cells interior to the basal layer divide, push, and then replace the neighbouring cells leaving the overall configuration unchanged except at the border where the abnormal cells exert a thrust of $\kappa - 1$ on their normal neighbours, that is $\frac{\mathrm{d}N}{\mathrm{d}t} = (\kappa - 1)n$ where n and N are respectively the numbers of peripheral and total abnormal cells. They found that dimensionality of fractal is involved and the dimension 1.1, instead of 1, must be assigned to the periphery, which means that n is proportional to $N^{0.55}$ not $N^{0.5}$. They found that abnormal cells push out faster in this order: the triangular, square, and hexagonal lattice. We may explain this, by looking at the coordination numbers of these three lattices, that the higher coordination number the lattice sites have, the slower the cluster expands. Added coordination means a higher degree-of-freedom the newly divided cells have to move about while still remaining local.

Percolation is a field full of far more open questions than discovered answers. This is one of the reasons it is more suitable to research than university examinations (*cf* Stauffer and Aharony, 1998). The fascination not so well hidden within the field is reflected in the number of research monographs written on this field that is close to percolating, which makes it impossible to give a full list of references. Percolation deals with the clusters formed by randomly occupying the each site of a very large lattice with probability p independent of its neighbours. Most of the studies to date concentrate on the critical phenomena, which exist in a very narrow range in each problem, and the scaling theory which tries to describe them. The contiguity criteria are easier to define in applications in physics and physical sciences, because here problems can be described by, or easily simplified into some definite geometry, whereas in other applications this may not be the case. For example, in economic modelling the nature of the geometry or even the number of dimensions of such structure is still not well understood (*cf* Tiyapan, 1997, KNT5(vii) and KNT5(viii)), let alone the problem of contiguousness. Traffic network (*cf* Tiyapan, 1997, KNT5(i) and KNT5(ix)) is another example. Here the flow is along channels within tubes, *i.e.* roads, both of which could be directed, which make the network a pool of tangled threads not only difficult to imagine but also to define such things as congestion and contiguousness. Even in geographical problems like that of forest fire, the spread of fire may be defined in at least three ways, namely by next-nearest neighbour, neighbour or double neighbour respectively in cases of a common corner, a common side or two neighbours. Reservoirs occur when the petroleum, formed in sedimentary rocks, migrates into permeable sedimentary rocks like sandstone. Around 70% of North Sea oil fields also date from the Jurassic period. These include Beatrice, Brent, Cormorant, Murchison and Tartan. Oil companies want to exploit reservoirs where $p > p_c$. Moreover, they want to tap into the largest cluster in these reservoirs. For this purpose bore hole samplings are carried out which collected within $L \times L$ frames and then the number of points, $m(L)$, belonging to the same cluster counted. It turns out that $m(L)$ is proportional to $L^{1.9}$. In general $m \propto L^d$ where d is either integral, in which case it is the Euclidean dimension, or nonintegral, in which case it is the fractal dimension introduced by Benoit Mandelbrot. There exists a correlation length ξ such that $m(L) \propto L^{1.9}$ for $L < \xi$, and $m(L) \propto L^2$ for $L > \xi$. This correlation length is a measure of the largest hole of the largest cluster and decreases as p is increased above p_c.

One way of finding the percolation threshold is by using the *ant in a labyrinth* algorithm shown here as Algorithm 4.1.

Algorithm 4.1 *Ant in a labyrinth. At* p_c, $k = 1/3$, *whereas* $k = 0$ *for a constant distance, and* $k = 1/2$ *for normal diffusion.*

```
for various $p$ do
  repeat a large amount of times
    occupy the sites with probability $p$;
    $sum \leftarrow 0$;
    for a large number of simulations $n$ do
      for various $t$ up to a large number do
        $sumr \leftarrow 0$;
        for $i = 1$ to $t$
          ant randomly occupies an occupied site;
          identify neighbours;
          move into an occupied neighbour randomly chosen;
          $r \leftarrow$ distance travelled;
          $sumr \leftarrow sumr + r$;
        endfor
        $sumrsq \leftarrow sumr^2$;
      endfor
    endfor
    $sum \leftarrow sum + sumrsq$;
    $R \leftarrow sum^{1/2}$;
    plot $R$ against $t$ in double logarithmic scales;
  endrepeat
  $k \leftarrow$ the slope of the straight line just plotted;
endfor
$p_c \leftarrow$ the $p$ such that $k = 1/3$;)                □
```

Also, the number of steps the ant takes, t, for the linear size of a region it visited, R, is fractal, that is $t \propto R^{1/k}$. They give several percolation probabilities for selected lattices, which are shown in Table 4.1, as well as the exact values of p_c's, which are for the square bond percolation 1/2, triangular site 1/2, triangular bond $2\sin(\pi/18)$, and honeycomb bond $1 - 2\sin(\pi/18)$. For the honeycomb site percolation, $p_c < 1/\sqrt{2}$. Site percolation on hypercubic lattices of high dimensions have $p_c = 1/(2d-1)$.

lattice	*site*	*bond*
honeycomb	0.6962	0.65271
square	0.592746	0.50000
triangular	0.500000	0.34729
diamond	0.43	0.388
simple cubic	0.3116	0.2488
body-centred cubic	0.246	0.1803
face-centred cubic	0.198	0.119
4-d hypercubic	0.197	0.1601
5-d hypercubic	0.141	0.1182
6-d hypercubic	0.107	0.0942
7-d hypercubic	0.089	0.0787

Table 4.1 *Percolation thresholds from Stauffer and Aharony (1998)*

There is no percolation in one dimension because in such case the percolating cluster must necessarily contain the whole space. But there are some applications where the critical blockage is important, for example the blockage of drainage grilles by pea shingle is one-dimensional. Here the critical amount of blockage depends on the critical rate of flow which in turn depends on the amount of water and the rate of accumulation of water to be drained. The maintenance of gullies and grilles is done by cleaning, flushing and grit bucket emptying (*cf* Harrison and Trotman, 2002).

For the Bethe lattice $p_v = p_e = 1/(z-1)$. In general, $p_e \leq p_v$. For a network of d dimensions, $zp_e \approx d/(d-1)$. In his review, Sahimi (1994) gave some of the current know p_c's which are listed in Table 4.2 here.

	d	z	p_e	zp_e	p_v
honeycomb	2	3	$1-\sin(\pi/18)$	1.96	0.6962
square	2	4	1/2	2	0.5927
kagome	2	4	0.522	2.088	0.652
triangular	2	6	$2\sin(\pi/18)$	2.084	1/2
diamond	3	4	0.3886	1.55	0.4299
simple cubic	3	6	0.2488	1.49	0.3116
body-centred cubic	3	8	0.1795	1.44	0.2464
face-centred cubic	3	12	0.198	1.43	0.119

Table 4.2 *Percolation probabilities, cf Sahimi (1994).*

The accessible fraction, f_α, is the fraction of occupied bonds that belong to the infinite cluster. The backbone fraction, f_β, is the fraction of those accessible bonds which belong to a transport path, *i.e.* a path with all dead ends excluded. The correlation length, λ, is the length scale over which the random network is macroscopically homogeneous. In a Monte Carlo simulation it is necessary that the size L of the network is sufficiently larger than this correlation length in order to obtain a p_c which is independent of L. The expected cluster size m is $\mathrm{E}(m) = \sum_m m^2 n_m / (\sum_m m n_m)$, where $n_m(p)$ is the expected number of clusters of size m per lattice site and mn_m the probability that a site belongs to an m-cluster. The fraction of the network which can accommodate flow has various names associated to each application, for example the effective electrical conductivity g_e, the effective diffusivity D_e and the hydrodynamic permeability k. The effective elastic moduli, G, are the elastic moduli of the network a fraction p of bonds of which are elastic elements, while the rest are rigid or stiff. The fraction of isolated occupied sites is $f_i(p) = p - f_\alpha(p)$. The universal scaling laws state that $p_c(p) \sim (p-p_c)^{b_p}$, $f_\alpha(p) \sim (p-p_c)^{b_p}$, $f_\beta(p) \sim (p-p_c)^{b_\beta}$, $\lambda(p) \sim |p-p_c|^{-\nu_p}$, $k(p) \sim |p-p_c|^{-\gamma_p}$, $g_e(p) \sim (p-p_c)^{\mu}$ and $G(p) \sim (p-p_c)^{f}$, where b_β, b_p, ν_p and γ_p are topological exponents, completely universal, depend only on the dimensionality but not the microscopic details of the system. The effective diffusivity $D_e(p) \sim (p-p_c)^{\mu - b_p}$ because $g_e \sim n_e D_e$ and $n_e \sim f_\alpha(p)$. Near p_c, $k(p) \sim (p-p_c)^{e}$. For network percolations $e = \mu$, but for the continuum percolation this may not be the case.

As $p \to p_c^-$, perfectly conductive clusters become larger and g_e increases. Then $g_e(p) \sim (p_c - p)^{-s}$ near p_c and diverges at p_c. In two dimensions, $\mu = m$. If p edges are totally rigid while the rest are elastic, then G diverges as $p \to p_c^-$ such that $G \sim (p_c - p)^{-\varsigma}$ For large m near p_c, $n_m \sim m^{-\tau_p} f\left[(p-p_c)m^{\sigma_p}\right]$, where τ_p and σ_p are universal and $f(0) \neq 0$. Some relations among the geometrical exponents are $\tau_p = 2 + b_p\sigma_p$ and $\nu_p d = b_p + 1/\sigma_p = 2b_p + \gamma_p$. Sahimi (*ibid.*) listed some values of the current critical exponents and fractal dimensions which we list again here as Table 4.3.

	$d = 2$	$d = 3$	Bethe lattices
b_p	5/26	0.41	1
b_β	0.48	1.05	2
ν_p	4/3	0.88	1/2
γ_p	43/18	1.82	1
σ_p	36/91	0.45	1/2
τ_p	187/91	2.18	5/2
D_c	91/48	2.52	4
D_β	1.64	1.8	2
$D_{\min}$	1.13	1.34	2
μ	1.3	2.0	3
m	1.3	0.73	0

Table 4.3 *Critical exponents and fractal dimensions, cf Sahimi (1994).*

The system is macroscopically homogeneous when $L \gg \lambda$ but not when $L \ll \lambda$ where the cluster spanning the sample is self-similar and fractal at all length scales up to λ. Its mass is $M \sim \lambda^{D_c}$, where $D_c = d - b_p/\nu_p$ is the fractal dimension of the cluster. For $L \gg \lambda$, $D_c = d$. Similarly the backbone is also a fractal object when $L \ll \lambda$, and it has the fractal dimension $D_\beta = d - b_\beta/\nu_p$. When $L < \lambda$ one should replace λ by L, at λ diverges at $p = p_c$ and the red bonds have $M_r \sim L^{D_r}$, $D_r = 1/\nu_p$, when $L \ll \lambda$. The minimal or chemical path between two points of a percolation cluster is the shortest path between the two points, and $L_{\min} \sim L^{D_{\min}}$ for $L \ll \lambda$.

Finite-size scaling is the effect of finite size on the critical properties. As $p \to p_c$, $\lambda \to L$, and the variations of a property P_L becomes $P_L \sim L^{-x} f(u)$, where $u = L^{1/\nu_p}(p - p_c) \sim (L/\lambda)^{1/\nu_p}$ and $f(0) \neq 0$. When $p \to p_c$ and $L \to \infty$, $P_\infty \sim (p - p_c)^\delta$ and $x = \delta/\nu_p$. There is also a shift in the percolation threshold, and $p_c - p_c(L) \sim L^{-1/\nu_p}$ where the percolation probability is p_c for the infinite system while its effective value for a finite system is $p_c(L)$. Correction should be made whenever the system simulated is small, that is $P_L \sim L^{-x}\left[a_1 + a_2 g_1(L) + a_2 g_2(L)\right]$, where g_1 and g_2 are the correction terms to the scaling. For transport properties, for example conductivity, diffusivity and elastic moduli, $g_1 = (\ln L)^{-1}$ and $g_2 = L^{-1}$ are often good enough.

Sahimi (1994) mentions a random distribution of inclusions, such as circles, spheres, or ellipses, in an otherwise uniform system or continua; a percolation on random polyhedra such as the VT; and also a random distribution of random conducting sticks of with given aspect ratio, or plates a given extent, which is important in modelling fracture networks in rocks. In the study of percolation in continua, one considers a continuous, random function $h(r)$, defined for all points r's in the entire space, such that $\langle h(r) \rangle = 0$. All regions of space where $h(r) < R$ are in a phase which is different from the rest, the volume of which goes from zero to infinite as R changes from negative- to positive infinite. As R becomes larger, so do the islands, and the phase grows and percolates at the critical R_c. For continua percolation, the critical occupied volume fraction is defined as $\phi_c = p_c f$, where f is the filling factor of a lattice when all its sites are occupied by an impermeable sphere in such a manner that those which are nearest neighbours touch each other. This volume fraction seems to be an invariant, approximately 0.45 for two- and 0.15–0.17 for three dimensions. If the spheres are permeable, then $\phi_c = 1 - \exp(-B_c/8)$ where $B_c = zp_b$ is the number of bonds per sites at p_c, and in continua percolation $B_c \to p_c z$ as $z \to \infty$. If ϕ_c is an invariant of continuous systems, then R_c is given by $\phi_c = \int_{-\infty}^{R_c} h(r)\mathrm{d}V$, where V is the volume of the system.

In this thesis it has been claimed that the thing most relevant to percolation is not the percolating threshold but the slow process which makes sure the network percolate. In other words, percolation is not an instance but a process. This is why the PhD course which goes on for a few years contains works of originality which are concentrated within the last few months, and those real contributions which cause the breakthrough occurring altogether amounting to a few days. In this case the time associated with the process itself compared with that of the breakthrough differ by at least two orders. This does not mean that a few years have been wasted, because without these none of those few days mentioned would probably have happened. This is also one of the reasons why the first few chapters contain materials seemingly eclectic. But to exclude this material would have meant to rid the readers of the flavour. Afterall this is a project about percolation, and what it claims has already been mentioned at the beginning of this paragraph.

Results from the homohedral tilings in Section 4.9, which starts from page 136, are useful for designing a truss. Though nobody has ever used these tilings for this purpose, such originality would be beyond the scope of this thesis. But the programme can be used for all kinds of non-random tiling. The Eden Project, whose framework is the honeycomb and its covering, represents one step along this line which could prove to define the direction of architecture at the beginning of this first century of the millennium.

There is a difference between phase and state of a networks. A road networks can have two phases, that is blocked or unblocked, but its number of states is three, namely free-flowing, congested and stand-still. Congested state is a limbo where either both cars and space have percolated, or neither has. Two approaches exist for studying and simulating the traffic. One of them looks at the movement of automatons, that is cars, while the other looks at the phase and the state of the roads, in other words bonds. The term 'percolation of cars' may sound automatonic but by it I only mean the percolation of blocked roads. Likewise, the term 'percolation of space' means percolation of empty or unblocked roads.

Bonds, cells, edges and vertices form lists in our percolation algorithms. These lists can become when required rows and columns of some sparse matrices. Therefore all the components are practically numbered, naturally according to their position on the list. Before simulation starts, the list of the component of interest is permuted and becomes the ordering of the blockages. Using the list two simulations are done, one going from front to rear and the other from rear to front.

§ 4.1 S-curves and the percolative phenomena

Before any abrupt change occurs there must be a graduation process leading up to it. In processes where the time constant is long, the characteristic s-shape is obvious, take for example a learning curve. But even where time constants are short, it is doubtful whether such thing as a strictly abrupt change exists. At the very point of transition, undoubtedly there may be a singularity, for instance at the Big Bang. But even the Big Bang can not exist alone by itself. To preserve the symmetry of nature, there must be another process at the other end leading up to it. It is only because such process must necessarily be on the other side from us and we can not see it from here. That side may belong to the antimatter or the anti-universe, but I believe whenever there is a singularity there must be a symmetry.

An earlier work by Tiyapan (2003, KNT8(iii)) can be used as an example. The per cent extraction curves which I drew then (Tiyapan, 1991, KNT1(i)) all show the change to be abrupt, starting off from zero time immediately with a positive gradient. This can not happen in the real world, so there must have been a foot of the characteristic s-curve at the beginning. It must have been that the sampling time used is too long, and anyhow the nature of the reaction may make it impossible to observe the development in detail with accuracy.

The predicted value of the heat capacity of monoatom solids according to the Debye theory, from de Podesta (1996), is shown in Figure 4.1. According to Debye, $C_V = (1944p/\Theta_D^3)T^3$ JK^{-1}mol^{-1} (*cf* de Podesta, 1996). The curve in Figure 4.1 shows the characteristic s-shape which should have appeared in Tiyapan (2003, KNT8(iii)).

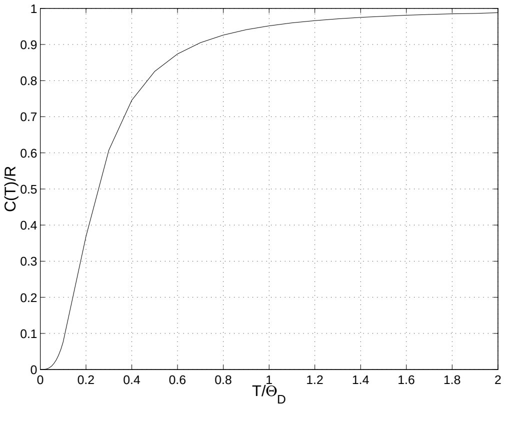


Figure 4.1 predicted value of the heat capacity of monoatomic solids according to Debye.

The foot of a positive s-curve has a positive, nonzero second derivative. This corresponds with the positive cooperativity of the product curve in studies of enzyme assay and kinetics. In enzyme assay, the product versus time graph shows a positive cooperativity characteristic when the Hill constant $h > 1$ in the equation $y_s = [s]^h/(K + [s]^h)$, where y_s is the fractional saturation of the enzyme with substrate while s the concentration of the substrate (Eisenthal and Danson, 2002). Cooperativity reflects the equilibrium binding of substrate or other ligand. The binding of a substrate molecule to an enzyme either facilitates, when the cooperativity is positive, or hinders, when the same is negative, the binding among molecules of the same substrate.

§ 4.2 Voronoi percolation in two dimensions

The percolation of Voronoi tessellation in two dimensions can be achieved by the following algorithm.

Algorithm 4.2 *Network percolation in 2-d.*

```
generate random points;
generate Voronoi tessellation and Delaunay triangulation;
find vertices within the square box bounded by lower and upper bounds;
find neighbours of all cells, bonds, vertices, and edges;
for unit = cell, bond, vertice and edge do
  find number of units;
  permute list of units;
  clear cluster list 1, cluster list 2, set of resultant clusters;
  cluster percolated ← false;
  for $i = 1$ to number of units do
    existing cluster joined = false;
    for $j = 1$ to number of clusters in cluster list 1 do
      if the $j^{th}$ cluster contains the $i^{th}$ unit in permuted list do
        merge the $i^{th}$ unit into the $j^{th}$ cluster ;
        existing cluster joined ← true;
      end
      if existing cluster joined is true
        move the $j^{th}$ cluster of cluster list 1 to cluster list 2 ;
        for $k = 1$ to number of clusters in cluster list 1 do
          if the $k^{th}$ cluster in cluster list 1 touches the cluster in cluster list 2 do
            merge the former into the latter;
          else
            append the former to cluster list 2;
          end
        end
        test percolation of the cluster just updated;
        move cluster list 2 to cluster list 1;
        clear cluster list 2;
        break;
      end
    end
    if existing cluster joined is false
      create a new cluster of size one and append it to cluster list 1;
    end
    append cluster list 1 to set of resultant clusters ;
  end
end
```

The Pc probabilities are found to be ${}^{Vn\ 2d}_{4(\mathbf{1}),14(\mathbf{2})}p_c = 0.5110 \pm 0.0856$, ${}^{Vn\ 2d}_{17(\mathbf{2})}p_b = 0.3095 \pm 0.0523$, ${}^{Vn\ 2d}_{16(\mathbf{2})}p_v = 0.7231 \pm 0.0616$ and ${}^{Vn\ 2d}_{16(\mathbf{2})}p_e = 0.6801 \pm 0.0468$.

And the coordination numbers are ${}^{Vn\ 2d}_{2(\mathbf{1}),7(\mathbf{2})}x_c = 5.2320 \pm 0.2436$, ${}^{Vn\ 2d}_{8(\mathbf{2})}x_b = 9.5022 \pm 0.2979$, ${}^{Vn\ 2d}_{8(\mathbf{2})}x_v = 2.8617 \pm 0.0259$ and ${}^{Vn\ 2d}_{8(\mathbf{2})}x_e = 3.8064 \pm 0.0382$.

The percolation theory in two dimensions has benefited much from discoveries regarding the Ising model in Physics. The Ising model is to date probably the most successful model in percolation theory. This is not only because it can describe in details the phenomena of ferromagnetism and antiferromagnetism, but also because it does so by replacing a noncrystalline solid in the atomic scale with a perfect lattice.

When a bar of iron is placed in a external magnetic field at a constant temperature, the field will induce some magnetisation in the bar. If the external field is slowly turned off, there are two scenarios possible. If $T < T_c$ the bar retains some of its internal magnetisation, but if $T > T_c$ the magnetisation completely disappears. The transition temperature T_c is known as the Curie point.

The Ising model represents the iron bar by lattices, for instance a square lattice. Then it defines two power series, namely the low- and the high temperature series. The low temperature series comes from a bivariate generating function which is generated from the number of colourings for which there are p black sites and q black-white edges. In other words if we let $A(p, q)$ be the said number of colourings, then our bivariate generating function becomes $a(x,y) = \sum_{p,q} A(p,q)x^p y^q$ and the power series is obtained from $\alpha(x,y) \lim_{N\to\infty}(1/N)\ln(a(x,x))$.

On the other hand, the high temperature series of the Ising model changes its form with the dimension. In one dimension it degenerates, while in two dimensions it is isomorphic to the low temperature case and comes from a bivariate generating function that is generated from the number of even polygonal drawings whose area is p and which has q edges. In three dimensions it comes from a univariate function that in turn is generated by the number of even polygonal drawings which have q edges. An even polygonal drawing on a lattice is a union of its subgraphs that uses each edge of the latter at most once and each site an even number of times. It is also known as an Eulerian subgraph and is indeed a union of simple, closed and edge-disjoint polygons which need not be connected.

In three dimensions if we let $B(q)$ be the number of even polygonal drawings mentioned, then our univariate generating function becomes $b(z) = \sum_q B(q)z^q$ and the power series can then be obtained from $\beta(z) = \lim_{N\to\infty}(1/N)\ln(b(z))$.

Two main problems remain unsolved regarding the Ising model, namely that of finding closed-form expressions, of $\alpha(x,y)$ for two dimensions and of $\beta(z)$ for three.

§ 4.3 Voronoi percolation in three dimensions

In three dimensions the procedure of finding p_c is similar to that used in the 2-d case. Algorithm 4.3 written was used in doing the simulation. Developed for running on Matlab, it does every thing in matrix form. As a consequence, there are various types of data all of which are matrices. These data structures is summarised in the following.

A *list* is a one-dimensional matrix whose dimension is the number of its members. An *index* is a matrix whose members are either one or zero. These index matrices in two dimensions are like a map or an array. They are normally one or two dimensions, and map the relationship between the members of one set and another. A *set* is an $m \times n$ matrix every one of the members a_{ij} of which is a matrix. A *pair matrix* is a square diagonal index matrix that maps among members within the same set. It contains the information about relationships like neighbourhood, group membership, connection, *etc.* A pair matrix or an index can also contain information other than the existence or membership flags, for example the edge length in the case of a vertices pair edge matrix. In a bond or an edge the mid point represents its position.

The programme can be divided into three parts, creating and arranging the Voronoi data, finding the neighbourhood matrix and the percolation simulation for p_c. Putting the vertices of a face in order amounts to juggling from one end of each stick, *i.e.* edge, to another. Edges are traced in one direction only until all the edges are successfully linked head to tail. Two lists receive the result from the tracing, vertices which match go to one of them while those which do not is put in the other and recycled.

Algorithm 4.3 *Managing the Voronoi data in three dimensions.*

```
$x \leftarrow$ create random points;
$(v_a, c_a) \leftarrow$ create Voronoi tessellation;
$t \leftarrow$ create Delaunay tessellation;
find $v_n$ such that for all $v_a$, $0 < v_a < 1$;
find $c_n$ such that all vertices of $c_a$ are in $v_n$;
$c \leftarrow c_n$;
for all  do
    $m_b \leftarrow$ find mid bond coordinates;
    $b_l \leftarrow$ find bond length;
endfor
for all pairs of cells do
    $f_a \leftarrow$ find shared vertices when either of the two cells is in $c_n$;
endfor
for all $v_n$, $v \leftarrow v_a$;
for all $c_n$, $f \leftarrow f_a$;
for all $f$'s do
    if it has three vertices then
        all the three possible pair combinations are neighbours;
    else
```

```
        $(a, b, c) \leftarrow$ find the face equation from three vertices;
        $\theta_i \leftarrow ka$ where $k = 1/(a^2 + b^2 + c^2)$ and $i = 1$, 2 and 3;
        max $\leftarrow 0$;
        for $j = 1$ to 3 do
          if $\theta <$ max then
            $j_m \leftarrow j$;
            max $\leftarrow \theta(j_m)$;
          endif
        endfor
        if $j_m = 1$ then
          $p \leftarrow y$; $q \leftarrow z$;
        elseif $j_m = 2$ then
          $p \leftarrow x$; $q \leftarrow z$;
        else
          $p \leftarrow x$; $q \leftarrow y$;
        endif
        $d \leftarrow$ find delaunay triangulation from $p$ and $q$;
        for all edges of all triangles of $d$ do
          record the number of times they occur;
        endfor
        neighbours $\leftarrow$ vertices of edges which occur only once;
        for all $f$'s do
          order all their vertices;
        end
    endif
endfor                                                                  □
```

The terms *vertices* and *edges* refer to the Voronoi tessellation only. The vertices and edges of the Delaunay tessellation are cells and bonds of the VT. The programme `perco3d.m` can find p_c for all of these. For each one of them we must find the neighbourhood matrix as well as lists of the upper- and the lower boundaries. The order of blockages is completely decided in advance by finding a permuted list of numbered items. The programme finds the history of clusters for the whole range of p, *i.e.* from 0 to 1.

To obtain the matrix of cell neighbours, first crop out between 5 and 10 per cent of the boundary in all directions. Then the neighbour matrix is found from the DT matrix.

The bond neighbour matrix is simply the rearrangement of the cell neighbour matrix obtained. We have already found the vertices neighbour matrix earlier while searching for vertices shared between cells. This is rearranged for the use in the percolation programme, and then again for the edge neighbour matrix.

Next is Algorithm 4.4 which carries out the percolation simulation. The cluster information is swapped alternately between three variables, A, B and *tmp*, to facilitate the flow; $A(i)$ is the i^{th} cluster in A. All variables are assumed to be cleared at the start of the algorithm. At the end of each run the value of p_c is computed, and the list p is reversed and reused for a second time if it was the first run.

Algorithm 4.4 *Voronoi percolation in three dimensions.*

```
for each permuted item $p(i)$ in the list do
    joined $\leftarrow 0$;
    for each cluster $A(j)$ in $A$ do
        if A(j) contains p(i) then
            $A(j) \leftarrow A(j) \cap p(i)$;
            $B(1) \leftarrow A(j)$;
            $tmp \leftarrow A$;
            $A \leftarrow tmp - A(j)$;
            for all $A(k)$ in $A$ do
                if $A(k) \wedge B(1)$ then
                    $B(1) \leftarrow A(k) \cap B(1)$;
                else
                    $B(++n_B) \leftarrow A(k)$;
            endif
          endfor
          percolated ?;
          $A \leftarrow B$;
          break;
        endif
      endfor
      if joined then
        $A(++n_A) \leftarrow p(i)$;
      endif
    endfor                                                                □
```

The resulted Pc probabilities from simulation are ${}^{Vn\;3d}_{12(\mathbf{2})}p_c = 0.2340 \pm 0.0448$, ${}^{Vn\;3d}_{2(\mathbf{2}),10(\mathbf{3})}p_b = 0.1178 \pm 0.0271$, ${}^{Vn\;3d}_{2(\mathbf{2}),10(\mathbf{3})}p_v = 0.2941 \pm 0.0831$ and ${}^{Vn\;3d}_{12(\mathbf{3})}p_e = 0.4311 \pm 0.0324$.

And the coordination numbers are ${}^{Vn\;3d}_{6(\mathbf{2})}x_c = 11.1231 \pm 0.3329$, ${}^{Vn\;3d}_{1(\mathbf{2}),5(\mathbf{3})}x_b = 23.0548 \pm 0.5996$, ${}^{Vn\;3d}_{1(\mathbf{2}),5(\mathbf{3})}x_v = 3.6926 \pm 0.0245$ and ${}^{Vn\;3d}_{6(\mathbf{3})}x_e = 5.5002 \pm 0.0432$.

Four different types of percolation have been performed, namely cell, bond, vertice and edge percolation. Vertices are inherently zero dimension, edges and bonds one dimension and cells, which in reality have three dimensions, is considered for this purpose as having none. When considering vertice- and cell percolation, the position of the vertices and nuclei determines whether they could connect somewhere to somewhere else. When the object the percolation of which we consider has a nonzero dimension and the dimension of the network is small, that is to say, in the same order as that of itself, p_c obtained will depend on the network dimension. In other words, there will be an influence from the size of the objects on, and distort, the space in which the percolation occurs.

Additional results on a larger network are, the Pc probabilities from simulation are ${}^{Vn\;3d}_{20(\mathbf{2})}p_c = 0.2039 \pm 0.0410$, ${}^{Vn\;3d}_{20(\mathbf{3})}p_b = 0.0963 \pm 0.0114$, ${}^{Vn\;3d}_{20(\mathbf{3})}p_v = 0.1659 \pm 0.0571$ and ${}^{Vn\;3d}_{28(\mathbf{3})}p_e = 0.4172 \pm 0.0242$, and the coordination numbers are ${}^{Vn\;3d}_{1(\mathbf{3})}x_c = 12.4177$, ${}^{Vn\;3d}_{,1(\mathbf{3})}x_b = 25.7077$, ${}^{Vn\;3d}_{1(\mathbf{3})}x_v = 3.8126$ and ${}^{Vn\;3d}_{1(\mathbf{3})}x_e = 5.7050$. In this case the network was originally built from 1,000 Poisson point generators, and for the simulation we have $n_c = 723$, $n_b = 4{,}489$, $n_v = 4{,}855$ and $n_e = 9{,}255$.

The algorithm used in the programme that simulates and gives the results above greatly differ from what was used by Tiyapan (1995, KNT3(iii).) We may call this KT Algorithm, and the other KNT Algorithm. Both of them exploit sparse matrices implemented on Matlab, but the latter, that is to say the KNT Algorithm, keeps all the results from every time instance. It could do this only by using sparse matrices of structures, for example a matrix within a matrix within a matrix.

§ 4.4 Percolation of 2-dimensional Voronoi sections

The programme in § A.4 does a 2-d section of the 3-d VT. Algorithm 4.5 describes how it works. It assumes that the Voronoi tessellation already exists. Here d_m means a denominator while n_r a numerator, V and C means vertices and cells of the 3-d Voronoi tessellation while v and c those of the 2-d section. In particular, $c \in C$

Algorithm 4.5 *Plane section of Voronoi in three dimensions*

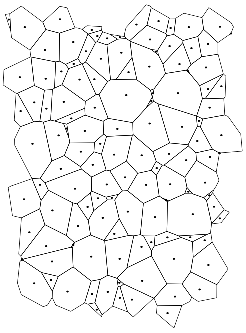
```
$(\Delta x, \Delta y, \Delta z)_i \leftarrow (x_2 - x_1, y_2 - y_1, z_2 - z_1)_i$ for all edges $i$;
for all edges $i$ do
  $d_m \leftarrow a\Delta x + b\Delta y + c\Delta z$;
  if $d_m$ nonzero then
    $n_r \leftarrow ax_1 + by_1 + cz_1$;
    $t \leftarrow -n_r/d_m$;
    if $0 \le t \le 1$ then
      $(x, y, z) \leftarrow (x_1 + \Delta x, y_1 + \Delta y, z_1 + \Delta z)_i$;
      $\{v_s\} \leftarrow (x, y, z)$;
    endif
  else
    $a \leftarrow a + \epsilon$;
    $d_m \leftarrow a\Delta x + b\Delta y + c\Delta z$;
    $n_r \leftarrow ax_1 + by_1 + cz_1$;
    $t \leftarrow -n_r/d_m$;
```

```
        if $0 \le t \le 1$ then
            $(x, y, z) \leftarrow (x_1 + \Delta x, y_1 + \Delta y, z_1 + \Delta z)_i$;
            $\{v_s\} \leftarrow (x, y, z)$;
        endif
        if $n_r = 0$ then
            $\{v_s\} \leftarrow (x_1, y_1, z_1)$;
            $\{v_s\} \leftarrow (x_2, y_2, z_2)$;
        endif
    endif
endfor
for all $c_s$ do
    find the Delaunay triangulation;
    count $n_e$ of all the triangles, add the numbers into a single list;
endfor                                                              □
```

The intersection of a plane $ax + by + cz + d = 0$ by the line which is defined by $x = x_1 + \Delta xt$, $y = y_1 + \Delta yt$ and $z = z_1 + \Delta zt$, where Δx, Δy and Δz are respectively $(x_2 - x_1)$, $(y_2 - y_1)$ and $(z_2 - z_1)$, is determined by $t = -(ax_1 + by_1 + cz_1 + d)/(ai\Delta x + b\Delta y + c\Delta z)$. If the denominator is zero the line is parallel to the plane, and if the nominator is also zero contained therein.

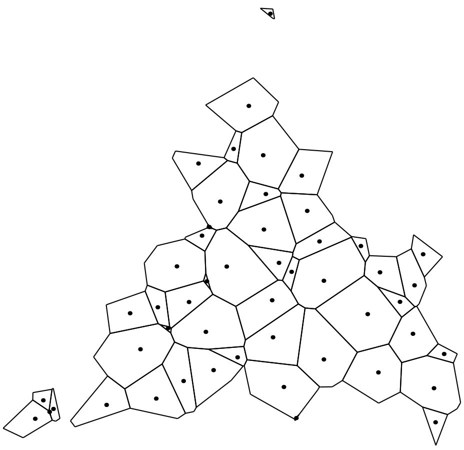
The results from the percolation simulation on the section $\mathcal{V}_3^2$ are ${}^{Vn\ (2,3)s}_{18(1),4(2)}p_c = 0.5494 \pm 0.1223$, ${}^{Vn\ (2,3)s}_{10(1),10(2)}p_b = 0.3515 \pm 0.0764$, ${}^{Vn\ (2,3)s}_{8(1),12(2)}p_v = 0.7557 \pm 0.0757$, ${}^{Vn\ (2,3)s}_{20(2)}p_e = 0.6210 \pm 0.0665$.

The coordination numbers are ${}^{Vn\ (2,3)s}_{8(1),1(2)}x_c = 4.6894 \pm 0.2212$, ${}^{Vn\ (2,3)s}_{5(1),4(2)}x_b = 8.8100 \pm 0.6021$, ${}^{Vn\ (2,3)s}_{4(1),5(2)}x_v = 2.7495 \pm 0.0387$, ${}^{Vn\ (2,3)s}_{9(2)}x_e = 3.6691 \pm 0.1118$.



When sectioned by the plane $(a, b, c, d) = (0.01,\ 0.5,\ 0.5,\ -0.5)$, our VT gives a picture as shown in Figure 4.2. Here the points shown are merely the average of the coordinates of all vertices in each cell. From ten simulations of these 118 cells, 300 bonds, 288 vertices and 405 edges having respectively the coordinate numbers of 5.0847, 9.7933, 2.8125 and 3.7333, we obtain $p_c = 0.5220 \pm 0.0966$, $p_b = 0.3107 \pm 0.0391$, $p_v = 0.7014 \pm 0.0573$ and $p_e = 0.6259 \pm 0.0603$.

Figure 4.2 Section by the plane (0.01, 0.5, 0.5, -0.5).



When sectioned by the plane $(a, b, c, d) = (0.5,\ -0.5,\ 0.5,\ 0.01)$, our VT gives a picture as shown in Figure 4.3. Here the points shown are merely the average of the coordinates of all vertices in each cell. From fourteen simulations of these 50 cells, 104 bonds, 140 vertices and 187 edges having respectively the coordinate numbers of 4.1600, 8.2500, 2.6714 and 3.5080, we obtain $p_c = 0.6914 \pm 0.1424$, $p_b = 0.4533 \pm 0.1108$, $p_v = 0.7760 \pm 0.0821$ and $p_e = 0.6929 \pm 0.0897$.

Figure 4.3 Section by the plane $(0.5, -0.5, 0.5, 0.01)$.

When sectioned by the plane $(a, b, c, d) = (-0.7,\ -0.3,\ 1,\ 0.01)$, our VT gives a picture as shown in Figure 4.4. Here the points shown are merely the average of the coordinates of all vertices in each cell. From twenty simulations of these 121 cells, 305 bonds, 291 vertices and 411 edges having respectively the coordinate numbers of 5.0413, 9.4426, 2.8247 and 3.7518, we obtain $p_c = 0.5413 \pm 0.1107$, $p_b = 0.3584 \pm 0.0494$, $p_v = 0.7077 \pm 0.0572$ and $p_e = 0.6749 \pm 0.0470$.

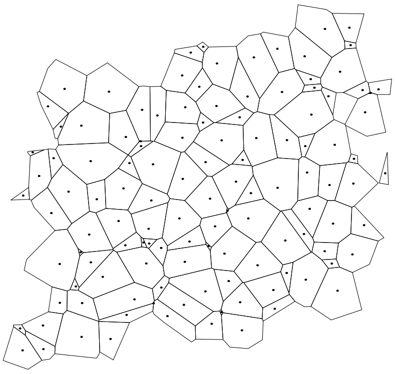


Figure 4.4 Section by the plane (−0.7, −0.3, 1, 0.01).

For the purpose of doing these simulations, the code for finding percolation has been rewritten into a function. To my surprise and delight, the same function works for both the 3–d VT and its 2–d section. In choosing a sectioning plane it is better if we choose the parameter d small, as the plane will then pass close to the origin. Also, choosing $a+b+c \approx 0$ seems to make a more wholesome section than otherwise. The codes for sectioning work well for oblique planes but do not like planes which are parallel to an axis. This shortcoming can be avoided if we make our plane only nearly parallel, when we want it to be parallel to an axis. Then to be able to view in a head on fashion such planes which have been plotted in three dimensions, we can look from the position (a, b, c), which is in effect the vector normal to the plane. The function mentioned above is listed as `perc.m` and is in § A.26.

As above mentioned, the programme used here in finding the percolation of the 2-d section of a 3-d Voronoi tessellation has been developed from the 3-d percolation programme. Therefore it uses a different algorithm from what has been used earlier by Tiyapan (1995, KNT3(iii)) which is only limited to two dimensions. Also the algorithm finds not only the cell- and edge Pc's, as is the case with the other algorithm, but also the bond- and vertice Pc's. And it keeps records of all the history of clusters in such a way that the results can be used again later for more extensive studies. If the same algorithm can be used for both 2-d and 3-d cases, it is hoped that it will work for those cases where the dimensionality exceeds three. This could be very useful for the study of percolation in economic models (*cf* Tiyapan, 1997, KNT5(viii).)

A 2-d section of a 3-d VT is not a Voronoi tessellation. While biological structures are often a 2-d VT, for example cells organise themselves like a honeycomb because they grow in layers, most inorganic material growing in three dimensions form a 3-d VT. One example of the usefulness of the study done in this section, then, is in those cases where we make a slice of, say, a piece of polycrystalline material, and then want to study what would happen if we pass electric current throught it. One way of doing this on the computer is an approach similar to what we have done, that is first create a 3-d VT, find a 2-d section of it, then represent this as a networks of resistors, and then simulate it as an electrical circuit.

Also when we pass liquid through a block of material, the flow is normally directional and its direction usually remains the same throughout. Often our block of material has a flat surface that lies at right angle to the direction of the flow. Then the blockage within the material is seldom homogeneous but varies with the distance along the flow direction. Moreover, if the sizes of the solid particles the liquid contains are large the particles may never enter the material but instead remain on its surface in a form of cake. If this cake is unwanted, we may need to wipe it off or do a back-flushing. The best time to efficiently do this is just before a percolation occurs on the surface, that is on a plane section of a 3-d Voronoi tessellation.

Not only is this the case when particle sizes are large, but it is also the case where the tiny suspended particles have sizes orders smaller than the pores. This is the case of percolation within percolation, that is to say, a continuum percolation within network percolation where tiny particles first form clusters in free space, then these clusters percolate to block the pore, and then the blocked pores cluster together to percolate the network. The first percolation in the equation here, that is the continuum percolation, depends on the solid concentration and the rate of flow of the colloid. This concentration, as well as the flow rate, may vary with the distance the liquid has travelled into the filter, creating thus perpendicularly to the flow plane sections of various percolation probabilities. Then again our percolation within percolation may not be a case of three- but of two dimensions.

According to this, the percolational studies of 2-d section of a 3-d VT may be even more important to our filtering membrane application than a similar study for the 3-d VT itself. Yet the three-dimensional percolation studies done are still necessary, because even if a plane section facing the flow has percolated first, the structure itself in this case remains unpercolated and may still be of some use until this is no longer true. On the other hand the investigation still remains to find out which of these two occurs first, that is whether a plane section percolate first, or the 3-d structure. This means that for the 3-d Voronoi tessellation both our studies of its 2-d and 3-d percolations (respectively § 4.4 and § 4.3) are directly relevant and important to the application of filtering membranes. The study of percolation of 2-d VT (§ 4.2) is important for the application of biological structures. The 3-d Voronoi Percolation results are also useful for the study of galaxy formation and evolution.

§ 4.5 Network percolation

In the study of networks an important parameter is the coordination number, which is the number of neighbours of an element which in the graph theory is usually the vertex. Each vertex or site of a graph is connected to each of its neighbouring vertices by a bond, so the coordination number of a graph is the number of bonds connected to a vertex.

Clusters and their various characteristics play an important role in the study of percolation of networks. With a material science application in mind, Levy *et al* (1982) numerically represent the shape of a cluster by the *shape parameter* S, defined as $S = b/N$ or more generally $S = (1/2)\sum_{i=1}^{z} i\nu_i / \sum_{i=1}^{z} \nu_i$ where b or i is the number of bonds and N or ν_i the number of elements in a cluster having b or i bonds respectively.

§ 4.6 Percolation statistics in literature

Rushbrooke and Morgan (1961) studied the Heisenberg and Ising ferromagnetics and found that the critical concentration p_c is the same for both Heisenberg and Ising problems. Moreover, this value is irrelevant to the magnitude of the spin s. By studying point-clusters and link-diagrams, they found approximations to the exact values of the percolation probabilities for, the face-centred cubic lattice $p_c = 0.18$, the body-centred cubic lattice $p_c = 0.22$, tthe simple cubic lattice $p_c = 0.28$, and the plane square $p_c = 0.48$. Only vertices of their point-clusters, not link-diagrams, need to have a bond between them in order to be neighbours.

Frisch *et al* (1961) give the site critical probabilities for various types of lattices in two- and three dimensions. In the terminology used here, these are p_c of vertices of the networks considered. Their results are shown in Table 4.4.

lattice	*dimension*	z	p_c *(N=1,000)*	p_c *(N=2,000)*
triangular	2	6	0.487 ± 0.021	0.493 ± 0.018
square	2	4	0.575 ± 0.017	0.581 ± 0.015
hexagonal	2	3	0.683 ± 0.020	0.688 ± 0.017
h.c.p.	3	12	0.204 ± 0.008	–
f.c.c.	3	12	0.199 ± 0.008	–
simple cubic	3	6	0.325 ± 0.023	–
tetrahedral	3	4	0.434 ± 0.013	0.436 ± 0.012
ice (quartz)	3	4	0.432 ± 0.013	0.433 ± 0.011

Table 4.4 *Critical probabilities from Frisch et al (1961).*

From their investigations they also realise that the critical probabilities of two homohedral tilings may not be the same. For instance, a plane lattice with $z = 3$ may have the vertex- and edge percolation probabilities different from those of the hexagonal lattice, even though the coordination number is the same for both.

Both Monte Carlo technique and series expansion method are means by which one can numerically study percolation problems. Dean (1963) used Monte Carlo method because, according to him, it provides more precise information. In a network of N sites pN of which are occupied, the probability of site occupation at the next time step becomes $q = p + 1/N$. The critical probabilities he found are shown in Table 4.5.

		lattice size		
	12 × 12	24 × 24	48 × 48	
lattice type	p_c ± s.d.	p_c ± s.d.	p_c ± s.d.	p_c ± s.d., (lattice size)
square (s)	0.507 ± 0.090	0.582 ± 0.032	0.580 ± 0.018	0.569, (78 × 78)
triangular (s)	0.435 ± 0.029	0.494 ± 0.037	0.486 ± 0.017	0.486, (84 × 84)
square (1st and 2nd) (s)	0.322 ± 0.047	0.381 ± 0.029	0.387 ± 0.014	0.401, (90 × 90)
honeycomb (s)	0.641 ± 0.061	0.675 ± 0.027	0.688 ± 0.015	0.679, ($\frac{3}{4} \times 84 \times 84$)
kagome (s)	0.609 ± 0.047	0.643 ± 0.028	0.635 ± 0.020	0.655, (72 × 72)
four-eight (s)	0.679 ± 0.039	0.718 ± 0.023	0.732 ± 0.015	0.739, (84 × 84)
square (b)	0.468 ± 0.049	0.469 ± 0.028	0.492 ± 0.011	0.498, ($\frac{3}{4} \times 96 \times 96$)
triangular (b)	0.279 ± 0.038	0.324 ± 0.046	0.329 ± 0.021	0.349
kagome (b)	0.419 ± 0.058	0.432 ± 0.045	0.449 ± 0.032	0.435, (84 × 84)
four-eight (b)	0.615 ± 0.050	0.649 ± 0.028	0.675 ± 0.027	0.661, ($\frac{3}{4} \times 78 \times 78$)

Table 4.5 *Critical probabilities and the standard deviation (s.d.), $p_c \pm$ s.d., for sites (s) and bonds (b), from Dean (1963).*

The effect of shape of arrays found by him is such that for array sizes of 6 × 96, 12 × 48, and 24 × 24 the percolation probabilities are respectively 0.707 ± 0.043, 0.594 ± 0.039, and 0.568 ± 0.032. The *modified second moment* of the cluster size distribution for a lattice has been defined as $\mu_2^* = \sum \sigma_i^2 / (\sum \sigma_i)^2$, where σ_i is the size of the i^{th}-cluster. And the percolation probability for the finite lattice has been defined to be the value of p at which $\Delta\mu_2^* / \Delta p$ is maximum, Δ being an increment of one step.

Tiyapan (1995, KNT3(iii)) finds from 27 simulations on 2-d Voronoi networks of between 70 and 500 cells the percolation probability of cells $p_c = 0.507$, and from 75 simulations on 2-d Voronoi networks of between 100 and 8,600 bonds p_c =0.658. For honey comb lattices between 100 and 3,000 bonds he finds the bond percolation probability $p_c = 0.640$, for Kagome lattices between 300 and 3,000 bonds $p_c = 0.517$, for square lattices between 200 and 2,000 bonds $p_c = 0.467$, and for triangular lattices between 300 and 3,000 bonds $p_c = 0.341$. These results from simulations are usually less than the exact values by a few per cent.

Mecke and Seyfried (2002) find pseudocritical threshold for different types of lattices and then extrapolate these by using finite-size scaling laws to obtain the critical probability for the infinite network. They use the Hoshen-Kopelman method to find the largest cluster of the system and then determine the percolation in fifteen steps by increasing or decreasing p by $\frac{\Delta p}{2}$ depending on whether the system percolates or not. Here $\Delta p = p_2 - p_1$, $p_2 > p_1$, such that the system only percolates at p_2 not p_1. In two dimensions the network percolates when two opposite sides connect with each other whereas in three dimensions the same is true when four opposite sides connect together. The pseudocritical percolation threshold is $p_c(L)$, where L is the network size. The finite-size scaling theorem is $p_c(L) = p_c(\infty) + aL^b$. In two dimensions they find p_c =0.59278(4), while in three dimensions $p_c = 0.31162(8)$.

In its early days the percolation theory concerned itself much with self-avoiding walks. The connective constant or the walk limit is defined as a measure of the connectivity of the lattice as $\ln \mu = \lim_{n \to \infty} (1/n) \ln c_n$, or sometimes as $\kappa = \ln \mu$. Shante and Kirkpatrick (1971) gives results from both the Monte Carlo and series method as shown in Table 4.6.

	z	μ	*Monte Carlo*		*series method*	
			p_e	p_v	p_e	p_v
honeycomb	3	1.8484	0.640	0.688, 0.679	0.6527	0.700
kagome	4		0.435	0.655		0.6527
square	4	2.6390	0.493, 0.489	0.581, 0.569	0.5000	0.590
triangular	6	4.1515	0.341, 0.349	0.493, 0.486	0.3473	0.5000
diamond	4	2.878	0.390	0.436	0.388	0.425
s.c.	6	4.6826	0.254	0.325	0.247	0.307
b.c.c.	8	6.5288			0.178	0.243
f.c.c.	12	10.0350	0.125	0.199	0.119	0.195
h.c.p.	12		0.124	0.204		

Table 4.6 *Critical probabilities and connective constants, Shante and Kirkpatrick (1971), where s.c. means simple cubic, b.c.c. body centred cubic, f.c.c. face centred cubic and h.c.p. hexagonal close packing.For critical probabilities, all the values given in four decimal places are exact where as those given in three decimal places are numerical.*

§ 4.7 Percolation of n-gons in continuum

Consider the continuum percolation of regular polygons in two dimensions. Polygons are placed on the plane randomly with regards to both their position and orientation. From their position one can find the Delaunay triangulation. Using the bonds of the triangulation obtained as reference axes, one for each pair of polygons, one then finds the orientation of any two neighbouring polygons with reference with the axis connecting them.

Unlike in the case of Voronoi percolation where the size of each cell depends on the density of the generating points, in continuum percolation the size of the polygons has been decided in advance. The expanse of the region of interest is then determined relative to this size. The area being simulated can then be imagined as lying within an infinite plane where the number density of polygons is homogeneous.

The area of the n-gon is $A = n \cdot 2 \cdot (1/2)yz$. But $y = R\sin\theta$ and $z = R\cos\theta$, where R is the radius of the circumscribing circle. Therefore we have $R = (A/(n\sin\theta\cos\theta))^{1/2}$. The radius of the inscribed circle is $r = z$.

A list is made of every angle that the rays perpendicular to the sides of the polygons make with the x-axis in the counterclockwise direction. Then the key procedure of the algorithm is to find for each pair the ray which lies closest to the reference line joining the two polygons. The angles that these two rays make with the reference axis determines whether or not the two polygons intersect.

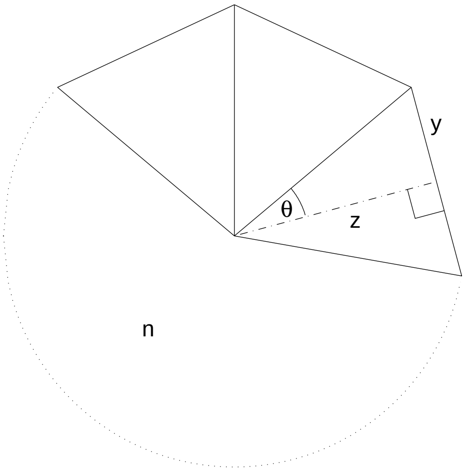


Figure 4.5

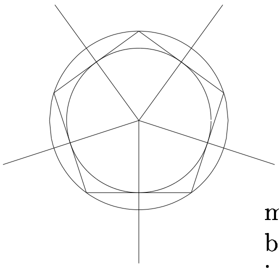


Figure 4.6

The inscribed circle and the circumscribing circle both play a role in deciding whether two polygons form a cluster. Let the radius of the former be represented by r and the radius of the latter R. If any two polygons are less than $2r$ apart, then they must overlap. They just touch if they are exactly $2r$ apart. And if the distance between them is more than $2R$, then they can never touch each other. When the distance lies between $2r$ and $2R$ there is a probability that they will touch. Whether this is the case or not depends on the orientation of the two polygons.

Among all the rays similar to those shown in Figure 4.6 one can find the one closest to the line between the centres of the two polygons, ie θ_1 and θ_2 of Figure 4.8 are a pair of minimum angles for c_1 and c_2. Let the centres of these polygons be c_1 and c_2, and the line joining them $\overline{c_1c_2}$. Two such lines obtained from the two polygons can fall into on of the two cases; they can either be on the same side of $\overline{c_1c_2}$, or they are on the opposite side of each other as shown in Figure 4.8 (a) and (b) respectively. In either case, the following theorem is true.

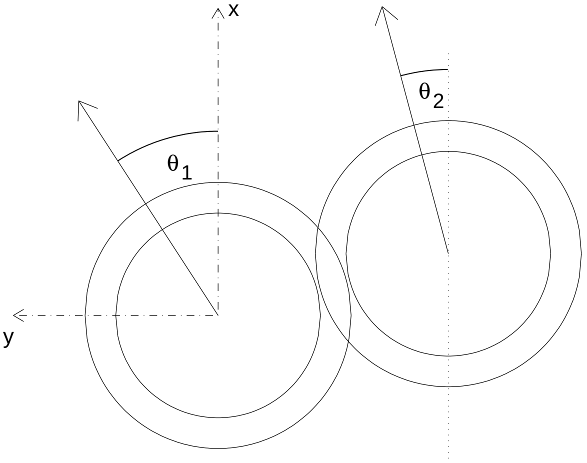


Figure 4.7

Theorem 4.1. *The line segments* $\overline{a_1a_2}$ *and* $\overline{b_1b_2}$ *always intersect* $\overline{c_1c_2}$.

Proof. Without losing generality, consider only the case of $\overline{a_1a_2}$. Suppose the above statement is false and $\overline{a_1a_2}$ did not intersect $\overline{c_1c_2}$. Then, because $\overline{a_1a_2}$ is a part of the circumference of a polygon, there would always be another ray d_k originating from c_1 which has $\overline{a_ia_j}$ connected to it, such that

$\overline{a_i a_j}$ intersects $\overline{c_1 c_2}$, i, j and k being positive intergers. But the two triangles $\triangle c_1 a_1 a_2$ and $\triangle c_1 a_i a_j$ are identical. Therefore the angle θ_k between $\overline{a_i a_j}$ and $\overline{c_1 c_2}$ would be smaller than θ_1 and we replace d_1 with d_k because the latter is closer to $\overline{c_1 c_2}$. □

Furthermore, between θ_1 and θ_2 the one which is greater would belong to an *aggressive* cell (see Definition 4.1 below,) while the other one is in a *passive* state. Suppose that $\theta_1 > \theta_2$, then c_1 would be the one which does the touching first. Theorem 4.2 states this in a more formal manner.

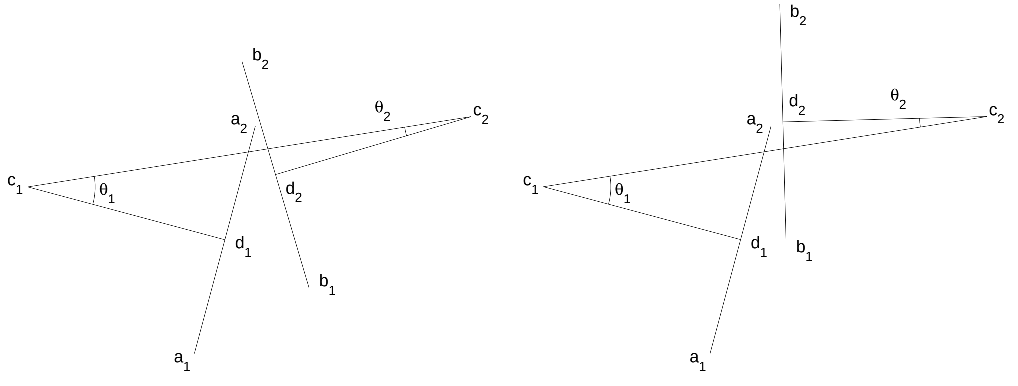
Definition 4.1. *A cell is aggressive if a part of it is closer to the other cell's centre of gravity than any part of that cell is from its own c.g.*

In the n-gons that we are interested in, the cell's c.g. is the same point as its nucleus and its geometrical centre. If how close one cell get to another is measured by the minimum distance between points on its surface and the c.g. of that cell, then Definition 4.1 says that among the two cells involved, the cell which gets closer to the other is aggressive. And since we say that it is aggressive, we also say that it causes the intersection (compare this with manuals on fencing.) We say touching is an aggression, and thereby the aggressive party contributes more to the touch. Imagining that polygons revolve helps understanding, but it is nothing more than an aid toward that purpose. Details regarding the revolution, for example the direction and speed, are irrelevant, since the spin is only imagined. Moreover, we only do calculation for each time step, which means that the trajectory of the polygons is not considered.

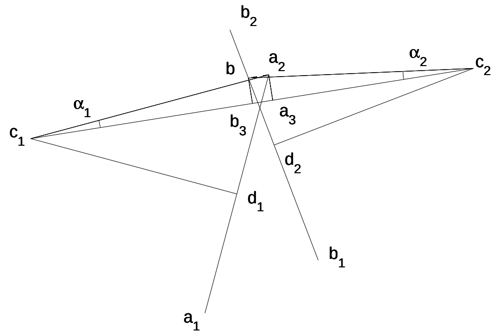
Theorem 4.2. *If the distance between any two identical regular polygons allows them a certain probability p of overlap, such that $0 \leq p \leq 1$, then the polygon whose ray is furthest from the line connecting their centres will be the one which causes the two to intersect.*

Proof. Let c_1 and c_2 be the two identical polygons in question. As mentioned earlier, there are two cases to be considered, which are represented by Figure 4.8 (a) and (b). Because the distance between them is more than the diameter of the inscribed circle, when $\theta_1 = \theta_2 = 0$ the two polygons do not touch each other. The two polygons being identical, it is suffice to consider only one of them. But because there is an overlap area between the two as shown in Figure 4.7 where there is a probability of colliding to occur, and as the distance from a centre c_1 increases as one goes from a midpoint d_1 of an edge to the next vertex d_2 closest to $\overline{c_1 c_2}$, the greater θ_1 is, the further away from c_1 is the point of intersection between $\overline{d_1 b_2}$ and $\overline{c_1 c_2}$, the more that polygon overlaps into the collision hazard zone mentioned above, and therefore the more contribution to the probability of the collision. □

Alternative proof. As shown in Figure 4.8 (c), with $\theta_1 > \theta_2$ it is the corner a_2 which collides with the flat of the side $b_1 b_2$ and penetrates into the second polygon.



(a) (b)



Imagine the polygons revolving around until they collide. At the point of collision if their vertices touch, then $\theta_1 = \theta_2$ and it is a special case where both contribute equally to the touch. The case of Figure 4.8 (b) where the two angles lie on the opposite side of each other follows a similar line of reasoning. □

In Figure 4.8 The distance between c_1 and c_2 is less than the diameter of the circumscribing circle but more than the diameter of the inscribed circle.

(c)

Figure 4.8 When two n-gons practise the Ayudhya sword together.

Figure 4.8 looks as though two n-gons were practising with each other the Ȧyudhyạ *d́ạḅ* (sword).

If $\theta_1 > \theta_2$, then $d\left(\overline{c_1 a_3}\right) > d\left(\overline{c_2 b_3}\right)$. Furthermore, a touch or penetration implies that $d\left(\overline{c_1 a_3}\right) + d\left(\overline{c_2 b}\right) = d\left(\overline{c_1 c_2}\right)$. If $d\left(\overline{c_1 a_3}\right) + d\left(\overline{c_2 b}\right) < d\left(\overline{c_1 c_2}\right)$ then the two polygons do not touch each other. The case when they merely touch at vertices is the case where $d\left(\overline{c_2 b}\right) = d\left(\overline{c_2 b_3}\right)$. On the other hand, the condition $d\left(\overline{c_1 a_3}\right) + d\left(\overline{c_2 b_3}\right) \geq d\left(\overline{c_1 c_2}\right)$ does not say anything much because a_2 may just pass under b_2 without touching the side in Figure 4.8 (a), or it may be the case where the two opposing sides lean towards the same direction as in Figure 4.8 (b). Therefore, we now have the following theorem which is later used in Algorithm 4.6 for finding overlaps.

Theorem 4.3. *Any two polygons p_1 and p_2 just touch each other if and only if $d\left(\overline{c_1 a_3}\right) + d\left(\overline{c_2 b}\right) = d\left(\overline{c_1 c_2}\right)$. If $d\left(\overline{c_1 a_3}\right) + d\left(\overline{c_2 b}\right) > d\left(\overline{c_1 c_2}\right)$, then they overlap.*

Proof. See explanation above together with Figure 4.8. □

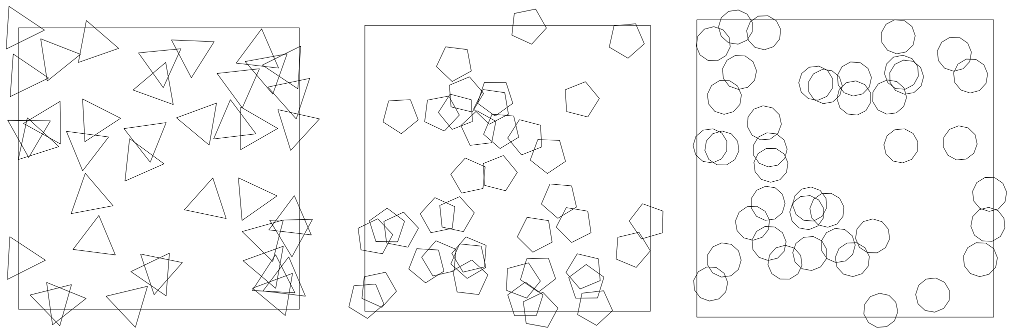
Figure 4.8 (c) may require some more explanation. The diagram has all the labels carrying a numeral subscript, since this is suitable for programming. For explanation purpose, however, it may become easier to understand if we label in place of c_1 and c_2, respectively c_a and c_b. Then c_a is the centre of the n-gon $\{a\}_n$, that is to say, an n-gon having a_i as its vertices where $i = 1$ to n in that order of consecutiveness of vertices. In fact this has always been the case on paper before the algorithm was put into code. Then after that Figure 4.8 was made in such a way as to keep accord with that programme. The corner a_2 is defined in such a way as is clearly seen in the figure, that is the corner of a ray with θ_1 where $\theta_1 > \theta_2$. The point a_2 is projected perpendicularly onto $\overline{c_1 c_2}$ at a_3. It is next projected parallel to $\overline{c_1 c_2}$ onto $\overline{b_1 b_2}$, and then from there perpendicularly onto $\overline{c_1 c_2}$ at b. Never used is b_3 in the picture, which is the projection of b_2 perpendicularly onto $\overline{c_1 c_2}$.

Theorem 4.3 is nicknamed here *D́ạḅ* Algorithm ‘d́ạḅ’ being the word for sword in Thai written using the Sanskrit-Pinyin-Tiyapan (SPT) one-to-one mapping system (Tiyapan, 2003, KNT8(iv)), and ‘Ȧyudhyạ’ mentioned in § 4.7 is both a capital of Siam and the place where the ancestor of the author came from. This name came after it has been realised that the problem is best reduced to the interaction between two pairs of ray and side. Before this was finally understood, however, the author first dreamt of spinning polygons moving in space. He thought it was wonderful; and the names originally given to the thing was ‘spinning wheels’, ‘spinning stars’, and ‘dancing stars.’ It is hoped that in the future it will be possible to do a similar thing using random polygons, and then randomly-shaped objects not necessarily convex.

Algorithm 4.6 *Find continuum percolation of regular polygons on a plane.*

```
place polygons randomly;
find radii of their inscribed circles and the circumscribing circles;
find coordinates of the vertices;
find Delaunay triangulation;
for $i = 1$ to number of triangles do
  for $j = 1$ to 3 do
    find distance between the $j^{th}$-pair vertices of the $i^{th}$ triangle;
    if this distance $\leq 2r$ then polygons overlap do
      record cells overlap;
```

```
        elseif $distance \leq 2R$ then polygons could overlap do
            record cells to find whether overlap;
        endif
    endfor
    find angle of each of the two polygons when viewed from the other
endfor
for every side of every polygon do
    find angles of the normal ray counterclockwise from the positive $x$-axis;
endfor
for every cell which could still overlap do
    find normal rays closest to the lines going towards its neighbours;
endfor
complete the overlap check for all remaining cells;
check percolation;                                                        □
```



(a) (b) (c)

Figure 4.9 **(a)** Randomly placed *triangles*. Here the programme tells us that the largest cluster has seven triangles, and the next largest one has six. There are 8 isolated triangles, 5 clusters of two, and three clusters of three. According to the programme, the two clusters of sizes six and two on the upper right corner do not quite touch. **(b)** The largest cluster in this case has got nine *pentagons*. **(c)** These 11-*gons* are also too sparsely placed to percolate. The largest cluster in this case has got 8 members. The number of all 11-gons is 40, the same as in the two previous cases, and six of which are isolated. There are one cluster of four, four clusters of three, and five clusters of two.

The programme in § A.5 uses the theorems above to find whether an aggregate of polygons percolate or not. Here each polygon has a unit area. The area of an n-gon is $nr^2 \cos(\theta/2) \sin(\theta/2)$, where r is the radius of its circumscribed circle. By setting this area to one we can find r, which, together with the decided orientation of each polygon gives the coordinates of its vertices. The programme is explained in Algorithm 4.6.

A percolation threshold normally means the critical percentage by number. But in the case of percolation in a continuum where there is no fixed amount of lattice sites to refer to, the per cent area covered may be a better candidate for p_c. Finding this area can become quite computationally intensive it is a problem of finding the union among sets which, for the case of three sets is $A \cup B \cup C = A + B + C - AB - AC + ABC$, for four sets $A \cup B \cup C \cup D = A + B + C + D - AB - AC - AD - BC - BD - CD + ABC + ABD + ACD + BCD - ABCD$, *etc.* For the general case where there are n sets intersecting one another, then, $\bigcup_n A_i = \sum_n Ai - \sum_{(n,2)} A_{ij} + \sum_{(n,3)} A_{ijk} - \cdots + \sum_n A_{i^{(n-1)}} + (-1)^{(n+1)} A_{i^{(n)}} = \sum_k^n (-1)^{(k+1)} A_{i^{(n)}}$, where $A_{ijk\dots}$ means $A_i \cap A_j \cap A_k \cap \cdots$, the subscripts (n,k) is the combination ${}^n\mathrm{C}_k = n!/\left[k!(n-k)!\right]$ and $i^{(n)} = ijk\cdots$ up to the n^{th} term.

Perhaps a better explanation for this section is needed. Then it must be this. First each n-gon may be represented by two circles, namely its circumscribing- and its inscribed circles. If two n-gons lie further apart from each other than the diameter of the former, they could never touch. But if they are less than a diameter of the latter apart, they definitely intersect. Ambiguities only come in when both are less than the former but more than the latter apart from each other. In such cases one must consider their mutual relative orientation. The location of an n-gon being the position of its mid-point, one may draw n lines starting from the mid-point and passing through the mid-point of each edge. For each pair of polygons fallen into this situation their mutual reference line is the line linking between their centres. For each of these two polygons determine a ray which makes the smallest angle with their reference line. Compare these two angles with each other. The smaller one of them implies that that corresponding ray is in an attacking position while the other one is in a defending one. One corner of the edge belonging to the attacking ray enters the other n-gon if and only if the two polygons intersect. An edge having two corners, such corner which is the only one in this case that could enter the other n-gon is the one that is closest to the reference line. This corner is important, because once we have identified it, projecting it parallel to the reference line onto the other edge, and then from there perpendicularly onto the reference line gives us our first point. Project the same corner again, this time directly and perpendicularly onto the reference line, gives us our second point. Both our first- and second points are on the reference line. Then these n-gons either touch or overlap, if and only if the distance from the first point to the other centre plus the distance from the second point to this centre amount to no less than the distance between the two centres themselves. This proof came well after the method has been found intuitively based on sword-fighting skill.

§ 4.8 Polygon percolation threshold

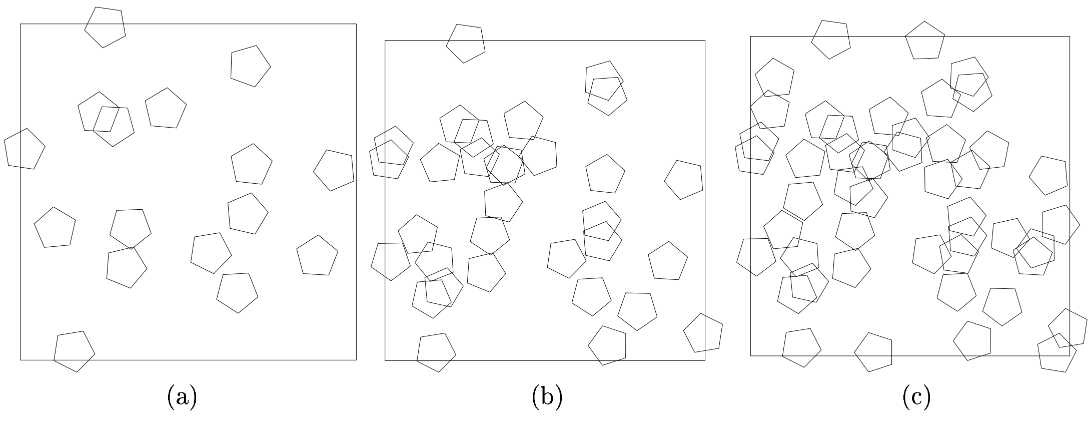
Because, unlike lattices however random, the continuum has neither underlying sites nor vertices and therefore has no definite number of these. To describe the percolation threshold in such situation we need to define the some new parameter other than our usual p_c, which not only depends on the number of polygons but also their area relative to that of the domain which they are in. We define the threshold area ratio as Definition 4.2. This definition disregards the intersection of the polygons, and therefore can only be used to compare networks which have the same type of distribution.

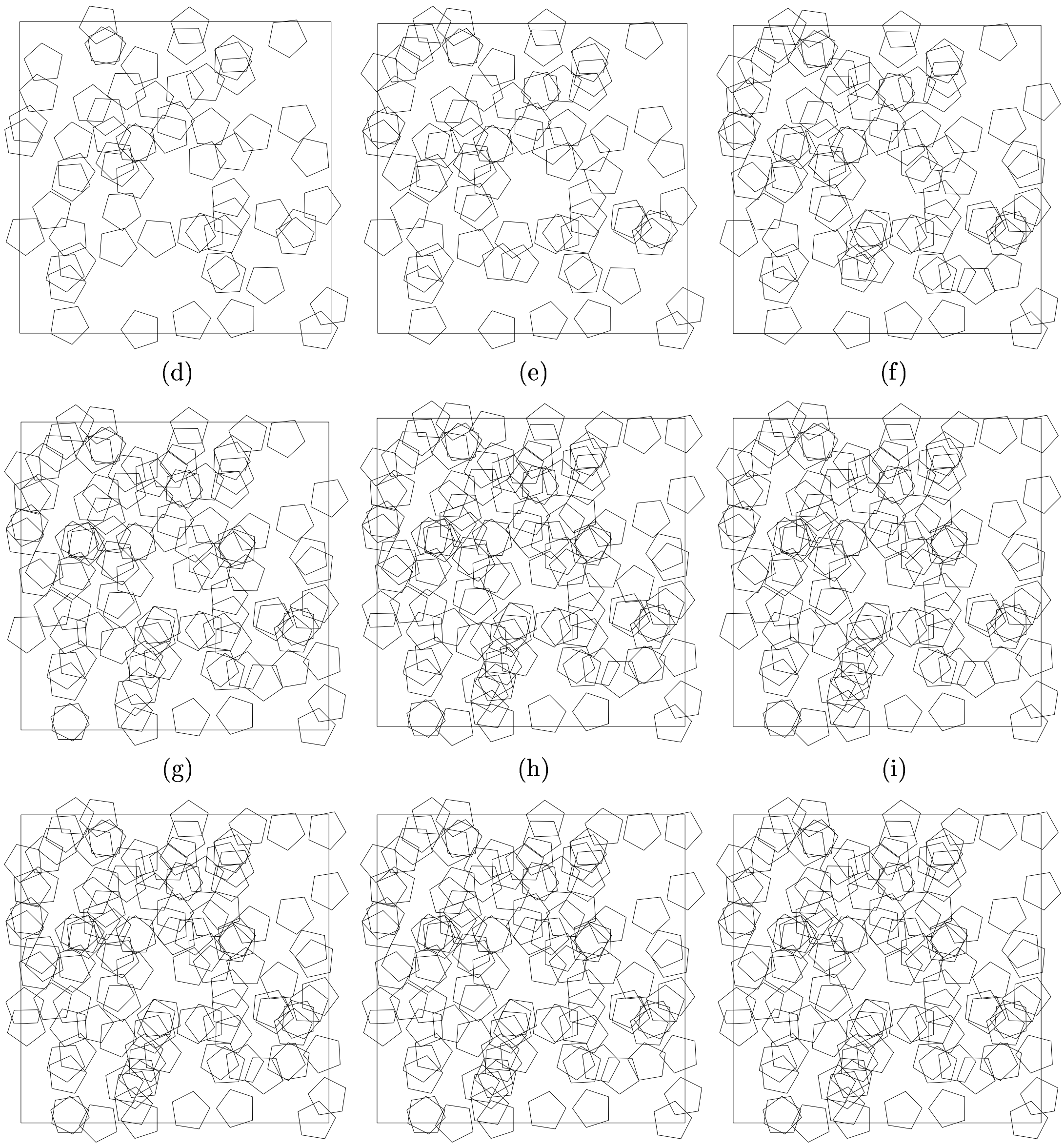
Definition 4.2. *For an aggregate of n-gons all the centres of which are Poisson points within a space of area A, the threshold area ratio is $\alpha_n = \sum_n |a_i|/A$, where a_i is the area of each n-gon. If all n-gons have an equal area, then $\alpha_n = na/A$.:*

The programme in § A.5 is transformed into a function which is then reused several times in the course of operation of another programme, listed in § A.33, which finds α_n. Both the programme, `ppgt.m`, and the function, `cmpc.m`, are here listed concatenated together whereas in practice they must separately exist.

The programme first proceeds to find whether percolation occurs for aggregates containing increasing numbers of pentagons in step of 16. This step size is in fact 2^k for some integral k, and can be chosen to be optimum for a certain size of the network. At the end of these steps we find whether our aggregate percolate. If it does, then we know that somewhere within the last increasing step exists the point where percolation sets in. The precise location of this point can then be found by doing binary searching. There are altogether k steps of these searches, at the end of each one of which there is again a test for percolation. If this test for the last and k^{th} step is successful, then the last picture represents the aggregate which starts to percolate, otherwise one needs to know how many consecutive percolation tests have failed up to that point. With the knowledge of that number, one can count backward and reach the picture in question. The programme `ppgt.m` fails to do this last job, but it has been added, together with the generalisation to 2^k steps, to make `ppgk.m`.

Figure 4.10 is the sample run on an aggregate of pentagons. Each of the pentagons is one unit area, and the space containing them, or rather the space which contains all their centres, is 10 × 10 in size.


(a) (b) (c)



(d) (e) (f)

(g) (h) (i)

(j) (k) (l)

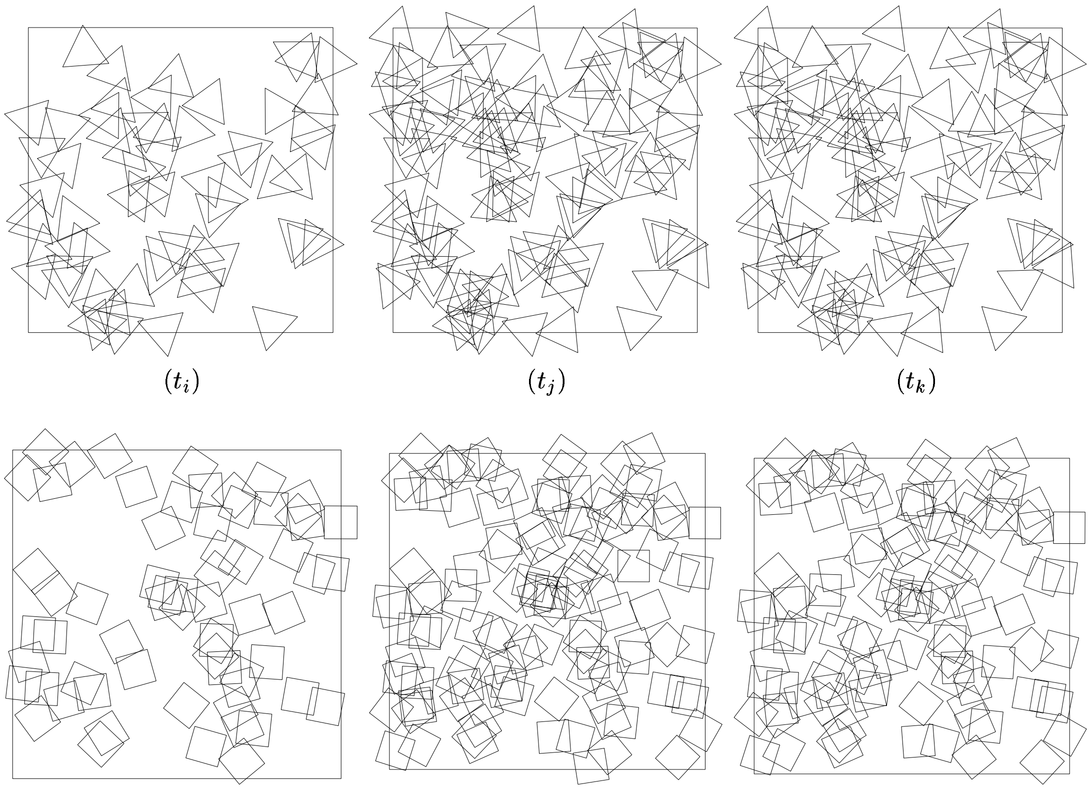
Figure 4.10 *Finding the threshold area ratio of pantagons. The number of pentagons from* (a) *to* (h) *increases in step of* $2^4 = 16$. *In* (i), (j), (k) *and* (l) *a binary search proceeds which divides the length of the last interval from 16 to 8, 4, 2 and 1 in sequence. This simulation gives* $\alpha_5 = 1.23$.

The programme `ppgk.m` in § A.33 uses $k = 4$ as explained above. Generalising this into a general k gives Algorithm 4.7. Here v_{ij} is the j^{th} vertex of the i^{th} n-gon. At the completion of the programme, if it does complete, we have the aggregate containing $m_{1/2}$ polygons, and our b^{th} percolation test before last has been on this aggregate.

Algorithm 4.7 *Threshold area ratio.*

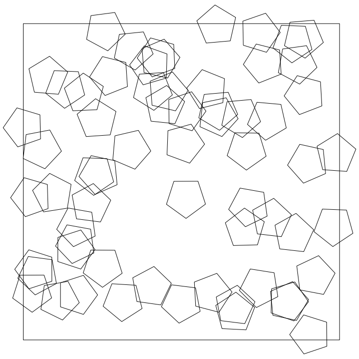
```
$s \leftarrow 2^k$;
$\beta \leftarrow 2\pi/n$;
$r \leftarrow [1/(n\sin(\beta/2)\cos(\beta/2))]^{1/2}$;
while percolation check fails do
    $\{(x,y)\} \leftarrow$ find $s$ more centre coordinates;
    $\{\theta\} \leftarrow$ find $s$ more orientational angles;
    for all these new centres and angles do
        $v_{ij} \leftarrow (x + r\cos(\theta_i + j\beta), y + r\sin(\theta_i + j\beta))$;
        check percolation all the $m$ centres and angles;
        endfor
    endwhile
    $(m_0, m_1]$ ← the last step above;
    $b$ ← 0;
    repeat $k$ times
        $m_{1/2}$ ← $(n_0 + n_1)/2$;
        check percolation all the first $m_{1/2}$ angles and centres;
        if percolated then
            $m_1$ ← $m_{1/2}$;
            $b$ ← 0;
        else
            $m_0$ ← $m_{1/2}$;
            $b$ ← $b + 1$;
        endif
    endrepeat                                                        □
```

Figure 4.11 shows the percolation of n-gons with various values of n, the results of applying Algorithm 4.7 above. Polygons up to the twelve-sided dodecagon are shown here. After this there are of course the triskaidecagon, tetrakaidecagon, ..., enneakaidecagon, icosagon, icosikaihenagon, icosikaidigon, icosikaitrigon, ..., icosiaienneagon, triacontagon, triacontakaihenagon, ..., triacontakaienneagon, tetracontagon, ..., pentacontagon, ..., hexacontagon, ..., heptacontagon, ..., octacontagon, ..., enneacontagon, ..., hectagon, *etc.* But all of these closely approaches the circle.

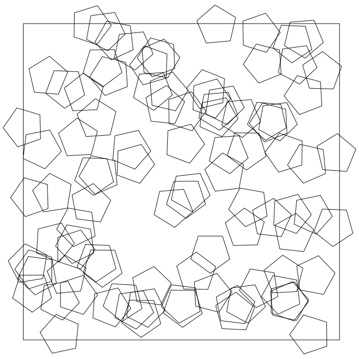


(t_i) (t_j) (t_k)

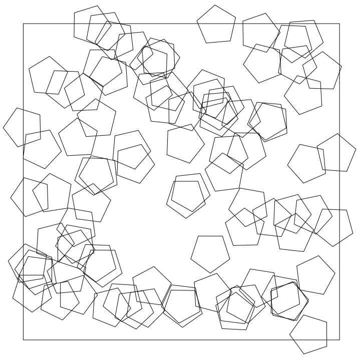
(q_i) (q_j) (q_k)



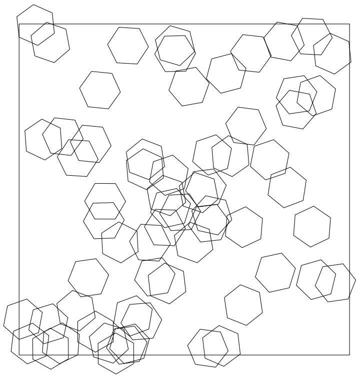
(p_i)



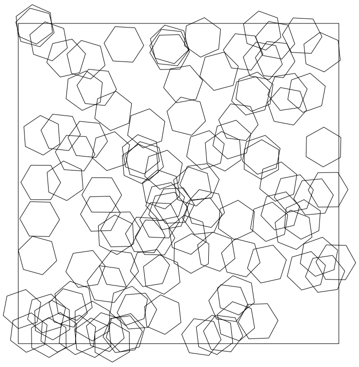
(p_j)



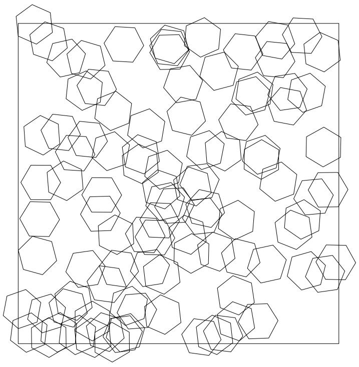
(p_k)



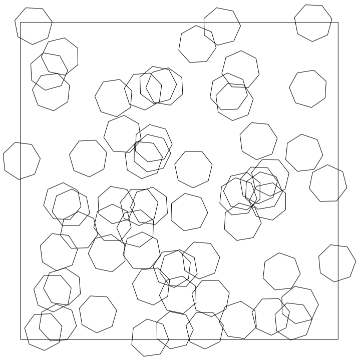
(h_i)



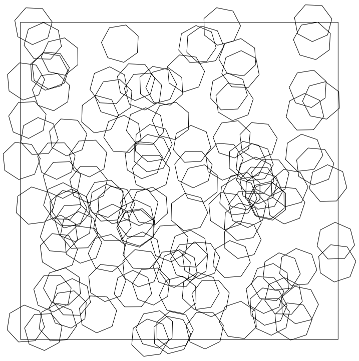
(h_j)



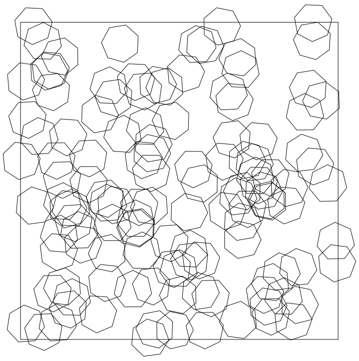
(h_k)



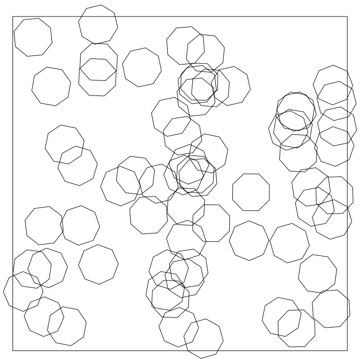
(k_i)



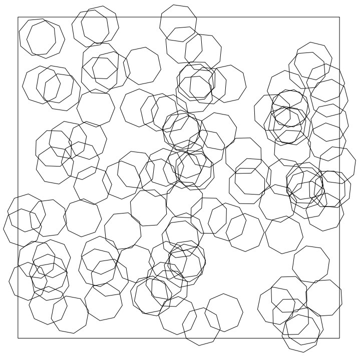
(k_j)



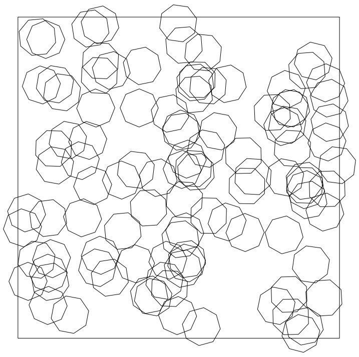
(k_k)



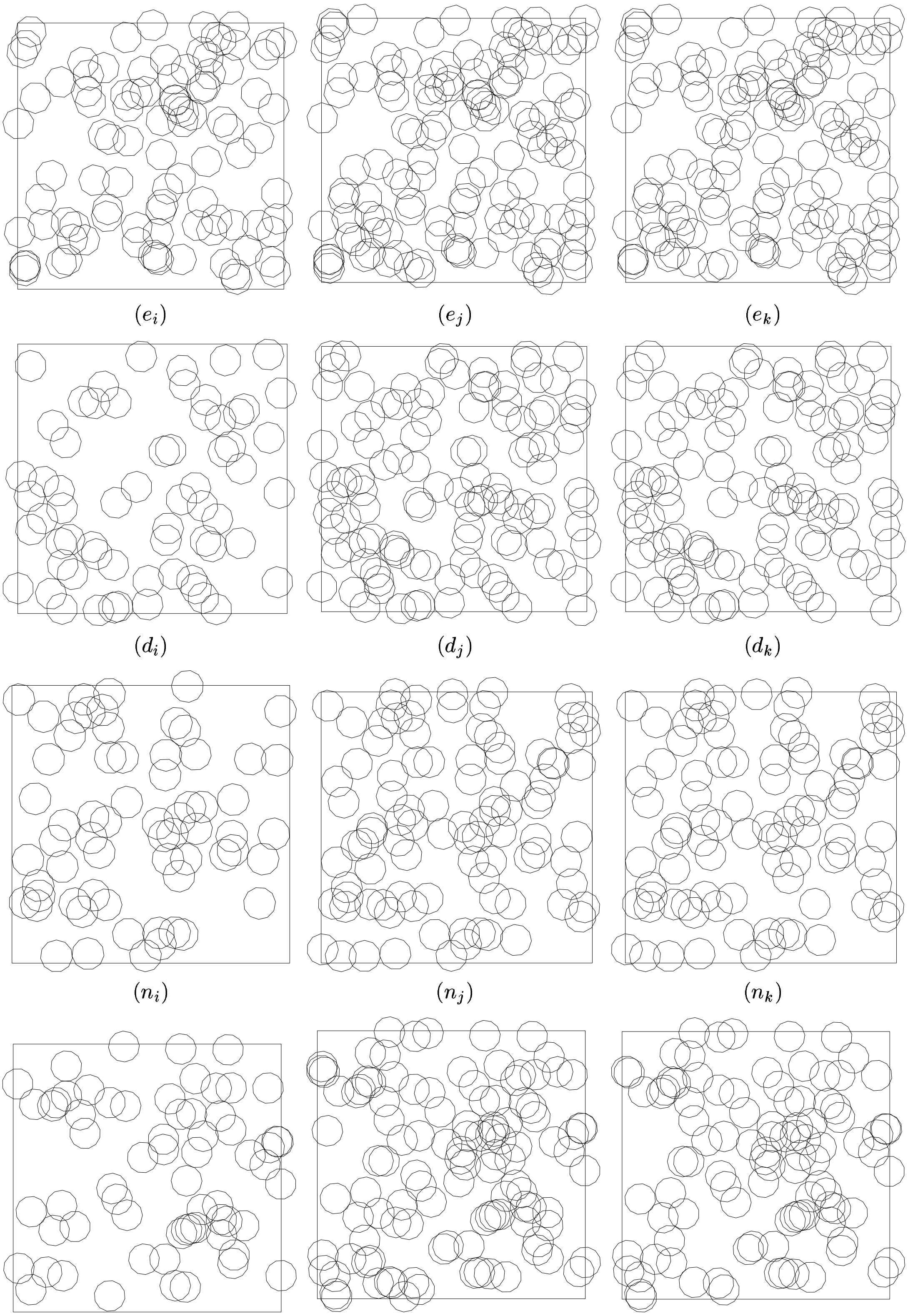
(o_i)



(o_j)



(o_k)



(e_i) (e_j) (e_k)

(d_i) (d_j) (d_k)

(n_i) (n_j) (n_k)

(c_i) (c_j) (c_k)

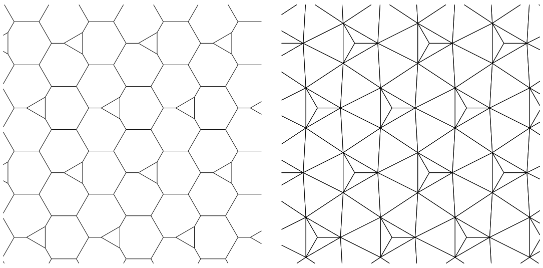
Figure 4.11 *Percolation of n-gons. Triangle, square, pentagon, hexagon, heptagon, octagon, enneagon, decagon, hendecagon and dodecagon are represented respectively by t, q, p, h, k, o, e, d, n and c. The subscripts i, j and k are respectively the fourth step of the programme, the over-percolated case and the critically percolated case.* $\alpha_3 = 1.03$, $\alpha_4 = 1.17$, $\alpha_5 = 0.89$, $\alpha_6 = 0.99$, $\alpha_7 = 1.23$, $\alpha_8 = 1.03$, $\alpha_9 = 1.23$, $\alpha_{10} = 1.07$, $\alpha_{11} = 0.91$ *and* $\alpha_{12} = 0.95$.

§ 4.9 2-homohedral tilings

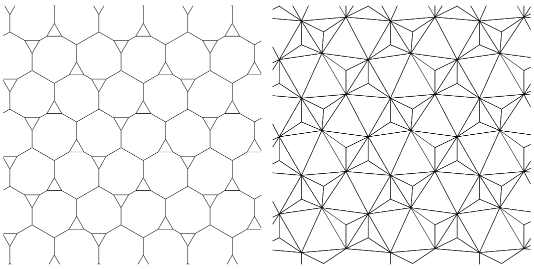
There are altogether 39 types of 2-homohedral tilings, ie. those which have vertices of the same valence. Of these, there are 26 types of valence 3, 10 of valence 4, and 3 of valence 5 (*cf* Grünbaum *et al*, 1987, or, for immediate further explanation, see text following Table 4.7 in page 145). The code and the data used in the simulations are in § A.6. In the programme, the variable o contains the vertex numbers which must be ordered by scanning the basic tile from left to right and gradually from bottom up. Unless the variable follows this ordering strange results will occur. The variables m and n are respectively the x- and y-coordinates of the vertices.

The results on percolation of the homohedral tilings are very important. Firstly the algorithm used for generating them can create any tiling whose components, that is vertices and edges, form a pattern that repeats itself. Tiyapan (1995, KNT3(iii)) developed an algorithm for finding the percolation threshold, and used it to study percolation on Voronoi Tessellation. The same programme was then used on several regular tilings in order to compare the results of these tilings with the former, in other words for bench-marking purposes. However, each of these regular tessellations had to be studied separately, and the algorithm for generating them thus developed may not be used to produce another type of tessellation. Such tasks are imaginably tiring and error-prone. The present algorithm changes all this. Also its percolation part finds not only one but four percolation thresholds, namely the bond-, cell-, edge- and vertice critical probabilities. The first two of these form a pair which belongs to its dual-, while the last forms one which belongs to the lattice itself.

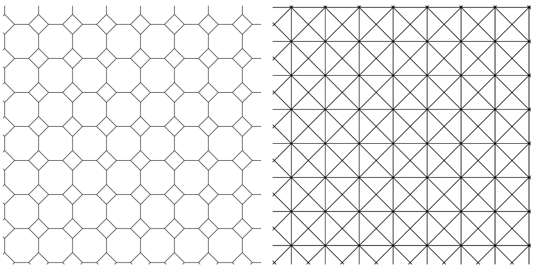
Not only this repeating-prototile algorithm a useful and original contribution for the purpose of mentioned, but also the results obtained from the homohedral tilings it has been tested upon are no less valuable. It is true one still needs to do more simulations before one would be able to conclude anything correctly, even this preliminary result can be used in the study of, for example, the weight distribution and stability of a truss structured using any one of these patterns. One way for studying this is to imagine each angle of the tile as a resistor forming a resistive networks. Percolation study of this network then analyse instead of force propagation in truss the electric current passing through the analogous resistive circuit. Another possible application is the resistive networks inside a semiconductor wafer. Here the chip's internal structure may resemble some repeating patterns which conduct electric current.



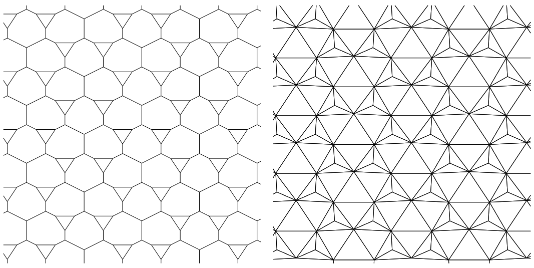
(1) $3_3[3^3]7_1[3^7]$



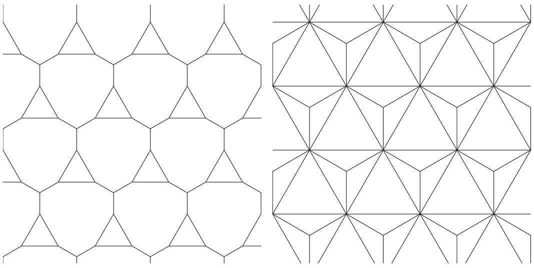
(2) $3_3[3^3]9_3[3^9]_{\text{II}}$



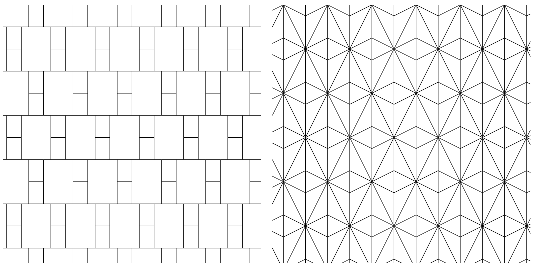
(3) $4_4[3^4]8_4[3^8]$

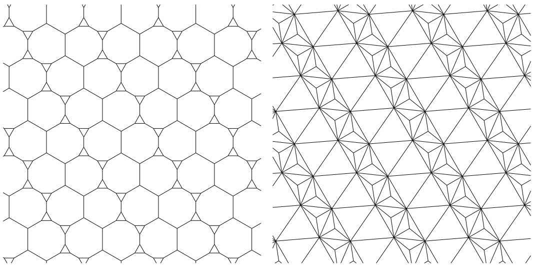


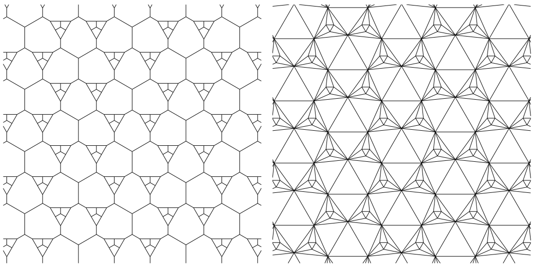
(4) $3_3[3^3]8_2[3^8]$

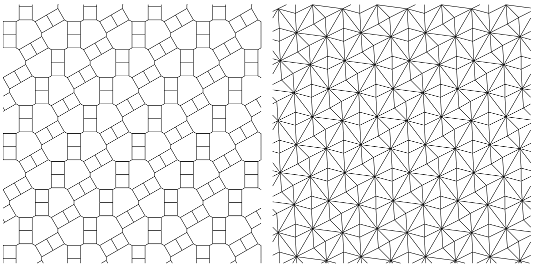


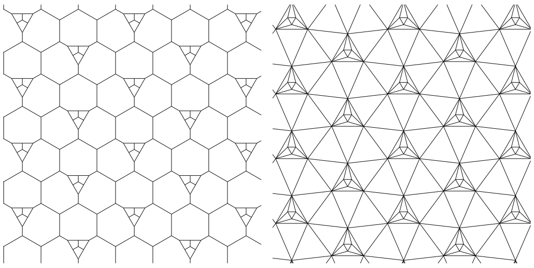
(5) $3_3[3^3]9_3[3^9]_{\text{III}}$

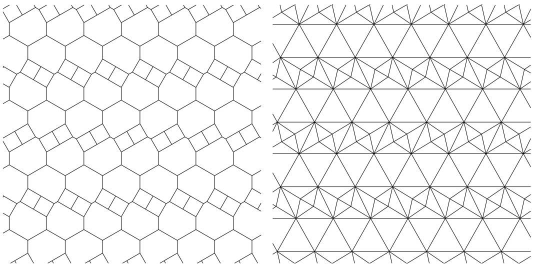

(6) $4_3[3^4]10_6[3^{10}]_{\mathrm{I}}$

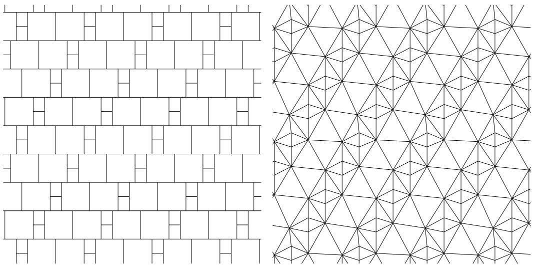

(7) $3_3[3^3]9_3[3^9]_{\mathrm{I}}$

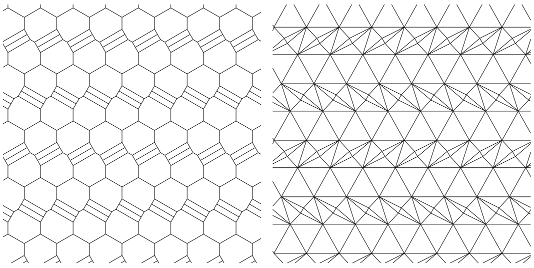

(8) $4_2[3^4]10_4[3^{10}]$

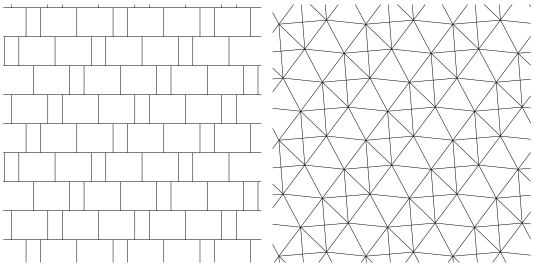

(9) $4_3[3^4]10_6[3^{10}]_{\mathrm{II}}$

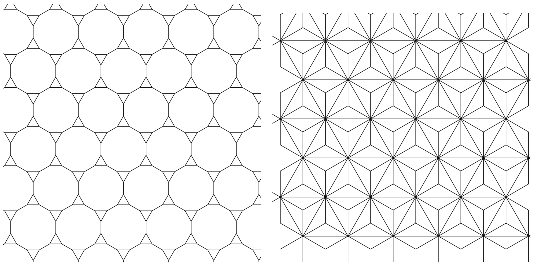

(10) $4_2[3^4]8_2[3^8]$

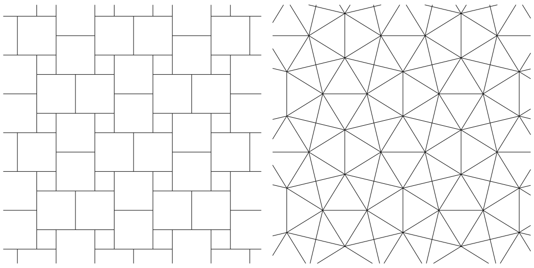

(11) $4_3[3^4]8_3[3^8]_{\mathrm{I}}$

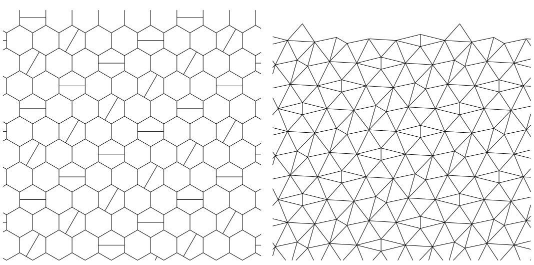

(12) $4_3[3^4]8_3[3^8]_{\mathrm{II}}$


(13) $4_3[3^4]8_3[3^8]_{\mathrm{III}}$


(14) $4_4[3^4]7_2[3^7]_{\mathrm{II}}$

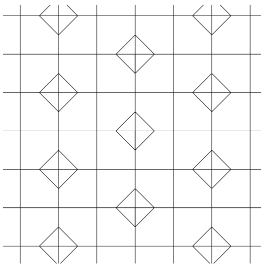

(15) $3_3[3^3]12_6[3^{12}]$

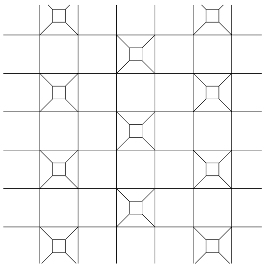


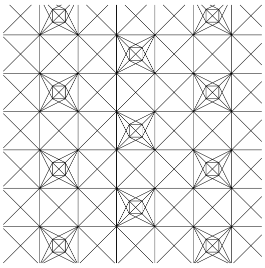


(16) $4_4[3^4]7_2[3^7]_{\mathrm{I}}$

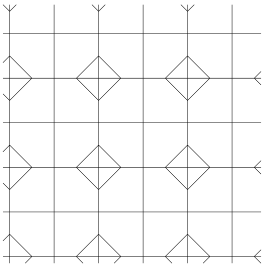
(17) $5_4[3^5]7_4[3^7]_{\mathrm{I}}$

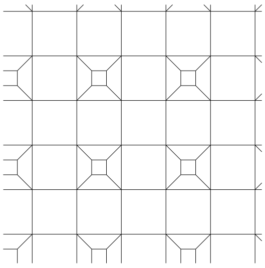


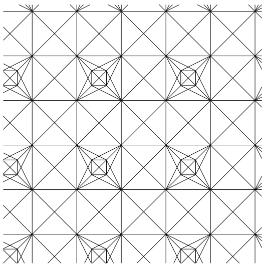




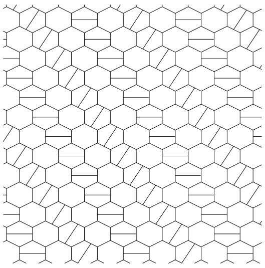
(18) $3_1[4^3]5_1[4^5]_{\mathrm{I}}$

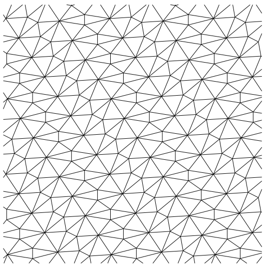




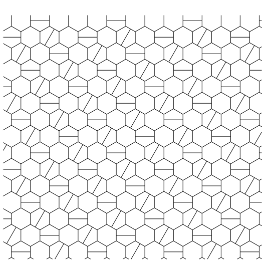


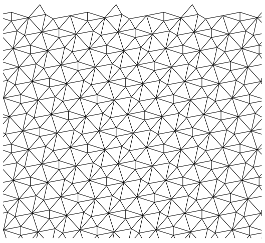
(19) $3_1[4^3]5_1[4^5]_{\mathrm{II}}$

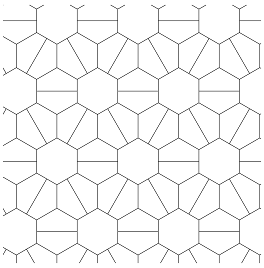


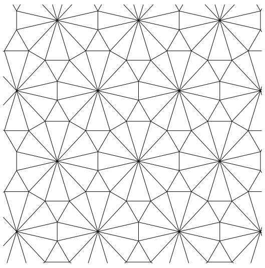


(20) $5_3[3^5]8_6[3^8]_{\mathrm{II}}$



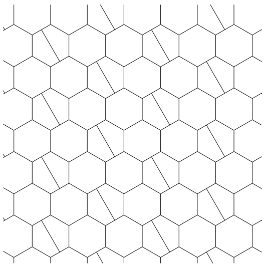


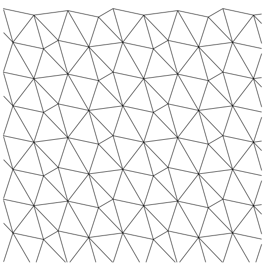


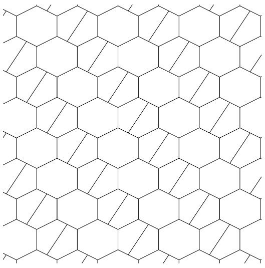


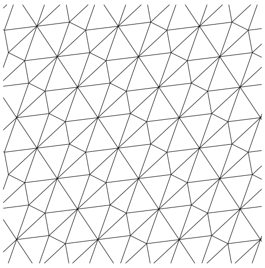
(21) $5_3[3^5]8_6[3^8]_{\mathrm{III}}$

(22) $5_2[3^5]12_{12}[3^{12}]$



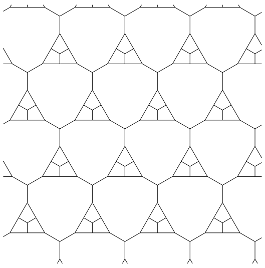


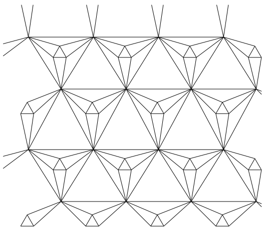


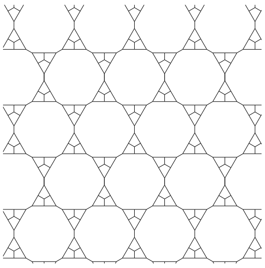


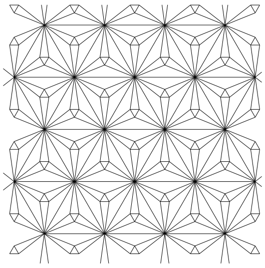
(23) $5_4[3^5]7_4[3^7]_{\mathrm{II}}$

(24) $5_3[3^5]8_6[3^8]_{\mathrm{I}}$

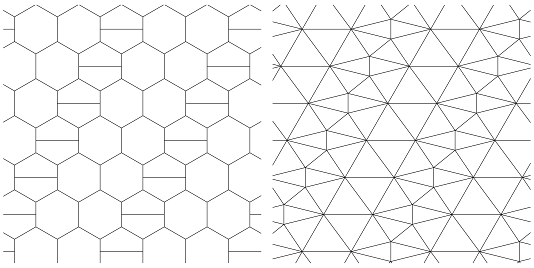




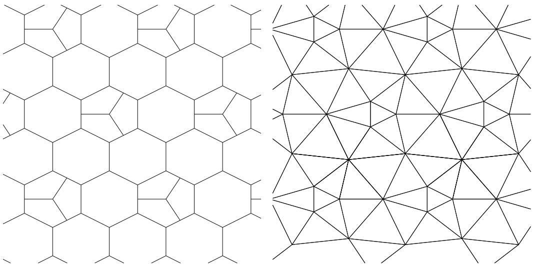




(25) $4_2[3^4]12_6[3^{12}]$

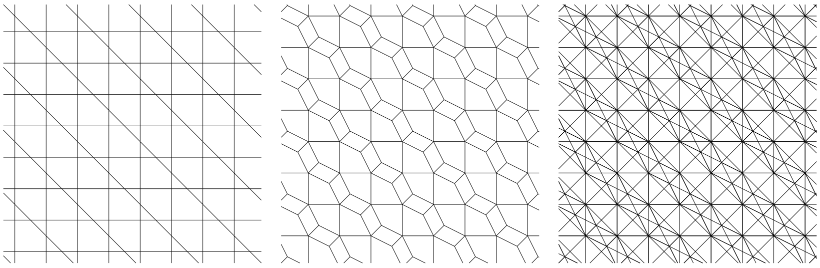


(26) $4_2[3^4]18_{12}[3^{18}]$

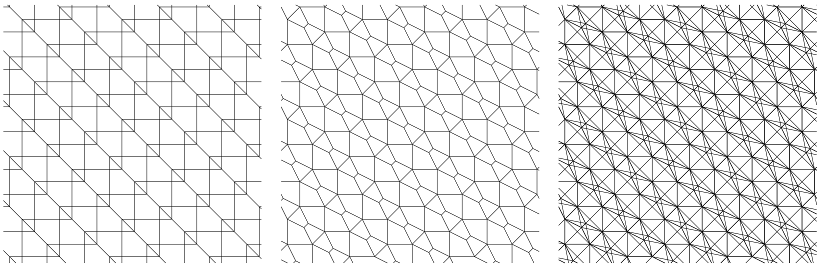


(27) $5_3[3^5]7_3[3^7]_{\mathrm{I}}$

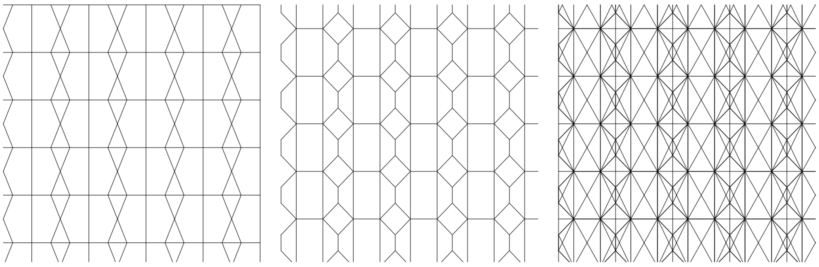
(28) $5_3[3^5]7_3[3^7]_{\mathrm{II}}$



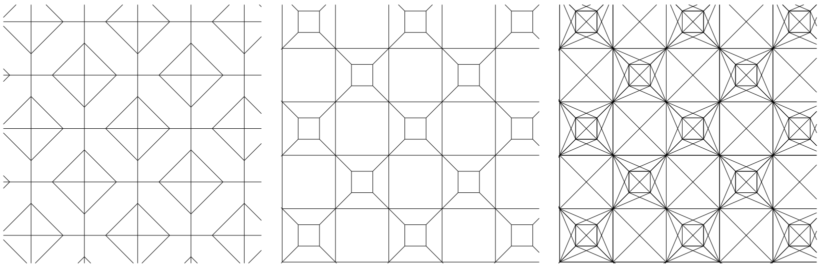
(29) $3_2[4^3]5_2[4^5]_{\mathrm{I}}$



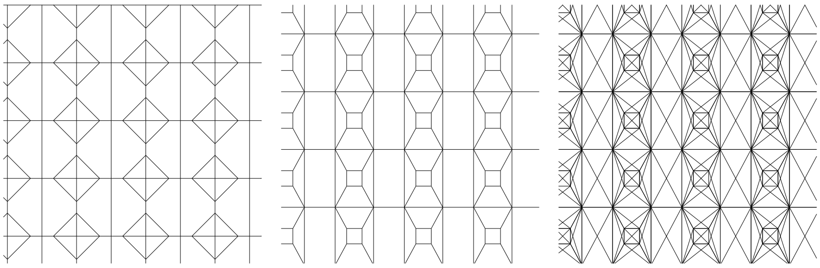
(30) $3_2[4^3]5_2[4^5]_{\mathrm{II}}$



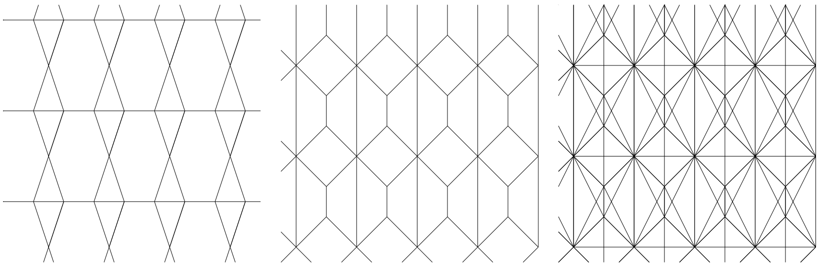
(31) $3_3[4^3]5_3[4^5]$



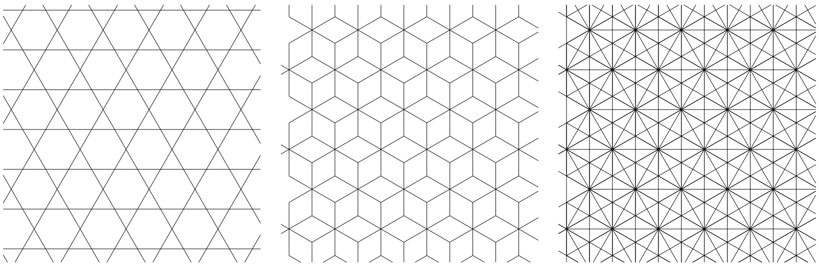
(32) $3_1[4^3]6_2[4^6]_{\mathrm{I}}$



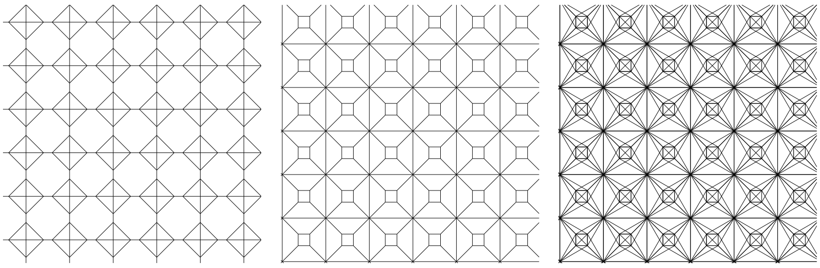
(33) $3_1[4^3]6_2[4^6]_{\mathrm{II}}$



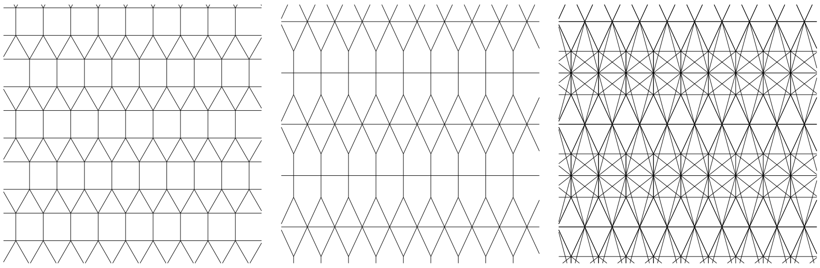
(34) $3_2[4^3]6_4[4^6]$



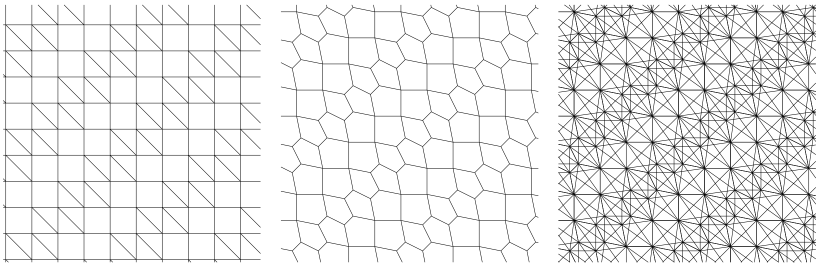
(35) $3_3[4^3]6_6[4^6]$



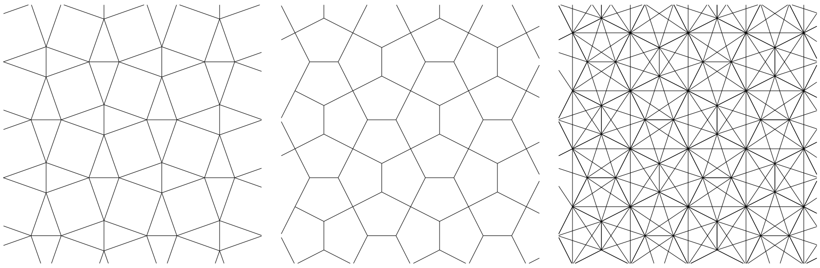
(36) $3_1[4^3]8_4[4^8]$



(37) $3_1[5^3]4_2[5^4]_{\mathrm{I}}$



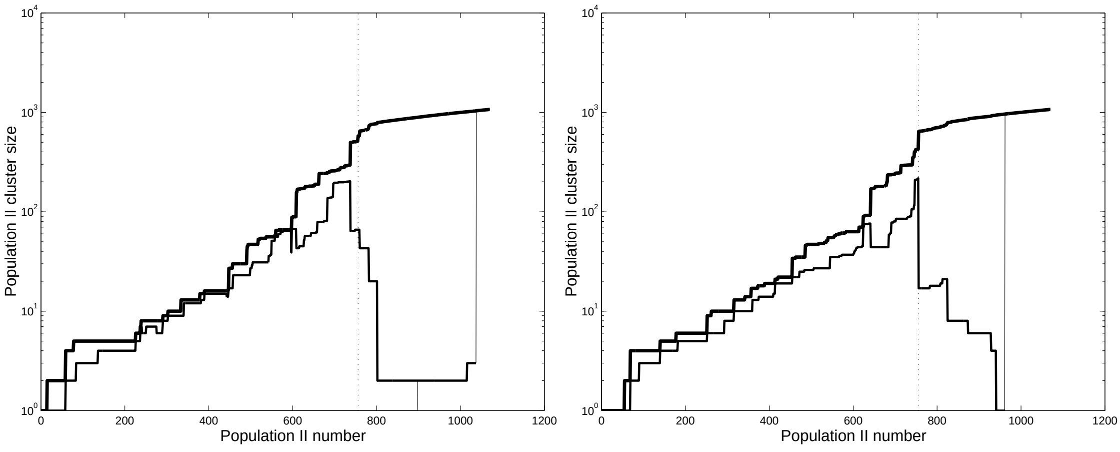
(38) $3_1[5^3]4_2[5^4]$



(39) $3_2[5^3]4_4[5^4]$

Figure 4.12 *The thirty-nine 2-homohedral tilings and their covering lattices.*

Once while doing the above simulations I came across an interesting batch of the $4_4[3^4]7_2[3^7]_{\mathrm{II}}$ 2-homohedral tiling where the percolations forward and backward in time gave the same value of p_c, that is there is an approximate time symmetry. In other words the p_c's of both the first and the second population are symmetric to each other. Each percolates when its population reaches 756 out of the combined total of 1070, which identically gives $p_c = 0.7065$. Large non-percolating clusters quickly disappears after the onset of percolation as shown in Figure 4.13. In Figure 4.13 (b) the size of the second largest cluster is reduced to one between the population II of 941 and 961, which correspond to the density of 0.8794 and 0.8981 respectively. Interestingly from the population of 961 until 1070 there is only one cluster despite the fact that there are still more than one hundred population I. Here *population II* always refers to any population which percolates or which is being observed with regard to percolation. On the other hand in Figure 4.13 (b) the second biggest cluster is the smallest cluster from population 898 to 1037. Again from the population of 1038 until the maximum 1070 no second clusters appear.

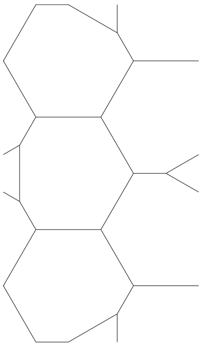


(a) (b)

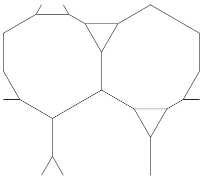
Figure 4.13 *The biggest, the second biggest, and the smallest cluster from a batch of the* $4_4[3^4]7_2[3^7]_{\mathrm{II}}$ *2-homohedral tiling where the percolation happens to be symmetric in time, that is (a) and (b) are time symmetric to each other. The vertical dotted line is where the percolation point occurs in each case. The the heaviest to the lightest lines are respectively the progression of the clusters which are biggest, second biggest, and smallest at each population density.*

The percolation considered is that which percolates from the lower bound to the upper bound, both of which are fixed. Because the networks must be constructed such that there would be duplicates of neither vertices nor links, there must be cells some vertices and edges of which belong to one or more of the basic tiles adjacent to them. As a result, those basic tiles on the boundary sometimes have a few of their cells missing, and depending on whether the basic tile in question is very complex or simple the position of the missing cell may merely lie on its boundary or may lie nearly halfway into its interior as is the case with the $4_3[3^4]8_3[3^8]_{\mathrm{I}}$ and the $4_3[3^4]8_3[3^8]_{\mathrm{III}}$ tilings. The upper and the lower boundaries, which decide the percolation, are chosen as straight lines parallel to the vertical axis and lying in the direction towards the interior of the network and at a fixed distance from the the maximum and the minimum horizontal positions respectively of all the vertices in the network. This fixed distance is defined for each network to be a fixed ratio of its overall width. The value of 0.05 had been used which was later changed to 0.1 due to a problem which occurred while simulating the $4_2[3^4]8_2[3^8]$ tessellation, missing cells formed a continuous band disconnecting all the cells on the boundary from the rest of the network, the upper boundary lay within this band when shifted by five percent from the upper limit but not when shifted by ten percent.

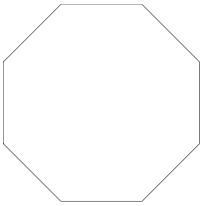
The tilings in Figure 4.12 have the basic units shown in Figure 4.14.



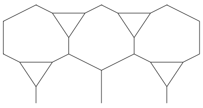
$3_3[3^3]7_1[3^7]$



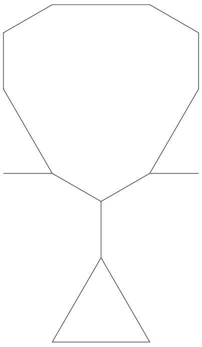
$3_3[3^3]9_3[3^9]_{\mathrm{II}}$



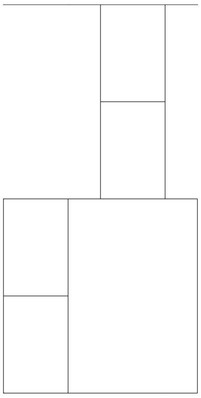
$4_4[3^4]8_4[3^8]$



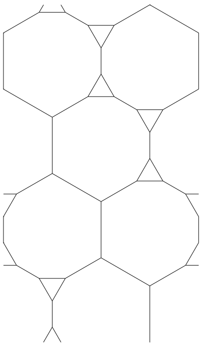
$3_3[3^3]8_2[3^8]$



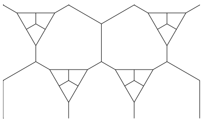
$3_3[3^3]9_3[3^9]_{\mathrm{III}}$



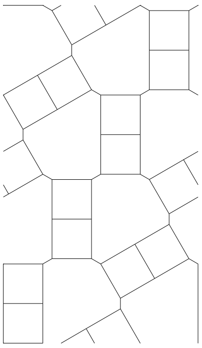
$4_3[3^4]10_6[3^{10}]_{\mathrm{I}}$



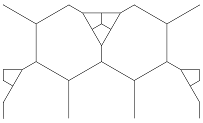
$3_3[3^3]9_3[3^9]_{\mathrm{I}}$



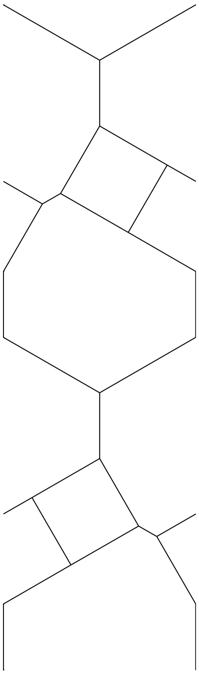
$4_2[3^4]10_4[3^{10}]$



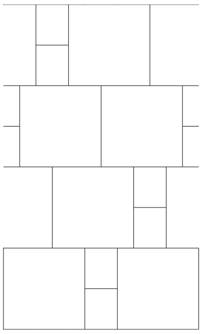
$4_3[3^4]10_6[3^{10}]_{\mathrm{II}}$



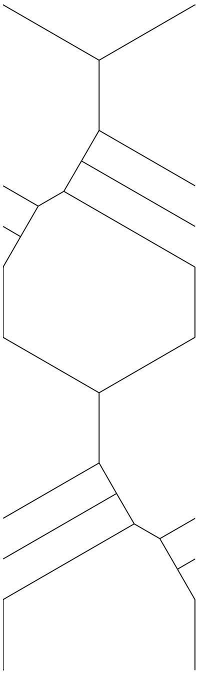
$4_2[3^4]8_2[3^8]$



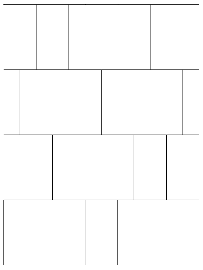
$4_3[3^4]8_3[3^8]_{\mathrm{I}}$



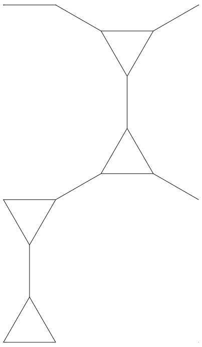
$4_3[3^4]8_3[3^8]_{\mathrm{II}}$



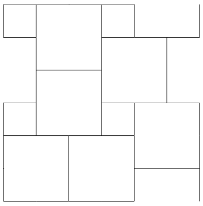
$4_3[3^4]8_3[3^8]_{\mathrm{III}}$



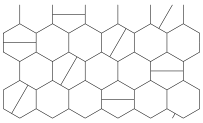
$4_4[3^4]7_2[3^7]_{\mathrm{II}}$



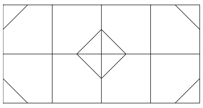
$3_3[3^3]12_6[3^{12}]$



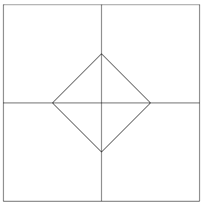
$4_4[3^4]7_2[3^7]_{\mathrm{I}}$



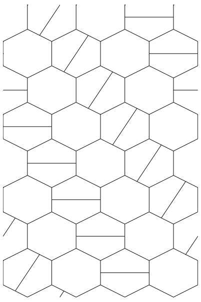
$5_4[3^5]7_4[3^7]_{\mathrm{I}}$



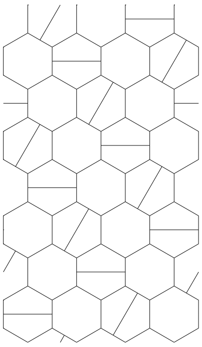
$3_1[4^3]5_1[4^5]_{\mathrm{I}}$



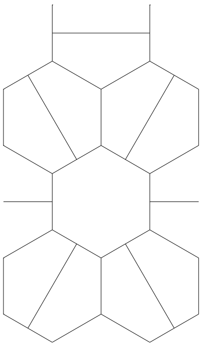
$3_1[4^3]5_1[4^5]_{\mathrm{II}}$



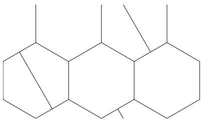
$5_3[3^5]8_6[3^8]_{\mathrm{II}}$



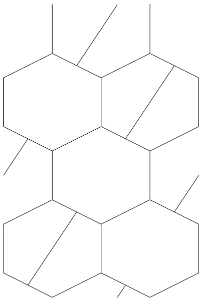
$5_3[3^5]8_6[3^8]_{\mathrm{III}}$



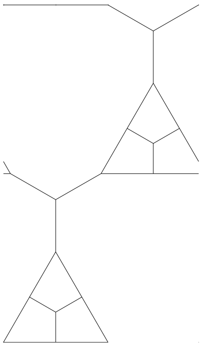
$5_2[3^5]12_{12}[3^{12}]$



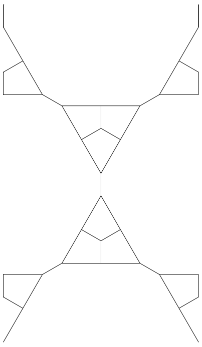
$5_4[3^5]7_4[3^7]_{\mathrm{II}}$



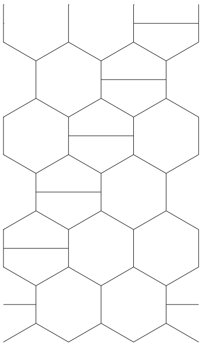
$5_3[3^5]8_6[3^8]_{\mathrm{I}}$



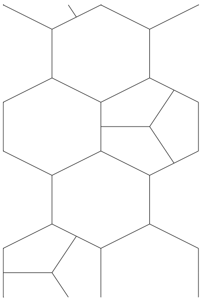
$4_2[3^4]12_6[3^{12}]$



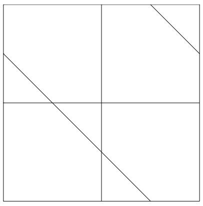
$4_2[3^4]18_{12}[3^{18}]$



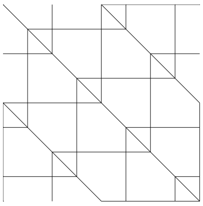
$5_3[3^5]7_3[3^7]_{\mathrm{I}}$



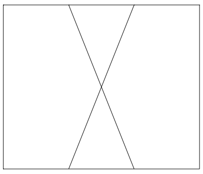
$5_3[3^5]7_3[3^7]_{\mathrm{II}}$



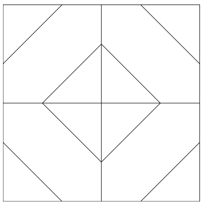
$3_2[4^3]5_2[4^5]_{\mathrm{I}}$



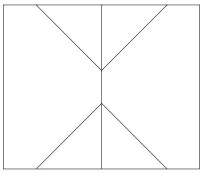
$3_2[4^3]5_2[4^5]_{\mathrm{II}}$



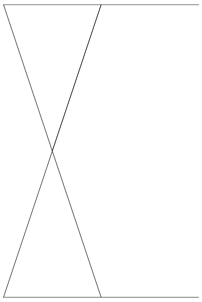
$3_3[4^3]5_3[4^5]$



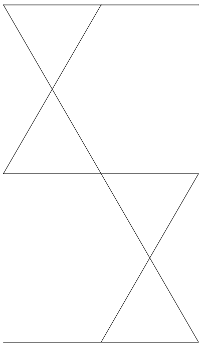
$3_1[4^3]6_2[4^6]_{\mathrm{I}}$



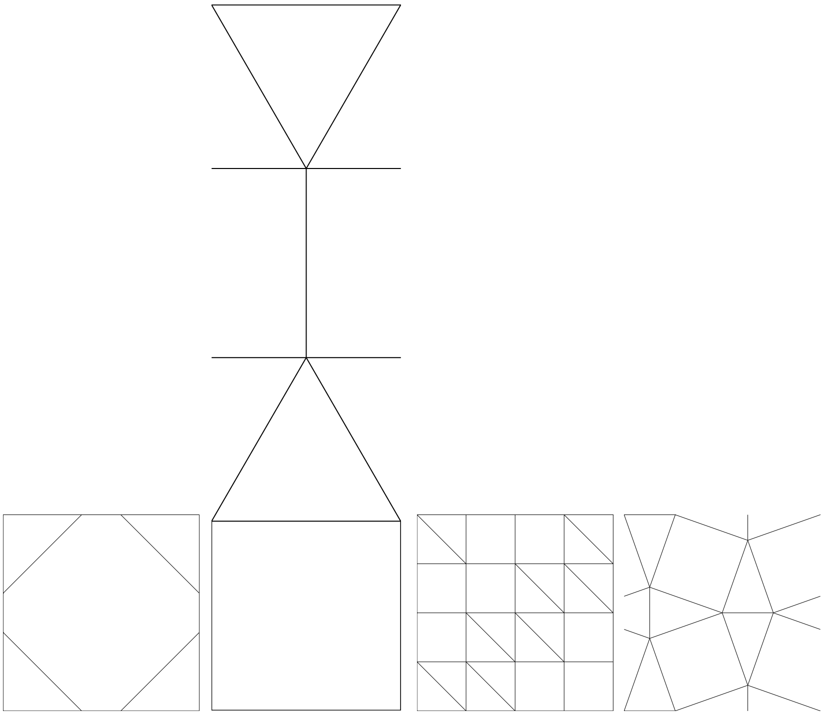
$3_1[4^3]6_2[4^6]_{\mathrm{II}}$



$3_2[4^3]6_4[4^6]$



$3_3[4^3]6_6[4^6]$



$3_1[4^3]8_4[4^8]$ $3_1[5^3]4_2[5^4]_{\text{I}}$ $3_1[5^3]4_2[5^4]$ $3_2[5^3]4_4[5^4]$

Figure 4.14 *The unit cells of the 2-homohedral tilings.*

In Table 4.7 there are six types of percolation probabilities. Two cells can be neighbours to each other under the criterion of having either one or two vertices in common. This gives rise to two different values of p_c's for cells, and another two for bonds. These can be termed 1-neighbours and 2-neighbours respectively. In Physics and other physical sciences 2-neighbours are more important because the greater the contact area is the stronger the bonding. This is not always the case in other fields, for example in Sociology a smaller point of contact sometimes implies the less duplications of neighbours' neighbours, and thus the larger the resultant networks. Because there is a cost for maintaining the connections, the smaller each contact point is and the larger the network the better, therefore here 1-neighbours seem to be more important.

1. $3_3[3^3]7_1[3^7]$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4714±0.0667	0.3041±0.0272			0.7212±0.0444	0.6771±0.0381
1.1551×10^{-6}	-3.3740×10^{-6}			-1.1124×10^{-4}	-5.2007×10^{-5}
4.0885×10^{-5}	1.1316×10^{-6}			1.9314×10^{-5}	9.5566×10^{-6}
5[99] , 4[195], 5.2525 5.4667 12[391] 5.6215	12[1099] 10.4149			4[256], 4[470], 2.7266 2.7957 12[894] 2.8523	4[349], 4[657], 3.7077 3.7869 12[1275] 3.8494

2. $3_3[3^3]9_3[3^9]_{\text{II}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4966±0.0655	0.2734±0.0206	0.5042±0.0574	0.3119±0.0417	0.7628±0.0228	0.7094±0.0435
-8.2288×10^{-5}	-5.5513×10^{-6}	-1.0585×10^{-4}	4.8211×10^{-5}	1.0042×10^{-6}	-2.7590×10^{-5}
3.7765×10^{-5}	5.9281×10^{-7}	2.1108×10^{-5}	7.5982×10^{-6}	7.3299×10^{-7}	7.7345×10^{-6}
4[176], 2[139], 5.2727 5.1942 2[305], 6[412] 5.4492 5.5243	6[831] , 6[1138] 12.0144 12.1564	2[305], 6[412] 5.0557 5.1165	2[771] , 6[1054] 11.0298 11.1556	4[415], 2[334], 2.8193 2.7964 2[692], 6[919] 2.8584 2.8770	4[585], 2[467], 3.8120 3.7859 2[989], 6[1322] 3.8544 3.8744

3. $4_4[3^4]8_4[3^8]$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4695±0.0858	0.3044±0.0429			0.7194±0.0209	0.6407±0.0296
-3.5574×10^{-5}	2.8967×10^{-5}			-5.8481×10^{-6}	2.8990×10^{-6}
1.4884×10^{-4}	6.8367×10^{-6}			6.5739×10^{-7}	2.0123×10^{-6}
2[181], 4[265], 5.5691 5.6453 12[313] 5.6741	4[748] , 12[888] 10.5294 10.5946			1[440], 2[624], 2.7273 2.7692 6[728] 2.7857	2[600], 2[864], 3.7333 3.7778 6[1014] 3.7949

4. $3_3[3^3]8_2[3^8]$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4903±0.1015	0.2926±0.0281			0.7447±0.0329	0.7019±0.0270
1.2205×10^{-4}	-1.8541×10^{-5}			2.1925×10^{-7}	-6.4679×10^{-6}
2.1262×10^{-4}	1.9682×10^{-6}			2.8427×10^{-6}	8.9933×10^{-7}
2[166], 4[301], 5.1928 5.3953 12[476] 5.5168	4[812], 12[1313] 11.0788 11.2749			2[381], 2[668], 2.8346 2.8743 6[1035] 2.8986	2[540], 2[960], 3.8444 3.8833 6[1500] 3.9067

5. $3_3[3^3]9_3[3^9]_{\mathrm{III}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4607±0.0586	0.2483±0.0258			0.7482±0.0264	0.7049±0.0195
7.9409×10^{-5}	4.8742×10^{-6}			2.4849×10^{-6}	-1.0461×10^{-5}
3.3840×10^{-5}	7.4553×10^{-7}			1.2161×10^{-6}	1.0354×10^{-6}
2[83], 4[357], 4.6988 5.3613 12[438] 5.4247	4[957], 12[1188] 12.0878 12.1785			2[212], 2[808], 2.6792 2.8317 6[980] 2.8469	2[284], 2[1144], 3.6620 3.8322 6[1395] 3.8480

6. $4_3[3^4]10_6[3^{10}]_{\mathrm{I}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.5286±0.0511	0.2991±0.0319			0.7789±0.0386	0.7173±0.0391
5.4350×10^{-5}	-8.0790×10^{-6}			8.7871×10^{-6}	-1.2114×10^{-4}
1.7402×10^{-5}	2.7687×10^{-6}			4.7653×10^{-6}	1.8197×10^{-5}
8[188], 4[295], 5.3085 5.4441 12[426] 5.5352	8[499], 4[803], 11.2826 11.5691 12[1179] 11.7574			4[429], 2[656], 2.6853 2.7439 6[931] 2.7841	4[576], 2[900], 3.8333 3.8667 6[1296] 3.8889

7. $3_3[3^3]9_3[3^9]_{\mathrm{I}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.5206±0.0551	0.2987±0.0308			0.7532±0.0296	0.7063±0.0222
-9.0646×10^{-6}	7.4580×10^{-6}			-8.9673×10^{-6}	-4.5643×10^{-6}
1.7837×10^{-5}	1.7780×10^{-6}			1.3152×10^{-6}	7.3255×10^{-7}
4[259], 4[473], 5.3977 5.5518 6[751] 5.6431	4[699], 4[1313], 11.9371 12.2224 6[2119] 12.3870			2[594], 2[1048], 2.8485 2.8855 6[1630] 2.9080	2[846], 2[1512], 3.8440 3.8836 6[2370] 3.9072

8. $4_2[3^4]10_4[3^{10}]$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4726±0.0684	0.2905±0.0633			0.7341±0.0316	0.6957±0.0303
1.0814×10^{-4}	5.5163×10^{-4}			-8.0603×10^{-6}	7.5952×10^{-7}
8.0038×10^{-5}	1.3835×10^{-4}			2.0508×10^{-6}	1.6121×10^{-6}
4[309], 4[556], 5.3722 5.5288 12[875] 5.6229	4[830], 4[1537], 11.3181 11.6695 12[2460] 11.8756			2[670], 2[1181], 2.9015 2.9263 6[1836] 2.9412	2[972], 2[1728], 3.9156 3.9363 6[2700] 3.9489

9. $4_3[3^4]10_6[3^{10}]_{\mathrm{II}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4958±0.0796	0.3020±0.0381			0.7490±0.0461	0.6751±0.0199
1.3909×10^{-4}	3.9193×10^{-5}			1.0834×10^{-4}	6.2752×10^{-7}
8.0344×10^{-5}	6.3920×10^{-6}			1.4492×10^{-5}	3.7299×10^{-7}
4[325], 4[325], 5.3292 5.3292 12[532] 5.4737	4[866], 4[866], 11.5727 11.5727 12[1456] 11.8187			2[748], 2[748], 2.8369 2.8369 6[1187] 2.8711	2[1061], 2[1061], 3.8624 3.8624 6[1704] 3.8920

10. $4_2[3^4]8_2[3^8]$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.5102±0.0568	0.3183±0.0405			0.7202±0.0493	0.6953±0.0307
4.1767×10^{-5}	-9.0691×10^{-6}			-1.1640×10^{-4}	1.5917×10^{-5}
3.0853×10^{-5}	7.8890×10^{-6}			2.1475×10^{-5}	1.8233×10^{-6}
4[259], 4[385], 5.2741 5.4026 12[535] 5.4916	4[683], 4[1040], 10.3309 10.5096 12[1469] 10.6331			2[613], 2[884], 2.8189 2.8507 6[1203] 2.8728	2[864], 2[1260], 3.8356 3.8651 6[1728] 3.8854

11. $4_3[3^4]8_3[3^8]_{\mathrm{I}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4577±0.0761	0.2910±0.0421			0.6887±0.0279	0.6532±0.0340
-2.5782×10^{-5}	-1.6419×10^{-5}			8.0593×10^{-7}	-2.5506×10^{-5}
8.4655×10^{-5}	7.8109×10^{-6}			7.8808×10^{-7}	3.2110×10^{-6}
4[195], 4[356], 5.3231 5.4944 12[487] 5.5647	4[519], 4[978], 10.2736 10.5562 12[1355] 10.6657			2[454], 2[797], 2.8282 2.8708 6[1074] 2.8864	2[642], 2[1144], 3.8349 3.8759 6[1550] 3.8903

12. $4_3[3^4]8_3[3^8]_{\mathrm{II}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.5030 ± 0.0603	0.3166 ± 0.0334			0.7385 ± 0.0255	0.6718 ± 0.0369
2.6397×10^{-5}	-1.3936×10^{-5}			-2.6605×10^{-6}	1.8952×10^{-6}
4.0377×10^{-5}	2.7675×10^{-6}			9.6100×10^{-7}	3.1533×10^{-6}
4[219], 4[419], 5.3151 5.4988 12[683] 5.6047	4[582] , 4[1152], 10.2818 10.5799 12[1914] 10.7461			2[517], 2[947], 2.7505 2.8131 6[1505] 2.8505	2[711], 2[1332], 3.8312 3.8769 6[2145] 3.9030

13. $4_3[3^4]8_3[3^8]_{\mathrm{III}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4931 ± 0.0560	0.2927 ± 0.0320			0.7294 ± 0.0348	0.6730 ± 0.0289
1.5620×10^{-4}	3.0191×10^{-5}			4.5084×10^{-6}	-2.7067×10^{-5}
3.1433×10^{-5}	6.4721×10^{-6}			2.5226×10^{-6}	3.4615×10^{-6}
4[212], 4[410], 5.3491 5.5268 12[594] 5.6061	4[567] , 4[1133], 10.3069 10.6019 12[1665] 10.7267			2[503], 2[927], 2.8151 2.8652 6[1317] 2.8853	2[708], 2[1328], 3.8107 3.8630 6[1900] 3.8832

14. $4_4[3^4]7_2[3^7]_{\mathrm{II}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4761 ± 0.0432	0.3174 ± 0.0396			0.7144 ± 0.0228	0.6440 ± 0.0371
-7.3074×10^{-5}	3.7498×10^{-5}			7.2116×10^{-6}	2.2191×10^{-5}
9.5057×10^{-6}	6.2114×10^{-6}			6.1119×10^{-7}	3.8797×10^{-6}
8[345], 12[486] 5.5710 5.6379	8[961] , 12[1370] 10.0208 10.1241			4[767], 12[1063] 2.7901 2.8222	4[1070], 12[1500] 3.8579 3.8800

15. $3_3[3^3]12_6[3^{12}]$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.5077 ± 0.0896	0.2640 ± 0.0333			0.8273 ± 0.0209	0.7269 ± 0.0313
4.5180×10^{-4}	2.5513×10^{-6}			1.3947×10^{-5}	-2.2183×10^{-6}
3.2202×10^{-4}	3.2552×10^{-6}			1.6592×10^{-6}	1.6715×10^{-6}
4[154], 4[223], 4.5065 4.7444 4[374], 12[520] 5.0214 5.1654	4[347] , 4[529] , 13.9135 14.2949 4[939], 12[1343] 14.7178 14.9248			2[359], 2[507], 2.8078 2.8402 2[827], 6[1133] 2.8730 2.8914	2[504], 2[720], 3.8333 3.8611 2[1188], 6[1638] 3.8889 3.9048

16. $4_4[3^4]7_2[3^7]_{\mathrm{I}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4774 ± 0.0624	0.3140 ± 0.0390			0.7305 ± 0.0263	0.6504 ± 0.0349
-1.6958×10^{-4}	3.5112×10^{-5}			7.4362×10^{-6}	1.7310×10^{-5}
4.8309×10^{-5}	9.6223×10^{-6}			1.0424×10^{-6}	3.0145×10^{-6}
8[403], 12[554] 5.4591 5.5379	8[1100], 12[1534] 10.0236 10.1213			4[1050], 6[1421] 2.4686 2.4828	4[1296], 6[1764] 3.8704 3.8889

17. $5_4[3^5]7_4[3^7]_{\mathrm{I}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.5112 ± 0.0412	0.3406 ± 0.0207			0.7029 ± 0.0507	0.6645 ± 0.0300
-5.3542×10^{-5}	2.1641×10^{-6}			-5.7814×10^{-5}	9.3355×10^{-6}
8.4833×10^{-6}	5.0113×10^{-7}			1.4169×10^{-5}	1.5205×10^{-6}
4[173], 4[450], 5.3179 5.5733 12[729] 5.6653	4[460], 4[1254], 9.4391 9.7895 12[2065] 9.9090			2[411], 2[1003], 2.7932 2.8654 6[1590] 2.8931	2[574], 2[1437], 3.8188 3.8859 6[2300] 3.9096

18. $3_1[4^3]5_1[4^5]_{\mathrm{I}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4003 ± 0.0534	0.2368 ± 0.0198	0.5786 ± 0.0400	0.4595 ± 0.0398	0.6184 ± 0.0356	0.5109 ± 0.0531
-1.4626×10^{-5}	-5.2284×10^{-6}	3.9800×10^{-5}	-2.9462×10^{-5}	-1.2175×10^{-6}	2.0112×10^{-5}
1.5617×10^{-5}	3.5502×10^{-7}	5.2273×10^{-6}	5.1123×10^{-6}	2.6728×10^{-6}	1.7919×10^{-5}
4[508] , 6[795] 7.37kk01 7.4943	4[1872], 6[2979] 14.1731 14.3411	4[508], 6[795] 3.7835 3.8264	4[961], 6[1521] 6.1748 6.2406	4[569], 6[871] 3.5993 3.6739	4[1024], 6[1600] 5.6738 5.7387

19. $3_1[4^3]5_1[4^5]_{\mathrm{II}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4065 ± 0.0646	0.2309 ± 0.0241	0.5898 ± 0.0633	0.4394 ± 0.0536	0.6423 ± 0.0381	0.5049 ± 0.0271
-9.8055×10^{-5}	2.2560×10^{-6}	-1.8568×10^{-6}	-1.3300×10^{-4}	-5.1659×10^{-5}	-9.8556×10^{-7}
3.7012×10^{-5}	6.6307×10^{-7}	3.3297×10^{-5}	3.6224×10^{-5}	7.7919×10^{-6}	8.8048×10^{-7}
4[512], 6[648] 7.6328 7.6728	4[1954], 6[2486] 14.2027 14.2912	4[512], 6[648] 3.8750 3.8889	4[992], 6[1260] 6.2016 6.2349	4[545], 6[685] 3.7578 3.7839	4[1024], 6[1296] 5.8145 5.8349

20. $5_3[3^5]8_6[3^8]_{\mathrm{II}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4631 ± 0.0471	0.3164 ± 0.0255			0.6800 ± 0.0384	0.6587 ± 0.0246
2.8969×10^{-5}	3.0266×10^{-6}			-9.9185×10^{-6}	1.5506×10^{-6}
1.3943×10^{-5}	1.2108×10^{-6}			4.4523×10^{-6}	6.3039×10^{-7}
4[260], 4[675], 5.4308 5.6474 12[1095] 5.7205	4[706], 4[1906], 9.8669 10.1815 12[3132] 10.2854			2[608], 2[1490], 2.8191 2.8846 6[2368] 2.9071	2[857], 2[2149], 3.8273 3.8920 6[3442] 3.9140

21. $5_3[3^5]8_6[3^8]_{\mathrm{III}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.5052±0.0474	0.3211±0.0274			0.7145±0.0299	0.6469±0.0242
-1.6178×10^{-6}	-9.4980×10^{-6}			8.1493×10^{-6}	-1.0841×10^{-5}
1.0710×10^{-5}	2.4172×10^{-6}			1.7096×10^{-6}	7.2095×10^{-7}
8[351], 12[814] 5.5328 5.6904	8[971], 12[2316] 10.0062 10.2366			4[804], 6[1782] 2.8458 2.8956	4[1144], 6[2580] 3.8531 3.9023

22. $5_2[3^5]12_{12}[3^{12}]$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4897±0.0718	0.3037±0.0257			0.7384±0.0258	0.6694±0.0297
-1.4823×10^{-4}	-1.3145×10^{-5}			3.3765×10^{-6}	1.5898×10^{-5}
1.0656×10^{-4}	1.4342×10^{-6}			8.4513×10^{-7}	2.4608×10^{-6}
4[192], 8[530], 5.1875 5.4981 12[803] 5.5890	4[498], 8[1457], 10.8072 11.2972 12[2244] 11.4332			2[442], 4[1156], 2.8371 2.8979 6[1724] 2.9165	2[627], 4[1675], 3.8405 3.9021 6[2514] 3.9204

23. $5_4[3^5]7_4[3^7]_{\mathrm{II}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.5001±0.0342	0.3327±0.0329			0.6973±0.0235	0.6688±0.0303
1.7585×10^{-5}	4.4678×10^{-5}			-7.2307×10^{-6}	2.4051×10^{-5}
3.4856×10^{-6}	5.2573×10^{-6}			5.9288×10^{-7}	2.0921×10^{-6}
8[371], 12[541] 5.5364 5.6155	8[1027], 12[1519] 9.7254 9.8328			4[835], 6[1194] 2.8623 2.8844	4[1195], 6[1722] 3.8661 3.8885

24. $5_3[3^5]8_6[3^8]_{\mathrm{I}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4867±0.0459	0.3265±0.0237			0.7108±0.0200	0.6720±0.0294
-1.0891×10^{-4}	-3.7362×10^{-6}			2.8929×10^{-6}	6.2044×10^{-6}
3.3218×10^{-5}	7.9808×10^{-7}			4.2792×10^{-7}	3.1424×10^{-6}
6[214], 8[535], 5.3925 5.6150 12[790] 5.6835	4[577], 8[1502], 9.8024 10.1278 12[2245] 10.2254			2[506], 4[1192], 2.8261 2.8859 6[1728] 2.9051	2[731], 4[1762], 3.9398 4.0204 6[2573] 4.0459

25. $4_2[3^4]12_6[3^{12}]$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.5117±0.0960	0.2637±0.0295			0.7861±0.0215	0.7273±0.0227
1.0580×10^{-4}	-4.6675×10^{-6}			7.1685×10^{-6}	1.1750×10^{-6}
2.1578×10^{-4}	1.9508×10^{-6}			4.3496×10^{-7}	7.1142×10^{-7}
4[200], 8[444], 4.6100 5.0495 12[629] 5.1924	4[461], 8[1121], 11.1714 12.1802 12[1633] 12.4850			2[474], 4[997], 2.8017 2.8646 6[1388] 2.8847	2[664], 4[1428], 3.8614 3.9048 6[2002] 3.9191

26. $4_2[3^4]18_{12}[3^{18}]$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.5249±0.1168	0.2493±0.0632			0.8497±0.0234	0.7743±0.0483
1.1603×10^{-3}	7.1820×10^{-4}			5.2173×10^{-6}	8.6384×10^{-5}
6.1479×10^{-4}	2.0594×10^{-4}			6.1183×10^{-7}	1.2578×10^{-5}
8[237], 4[428], 4.8608 5.1449 12[675] 5.3156	8[576], 4[1101], 15.0868 15.8656 12[1794] 16.3155			4[529], 2[929], 2.8696 2.9020 6[1441] 2.9216	4[774], 2[1376], 3.9664 4.0087 6[2150] 4.0344

27. $5_3[3^5]7_3[3^7]_{\mathrm{I}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4882±0.0494	0.3335±0.0310			0.7149±0.0273	0.6495±0.0278
-4.4634×10^{-5}	-1.2425×10^{-5}			1.6306×10^{-6}	5.9525×10^{-6}
1.5462×10^{-5}	2.4243×10^{-6}			9.8464×10^{-7}	1.4733×10^{-6}
4[170], 4[399] 5.3529 5.5739	4[455], 4[1112] 9.4110 9.7392			2[413], 2[907] 2.7797 2.8512	2[574], 2[1293] 3.8084 3.8716

28. $5_3[3^5]7_3[3^7]_{\mathrm{II}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4345±0.0489	0.2930±0.0243	0.4785±0.0354	0.3239±0.0187	0.6900±0.0283	0.6421±0.0289
8.3482×10^{-5}	-9.3058×10^{-6}	4.7566×10^{-5}	-4.1986×10^{-6}	-7.8267×10^{-6}	3.4481×10^{-6}
4.0009×10^{-5}	1.1454×10^{-6}	7.5326×10^{-6}	3.8717×10^{-7}	1.5860×10^{-6}	1.8973×10^{-6}
2[167], 2[61], 5.8443 5.2787 6[535] 6.2430	2[161] 6[1670] 9.8137 11.3257	2[167], 2[61], 5.7485 5.2131 6[535] 6.1234	2[480], 2[159], 10.4458 9.5849 6[1638] 11.0000	2[386], 2[161], 2.7720 2.6460 6[1124] 2.8665	2[535], 2[213], 3.8019 3.6901 6[1611] 3.8845

29. $3_2[4^3]5_2[4^5]_{\mathrm{I}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4099±0.0264	0.2271±0.0154	0.5687±0.0469	0.5028±0.0283	0.6228±0.0346	0.5022±0.0343
4.7650×10^{-6}	2.1524×10^{-6}	2.6238×10^{-5}	1.1978×10^{-5}	-1.1730×10^{-5}	3.4916×10^{-5}
1.0783×10^{-6}	1.2777×10^{-7}	8.5056×10^{-6}	1.1260×10^{-6}	5.5785×10^{-6}	3.2050×10^{-6}
2[277], 2[379], 7.3285 7.4248 6[497] 7.4970	2[1407], 6[1863] 14.0640 14.1857	2[277], 2[379], 3.7473 3.7836 6[497] 3.8109	2[519], 2[717], 6.0308 6.1004 6[947] 6.1521	2[325], 2[435], 3.5446 3.6046 6[561] 3.6506	2[576], 2[784], 5.6285 5.6811 6[1024] 5.7207

30. $3_2[4^3]5_2[4^5]_{\mathrm{II}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4322±0.0512	0.2344±0.0171	0.5638±0.0590	0.4749±0.0440	0.6260±0.0368	0.5468±0.0270
5.8373×10^{-5}	-1.5204×10^{-6}	4.8226×10^{-6}	-5.1510×10^{-5}	-2.9282×10^{-5}	8.1058×10^{-6}
1.9444×10^{-5}	2.2157×10^{-7}	1.5253×10^{-5}	1.0399×10^{-5}	6.7165×10^{-6}	1.6103×10^{-6}
2[276], 2[496], 7.1812 7.3831 6[780] 7.5051	2[1831], 6[2927] 14.1038 14.2890	2[276], 2[496], 3.6739 3.7540 6[780] 3.8026	2[507], 2[931], 5.9961 6.1267 6[1483] 6.2036	2[313], 2[545], 3.6422 3.7284 6[841] 3.7812	2[570], 2[1016], 5.6702 5.7520 6[1590] 5.8013

31. $3_3[4^3]5_3[4^5]$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.5029±0.0759	0.2504±0.0233	0.6238±0.0601	0.5029±0.0359	0.6645±0.0414	0.5801±0.0306
8.6335×10^{-6}	-3.9666×10^{-6}	-1.3226×10^{-4}	1.1319×10^{-4}	1.1038×10^{-4}	-3.3375×10^{-5}
1.4737×10^{-4}	8.6134×10^{-7}	3.8526×10^{-5}	1.9717×10^{-5}	1.6270×10^{-5}	3.6510×10^{-6}
4[161], 2[315], 6.9565 7.2317 6[473] 7.3700	2[1139], 6[1743] 13.8191 14.0425	6[161], 2[315], 3.5404 3.6698 6[473] 3.7294	4[285], 6[578], 5.8175 6.0104 6[882] 6.1020	4[200], 2[369], 3.3600 3.5122 6[539] 3.5918	4[336], 2[648], 5.4881 5.6204 6[968] 5.6880

32. $3_1[4^3]6_2[4^6]_{\mathrm{I}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4339±0.0259	0.2335±0.0200	0.5468±0.0321	0.4235±0.0152	0.6418±0.0638	0.5715±0.0431
4.6901×10^{-7}	-3.2315×10^{-6}	3.4990×10^{-5}	2.5190×10^{-6}	-1.1180×10^{-4}	4.9165×10^{-5}
6.9917×10^{-7}	3.6617×10^{-7}	4.5289×10^{-6}	1.5119×10^{-7}	4.6849×10^{-5}	7.7330×10^{-6}
2[760], 6[1190] 7.5263 7.6202	2[2860], 6[4534] 15.2503 15.4027	2[760], 6[1190] 3.8342 3.8672	2[1457], 6[2301] 6.7138 6.7718	2[833], 6[1281] 3.6879 3.7471	2[1536], 6[2400] 5.7513 5.8008

33. $3_1[4^3]6_2[4^6]_{\mathrm{II}}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4740±0.0681	0.2803±0.0276	0.6333±0.0534	0.4785±0.0260	0.6961±0.0552	0.6132±0.0465
-1.5376×10^{-5}	-1.1319×10^{-5}	-1.4476×10^{-4}	1.8163×10^{-5}	2.5890×10^{-5}	6.0868×10^{-5}
4.0570×10^{-5}	1.3703×10^{-6}	2.7149×10^{-5}	1.3662×10^{-6}	2.2393×10^{-5}	1.1922×10^{-5}
2[245], 2[477], 7.0531 7.3040 6[715] 7.4294	2[1742], 6[2656] 14.8576 15.0715	2[245], 2[477], 3.6408 3.7400 6[715] 3.7874	2[446], 2[892], 6.3184 6.5135 6[1354] 6.6041	2[284], 2[532], 3.5493 3.6541 6[782] 3.7136	2[504], 2[972], 5.6349 5.7243 6[1452] 5.7741

34. $3_2[4^3]6_4[4^6]$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4540±0.0687	0.2265±0.0369	0.6160±0.0685	0.4908±0.0346	0.6320±0.0532	0.5527±0.0434
7.8079×10^{-5}	-1.5535×10^{-5}	-1.8755×10^{-4}	-1.2315×10^{-5}	1.0557×10^{-4}	1.0264×10^{-5}
4.6624×10^{-5}	2.8752×10^{-6}	5.5163×10^{-5}	2.5752×10^{-6}	2.0627×10^{-5}	1.0088×10^{-5}
2[203], 2[440], 6.9754 7.2955 6[550] 7.3673	2[1605], 6[2026] 14.9533 15.0642	2[203], 2[440], 3.6059 3.7318 6[550] 3.7600	2[366], 2[821], 6.4645 6.6358 6[1034] 6.6731	2[238], 2[491], 3.5294 3.6660 6[607] 3.6969	2[420], 2[900], 5.8095 5.9689 6[1122] 6.0036

35. $3_3[4^3]6_6[4^6]$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4042±0.0580	0.2281±0.0277	0.5958±0.0567	0.4698±0.0313	0.6353±0.0449	0.5210±0.0370
3.6150×10^{-5}	2.6810×10^{-6}	-4.2796×10^{-5}	3.1642×10^{-5}	-2.0589×10^{-5}	7.9238×10^{-6}
1.8890×10^{-5}	7.5233×10^{-7}	1.4656×10^{-5}	4.6775×10^{-6}	8.7591×10^{-6}	2.4104×10^{-6}
2[230], 2[490], 6.9391 7.2735 6[656] 7.3720	2[1782], 6[2418] 14.8911 15.0488	2[230], 2[490], 3.5304 3.6816 6[656] 3.7256	2[406], 2[902], 6.5714 6.7095 6[1222] 6.7496	2[267], 2[543], 3.5955 3.7127 6[717] 3.7490	2[480], 2[1008], 5.6417 5.7500 6[1344] 5.7827

36. $3_1[4^3]8_4[4^8]$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4525±0.0780	0.2122±0.0230	0.5237±0.0627	0.3807±0.0401	0.7010±0.0470	0.6086±0.0679
7.1913×10^{-4}	-9.6553×10^{-6}	-1.0143×10^{-5}	3.5308×10^{-6}	-7.5034×10^{-5}	1.6014×10^{-4}
1.4678×10^{-4}	8.2295×10^{-7}	3.2798×10^{-5}	5.4378×10^{-6}	1.2495×10^{-5}	3.8208×10^{-5}
2[238], 2[490], 7.0336 7.3224 6[708] 7.4350	2[1794], 6[2632] 16.9197 17.1071	2[238], 2[490], 3.6555 3.7592 6[708] 3.7994	2[435], 2[921], 7.3931 7.5831 6[1345] 7.6550	2[288], 2[561], 3.4028 3.5651 6[793] 3.6318	2[490], 2[1000], 5.4898 5.6420 6[1440] 5.7014

37. $3_1[5^3]4_2[5^4]_\mathrm{I}$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.4047±0.0637	0.2185±0.0212	0.6180±0.0491	0.4930±0.0327	0.5516±0.0469	0.4101±0.0301
6.1235×10^{-5}	5.5289×10^{-6}	6.5660×10^{-5}	1.4895×10^{-5}	4.3305×10^{-5}	1.2816×10^{-5}
3.1306×10^{-5}	5.4838×10^{-7}	1.1080×10^{-5}	2.1733×10^{-6}	1.3841×10^{-5}	2.1196×10^{-6}
2[252], 2[570], 8 8.3298 6[777] 8.4247	2[2374], 6[3273] 16.0126 16.1577	2[252], 2[570], 3.7143 3.8105 6[777] 3.8378	2[468], 2[1086], 6.3932 6.5157 6[1491] 6.5500	2[281], 2[613], 4.4982 4.6591 6[827] 4.7062	2[632], 2[1428], 7.4525 7.6345 6[1946] 7.6865

38. $3_1[5^3]4_2[5^4]$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.3457±0.0484	0.1820±0.0200	0.6226±0.0443	0.5695±0.0477	0.5527±0.0393	0.4162±0.0419
8.5340×10^{-5}	-5.3816×10^{-6}	-5.2469×10^{-5}	-1.2139×10^{-4}	-2.9005×10^{-5}	-3.2254×10^{-5}
2.5074×10^{-5}	6.0154×10^{-7}	1.2122×10^{-5}	1.8902×10^{-5}	1.0732×10^{-5}	7.5538×10^{-6}
8[213], 2[380], 8.9202 9.1842 6[595] 9.3445	6[950], 2[1745], 16.8147 17.2218 6[2780] 17.4626	8[213] , 2[380], 3.1080 3.1632 6[595] 3.1966	8[331] , 2[601], 4.4411 4.5324 6[951] 4.5868	8[169] , 2[289], 4.2604 4.4291 6[441] 4.5351	8[360] , 2[640], 7.3444 7.5062 6[1000] 7.6040

39. $3_2[5^3]4_4[5^4]$

Cell	*Bond*	*cell*	*bond*	*vertex*	*edge*
0.3672±0.0405	0.1766±0.0243	0.6265±0.0398	0.5822±0.0505	0.5413±0.0515	0.4101±0.0536
-6.6453×10^{-5}	-3.2608×10^{-6}	7.1192×10^{-5}	2.3922×10^{-5}	-9.8715×10^{-5}	9.3126×10^{-5}
1.1174×10^{-5}	1.0035×10^{-6}	7.0957×10^{-6}	2.8516×10^{-5}	1.9319×10^{-5}	3.1517×10^{-5}
4[168], 4[270], 8.7143 8.9778 6[396] 9.1515	2[732], 2[1212], 16.5929 16.9868 6[1812] 17.2395	4[168], 2[270], 3.0357 3.0963 6[396] 3.1364	4[255], 2[418], 4.4078 4.4928 6[621] 4.5475	4[141], 4[216], 4.3688 4.4907 6[307] 4.5733	4[308], 4[485], 7.2662 7.4144 6[702] 7.5128

Table 4.7 *Percolation statistics of the 2-homohedral tilings. Each item of the statistics are, from top to bottom, mean ± standard deviation, second moment, and third moment. Similarly each of the items in each box contains the number of runs [the number of network components] and the coordination number.*

Let us venture to explain the information and results contained in Table 4.7. The easiest way to do this is to take only one case as an example. For this purpose let us choose the last item in the table, that is the edge percolation of the $3_2[5^3]4_4[5^4]$ tiling. If we read the tiling's title backwards, it tells us the following.

1. There are four vertices of valence 5 in each of the four four-cornered polygons.
2. There are three vertices of valence 5 in each of the two three-cornered polygons.
3. Consequently there six different polygonal prototiles. These include two that have three vertices, and four that have four vertices.
4. The valence of this homohedral tiling is five.

The polygonal prototiles repeat themselves through some non-rotational translations.

The percolation results can be read from top to bottom thus. The critical probability averaged from all the different simulations done is 0.4101, its variance 0.0536, the second moment 9.3126×10^{-5} and the third moment 3.1517×10^{-5}. Simulations have been done four times on a networks containing 308 edges (since we are talking about edge percolation.) The average coordination number of this network of edges is 7.2662. Four other runs have been done on another networks of 485 edges, and the coordination number in this case is 7.4144. Another six runs on another networks containing 702 edges the coordination number of which is 7.5128. Notice that the number of simulations run is always an even number as each run of the programme does two different simulations, that is one forwards and the other backwards. This numerous results tell us that the programme is easy to use and the algorithm reasonably efficient.

The advantage of sparse- over dense networks described by Burt (1992) is an idea similar to that of the benefit of decentralisation and local autonomy in politics. According to him, size still matters but the cost of maintaining the size is also important. The idea can be very general; dense networks are virtually worthless monitoring devices, while sparse networks give more information benefits. Taking opportunity costs into account, the rate of return of a dense network is lower than that of a sparse network. The idea can be applied to a strategic network expansion which is crucial for sales persons and jobs hunting alike.

Something similar to this comes up in a variety of fields. In Sociology it is sometimes called the *strength of weak ties.* A tie between two things is weaker the less each of them has in common with the other. It is well known, *i.e.* the origin unknown, that best friends are usually those who have least in common with each other. It would be interesting to trace this idea back to where it first appeared in literature.

The study above shows that the percolation probability is not a function of the coordination number of the network alone. I suspect that it is a function of both this and the coordination number of the dual lattice of the network, that is $p_c = \mathrm{f}(x_c, x_v)$. In other words, p_c may be a function of not only the connectivity but also the tortuosity of the network.

§ 4.10 Cosmology

The universe is made up of superclusters of galaxies, each of which contains local clusters. Each local cluster contains galaxies, each of which contains stars. When stars explode, gas bubbles are formed which expand to meet one another, forming walls of materials in the manner of the Voronoi tessellation in three dimensions. Within the planets around each stars there exist yet other structures which can be similarly represented by the Voronoi tessellation.

Superclusters are like endless cobwebs covering the space. The Andromeda galaxy, the Sculptor group, the Virgo cluster, and the M81 group are examples of the members of the Local Supercluster in which our Milky Way galaxy reside. The Pisces-Perseus supercluster is 250 million, and the Hercules supercluster 500 million light-years away from us.

Assuming the big bang origin of the universe, the defects originated in the phase transition at the Big Bang could play an important role in the formation of galaxies and their clusters and superclusters. Tom Kibble predicted this in 1976 (*cf* Croswell, 1995), and gave three types of possible defects, namely the point-like monopoles, the line-like cosmic strings, and the plane-like domain walls.

Van de Weygaert and Icke (1989) studied the statistics of 3-d Voronoi tessellations for the purpose of understanding the structure of the universe. Here *neighbours* are defined by a common polygonal face whereas *full neighbours* have an additional requirement that the line joining nuclei intersects this common face. Taking a linear section through d-d, $d \geq 2$, the mean length is

$$\langle\lambda\rangle = \frac{d\Gamma\left(d-\frac{1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)^2}{(d-1)!\rho^{\frac{1}{d}}2\Gamma\left(\frac{d}{2}+1\right)^{2-\left(\frac{1}{d}\right)}\Gamma\left(2-\frac{1}{d}\right)}, \qquad \mathbf{(20)}_{\mathrm{iv}}$$

where ρ is the number density, in other words nucleus density.

In three dimensions we need at least two points in order to establish the distance of an object by a geometrical method. Because the cosmic scale is so large compared with the scale of the solar system and Earth, all the observation points we may choose become the same and only one point in practice. Therefore the determination of the distance to extragalactic objects, which is crucial in cosmology, can only be achieved by the various means of observations from a single point in space. This is a limitation that has caused the establishment of distance to be a hot, long-standing controversial issue. As the universe is expanding with the receding velocity of objects increases with their distance away from us, and as this velocity can be accurately determined by the amount of redshift in the light of these objects observed, the study of distance becomes the study of a single, universal constant called the Hubble constant. This is only a constant in theory not in practice. All numerical values of the Hubble constant used in literature are fictitious to some degree. If its exact value is known, then the accurate distance to any cosmic object is simply its radial velocity, *i.e.* its velocity away from us, divided by this Hubble constant. The reciprocal of the Hubble constant is called the Hubble time. The importance of the Hubble constant, together with the difficulty in finding it, make the observational and the theoretical parts of cosmology inseparable from each other. They are also the cause of the proliferation of the modern literature on Astrophysics. One example of studies of the extragalactic distance scale is that reported by de Vancouleurs (1993) which includes among the objects studied the following objects belonging to the Local Group, namely the Large Magellanic Cloud, M31, M33, NGC 3109, as well as the objects beyond the Local Group, namely NGC300, IC4182, NGC2403, NGC5128, M81, M101, M104 (NGC4594), and NGC4571.

§ 4.11 CCTV, forest fire, the navy and porcupines

Voronoi graph can be used to aid the decision on the locations of Closed-Circuit Television (CCTV) cameras. CCTVs have proved to be very effective in reducing the number of crimes in Manchester as well as other places within the UK. In order to be effective, it is important to have no large gaps within a control area. Strategically placed, the cameras will give the best cost-effective solution while covering the largest possible area.

Within crucial areas cameras could be positioned in such a way that their ranges of operation intersect without gaps. The points of intersection represent the vertices of the Voronoi regions centred around two or three neighbouring cameras. The design objective is to produce the maximum covered area using the least number of cameras.

Outwards from each crucial area extends space not being covered by cameras. Similarly around other crucial areas surrounding this area also extend such unobserved space. The next step in the design is perhaps to make sure that people walking from one covered area to another are safe. To do this one can consider the whole covered areas as centres of yet another set of Voronoi tessellation orders of unit larger in size. The purpose now is to make sure to position cameras at the positions of vertices of the Voronoi regions of this second set of tessellation.

Both the first and the second tessellations will be subjected to geographical constraints. The former would be predominantly influenced by the shape and the position of buildings and surrounding structures, the latter by the paths connecting the covered regions. Here a vertex of these geographically-constrainted Voronoi regions could be a point which is equally far from three covered regions. Another set of positions to be considered for placing CCTVs is one of those points equally far from two covered regions along the shortest path between them.

At a graduate development programme on 16^{th} November 2001 I gave a presentation for ten minutes on a title *division of space.* In it I described the current project of mine as concerning porous media and division of space in general. The part presented is from a section for miscellaneous applications. Mathematical models of partitionings, for example the Voronoi tessellation pioneered by G. F. Voronoi and G. L. Dirichlet in the 19th century, help towards understanding physical Euclidean and non-Euclidean world. Procedures based on these models help solve many physical and strategical problems. When I was fourteen years of age I was trained in a course for those who volunteer to fight forest fires in the area around the Sudẹb mountain, close to where I used to live. Although I never used it afterwards, the knowledge obtained from that course has given me the interest in forest fire fighting.

To mention but one of the relevant topics contained in that course, one can imagine a forest fire station as being a nucleus of a Voronoi partition of area under the protection of that stations. Here the distance from a station to any point on the forest ground is not a straight line but the time it takes a fire fighter from that station to reach the point in question. This time is affected by the *tortuosity* of the path, in other words the degree of winding, as well as the difficulty of the climb. There normally is an observation tower at each fire station, so we may safely assume that every nucleus is equipped with one. Then the exact location of a fire observed can be located on the map from the intersection of lines of sight drawn from the neighbouring towers towards the direction of the fire.

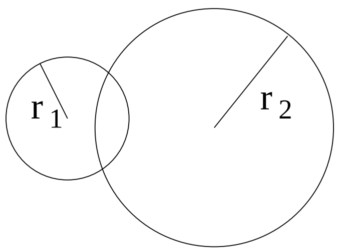


Figure 4.15. Range of observatory towers.

These lines of sight only give the position of the fire on the map. The distance measured along such a line does not necessarily correspond with the Voronoi distance which has a unit of time. The area covered by a tower depends on its height; a higher tower can report a fire occurring at a further distance. The area of covering is circular if the forest is a flat plain. But such a case is rare and in general this area is a distorted circle the degree of distortion of which depends directly on the contour of the land around it. In Figure 4.15 if h is the height of a tower and r its radius, then $r_1 < r_2$ implies $h_2 > h_1$. Suppose the dashed lines are boundaries of Voronoi regions, solid boundaries the observation range of the towers, and straight lines from nuclei the lines of sight. In Figure 4.16 the Voronoi boundaries are not straight lines as a result of tortuosity. Observations from only two towers suffice for finding location of the fire, but an observation from the third tower would reduce the probability of an error.

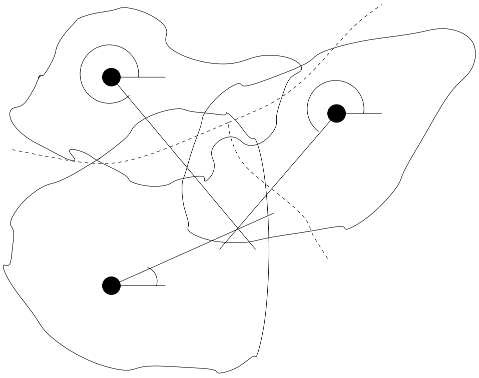


Figure 4.16. Tortuous boundaries.

I also talked about strategic locations. At a job presentation by Tesco we learnt that the company uses the Geographical Information System for placing their new stores in strategical locations.

The planning department at Tesco uses the so called the *porcupine diagram* which looks like the rays of light emanating from a star or the quills from a porcupine. The rays radiate from a supermarket and their length represents the drive times required to get to the store from the other end of the line. Factors affecting the locating of new stores include for example drive time, competition, logistic, parking, and traffic. The porcupine diagram superimposed on a map. Triangles are Tesco supermarket stores while circles are those belonging to their competitors. The spatial boundary of the Voronoi diagram is in general different from its drive time boundary, as shown in Figure 4.17 where broken boundary lines represent Voronoi partitions when the distance is the drive time while solid boundary lines that when the distance is the spatial distance.

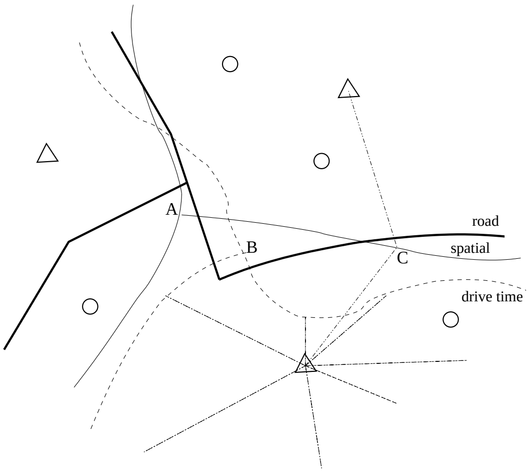


Figure 4.17. Porcupine diagram superimposed on a map.

A similar approach to the above can be applied to help locate the Closed Circuit TeleVision cameras in a strategic location. Suppose that now the triangles in Figure 4.17 are important places, for examle business centres, schools, or universities; let us call these *sensitive areas.* Then security cameras should be placed at both points A and B as well as at the sensitive places. Here the solid boundary lines meeting at the point A represent the Voronoi boundary for walk time instead of the actual spatial distance. Moreover, a camera should be placed at the point C which is the point where the shortest walkway linking two neighbouring sensitive areas meets with the Voronoi boundary of the walk time, in other words a point which lies midway along the shortest path that joins nextdoor sensitive areas. The reason for this is that these points are likely to be frequented by people going from one sensitive area to another, therefore they are prospectous for robberies. On top of that if a robber wants to flee from one sensitive area to another his sensible choice would pass through C while A with its connecting three different routes would be a clever choice to choose to confound a pursuer, or likewise B if he is driving.

At a police headquarter and at the city council office I wanted to know whether security cameras have been placed at locations analogous to these points A, B, and C, but was told that the information is sensitive. The answer is understandable because at the time M. Bush was trying to get his hand on M. Bin Laden and M. Blair had just declared Britain at war and alert. However this was not good for my theory since we had no way of knowing whether this approach is new, or whether it could be used to improve the security in the city. Hopefully most of these strategic locations A, B and C have already been covered by security cameras even if the decision to do so had been arrived at by some other theories different from the one I suggest. Unlike in the case of sensitive places, the cost of the cameras to be placed at these strategic places may have to be the responsibility of the city council because their importance is not immediately obvious owing to their locations. In contrast, most schools, universities or businesses should be willing to help with the cost of such installation from which they would directly benefit.

On one slide I explain how one can divide transformations into affine and non-affine ones. The affine transformations are governed by the equations $x' = ax + by + p$ and $y' = cx + dy + q$. As a conclusion, mathematical models are beautiful because they give us understanding, and they are useful because they stem ideas and procedures necessary for solving problems. The presentation has gone from forest fire fighting to crime prevention without my knowing it, so the future works are to collect real data and practices, to analyse and compare with the theory and simulations, and then to implement and monitor the operations which I think unlikely to happen because it is impossible for me to become a member of the Greater Manchester police force. Dr. Jim Boran who was present gave *good*'s for project understanding, overview and structure, summary, examples, use of colour, connection with drawings, enthusiasm, pointing at screen, and familiarity with the material. He wrote that the explanations were clear, the content technical but with good personal view, but that I looked at the screen too often and at times was difficult to hear. Bigger labels are suggested.

Naval forts are positioned with an idea similar to forest fire stations. Each fort forms a circle of effective defence around itself. These circles form a chain like beads on a necklace. One example (*cf* Petersburg, 2001) of this are the forts of Kronstadt near Kotlin Island in the Finnish Gulf, 20 kilometres from Saint Petersburg, which have been used as a base for the Baltic Fleet and is now controlled by the Russian Navy. Here seventeen forts had been built between 1704 and 1896 which form a guarding barrier for the city of Saint Petersburg. Examples of these forts are the Totleben Fort, founded 1886–1913, and the Alexander I fort, founded 1836.

In the study of traffic in § ℵ we consider the percolation of roads, which are edges of the structure. We can also use the same structure for the study of fire protection and evacuation in cities, in cases of disaster, but now it is instead the percolation of cells that interests us. Roads form effective fire barriers as well as divide the conflagration into partitions. By carefully managing these, for example putting more efforts on the key or critical partitions, the flames can be contained more quickly and the damage minimised. These zones are also important for evacuation planning, for example that made by the Greater Manchester Police. In Tokyo where there is always a fear of big earthquake, informations are given out in plenty to tell people what to do in case such a disaster as the Great Kanto Earthquake which occurred on 1^{st} September 1923 should struck. School grounds are usually assigned as evacuation centres. And as the distance one has to walk to get to one of these is crucial for one's safety, the evacuation zones assigned by the city councils there naturally divide the city into domains that approximate the Voronoi tessellation. In 1998 when I was in Tokyo the fear reached its height, since it was generally believed that big earthquakes in Tokyo occur at a period of approximately seventy-five years.

§ 4.12 Fractals

In doing simulations on filtering membranes, one usually assume that the blocking particles are all of the same size. Or one may apply more intuition and experience, and say that particles should have a normal distribution. Any mineral engineer will be able to tell that it is normally the case that the distribution of aggregates according to their sizes is normally normal. But one must keep in mind that the normal distribution here is by weight not number of particles. Therefore it is unlikely, except in a few special cases, that simulations which assume a normal number distribution of blocking particles will represent the real process.

A pile of aggregates with a normal distribution is likely to have a fractal number distribution (*cf* Liebovitch and Scheurle, 2000). Such fractal distribution is hyperbolic towards the lower size ranges. But the overall distribution is more complicated, that is to say, a mixture of normal- and hyperbolic distribution.

So far we have only considered percolation by competition between two phases. We have looked at networks in a cubic box and have assumed that the network is initially available in its entirety. But what if one has a competition of three phases or more? Or what happens when there are only parts of the network from the start? The picture of this can not be clearer than that of politicians arguing for votes, or the opinion polls of people across the social cross-section. Lustick and Miodownik (2000) consider this problem in the context of politics where the agreement clusters necessarily appear in various colours not merely black and white.

Applying this to percolation and a whole new dimension opens up for investigation. One possible approach is to consider only an arbitrary part of the network within our cubic box. Clearly this is only meaningful if the part being considered is itself a percolating cluster. Carry this one step further and we can have percolation within percolation, when each cluster within the network is itself subject to another percolative process, threatening to destroy it for instance. Such a dynamic scenario is a yet untread water which should open up much ground for investigation.

In the early days of the theory, literature on percolation only concerned itself with fluid wetting or blocking bonds or sites in the kind of situation that is said to be dual to the diffusion problem (Broadbent and Hammersley, 1957; Hammersley, 1957, 1961). The subject generally looked at self-avoiding walks and a measure of connectivity, the connective constant. From then on the appearance of the subject has somewhat changed, and it has become connected to Physics and fractals. The reason why there are very few works on percolation of a random network like the Voronoi tessellation is probably because there seems to be no needs for such study since the applications in Physics have told us that the regular lattices can represent the noncrystalline structure of the metal well. Moreover the only tools we have now, namely the Hamiltonian and power series method, can not be applied in its present form to random tessellations since there are no modulus relationships among the vertices.

5

§ 5. Porous media

Patrick *et al* (1999) study the propagation in random Delaunay lattice. Particle on arriving at a site in the lattice deflects over the largest possible angle to either the right or the left, depending on the right- or the left scattering nature of the site. After the particle has passed through the site the latter goes into the reverse state. When a similar study is done on the triangular lattice, the entire trajectory quickly becomes confined to a particular strip which is bounded by two adjacent parallel lines of the lattice. They explain the propagation as being due to a blocking mechanism which prevents the particle from moving in a direction opposite to the propagation direction for more than a few steps.

The flow of fluid in porous media follows the Darcy's Law which states that the rate of flow through such a medium is proportional to the potential energy gradient within that fluid. The constant of proportionality is the hydraulic conductivity, which is a property of both the porous medium and the fluid moving through it, $v = Q/A = -k\mathrm{d}h/\mathrm{d}l$.

The average velocity over the entire cross section is called the superficial velocity, $v_s = Q/A = \dot{m}/\rho A$. The velocity that is based on the actual open space within the porous media is called the interstitial velocity, $v_n = Q/(\varepsilon A) = \dot{m}/\varepsilon\rho A$, where ε is the porosity, $\varepsilon = V_v/V_s = (V_t - V_s)/V_t$, V_v, V_s and V_t are respectively the void, solid and total volume.

The definition of porous media is different from that of porous materials. A porous medium is a medium through which other substance may pass, whereas a porous material merely means some certain kind of material the internal structure of which is filled with pores. Most of porous materials can be used as porous media. All of them are useful because of their internal structure, and though we do not need to know this to be able to use them, we do have to understand it if we want to use them efficiently, or if we want to improve upon some of their particular properties.

The outer bark of Quercus suber L., for example, has a property that is ideal for its commercial use as cork. Its material property is such that it neither shrink nor expand when stretched or compressed. This makes it an excellent insulator of both heat and sound as well as good for damping vibration (*cf* Ashby, 1990). It has high frictional coefficient, and is both impervious to liquids and chemically stable. These properties make it ideal as wine corks. By understanding its internal structure and mechanism that makes it stay in the same shape when experiencing external forces, scientists have succeeded in making a synthetic material that shrink, instead of expand radially when pressed. When such material is used as cork, it can be easily pushed inside the neck of a bottle. When we release the force it will expand to its full size and fit snuggly where we placed it.

Pores introduced into a fabric can also increase its commercial value by giving it a lighter weight, higher heat retention rate, which makes it ideal for both ski and snow-board wearer and casual wear (JETRO, 2001).

§ 5.1 Zeolites

Zeolite is the name given to a group of minerals with a porous structure. It is used as molecular sieves as well as in many chemical engineering processes. It may be an important ingredient responsible in lubricating the fault lines in earth crusts, thus making earthquakes less severe. Evans *et al* (1995) found foliated cataclasite and ultracataclasite in the San Andreas faults. These are composed of clay and zeolite. In the case of the ultracataclasite, there are fragments of 20–100 μm diameter feldspar and quartz embedded in a matrix of clay and zeolite which has grains of sizes smaller than 10 μm.

The chemical composition of zeolites is that of sand, *i.e.* aluminosilicate. Their general formula is (Dana and Dana, 1997)

$$\mathrm{(Na, Ca, Ba)_{(1-2)}(Al, Si)_5O_{10} \cdot nH_2O},$$

or, as given by Gottardi and Galli (1985),

$$\mathrm{(Li, Na, K)(Mg, Ca, Sr, Ba)_d[Al_{(a+2d)}Si_{n-(a+2d)}O_{2n}] \cdot mH_2O}.$$

Normally $m \leq n$. They differ from sand in that they have large internal cavities. These cavities increase their internal surface area as a result of which they can be used as catalyst or molecular sieves with both shape- and size-selectivity. One example is the use of zeolite as sieve to separate iso octane, which has high antiknock property, and iso pentane from octane and pentane. Zeolites can not withstand high temperature because they rapidly lose water. Even though this process is reversible, their molecules collapse at temperature higher than 600°C. Around 41 types of zeolite occur in nature. Linde Division of the Union Carbide Corporation produced the first synthetic zeolites in 1950s.

The name 'zeolite' was coined by A. F. Cronstedt in 1756 from the Greek *zein*, to boil, and *lithos*, stone. The structural units of zeolite comprises of the primary building units of silicates in the form of XO_4 tetrahedra, where X is mainly Si. Other basic units are the chain of fibrous zeolites, the singly connected 4-ring chain, the doubly connected 4-ring chain, the 6-ring either single or double, the hexagonal sheet with handles and the heulandite unit.

§ 5.2 Crystalisation

In the diffusion theory of crystal growth, the overall rate of crystallisation is determined by the rates of two processes occurring one after the other, that is diffusion of solute from the bulk solution to the interface between solution and the crystal, followed by integration of solute atoms into the crystal lattice. The rate of diffusion of the former is $R_g = k'_d(c - c_i)$, while the rate of surface reaction or integration is $R_g = k_r(c_i - c')^n$, where c_i is the interfacial concentration. The value of the diffusion mass transfer coefficients k'_d when estimated from the growth is considerably different from the same value that is obtained from the dissolution experiment. But if we assumed that they are equal, then $R_g = k_r(\Delta c - R_g/k_d)^n$, which, when $n = 1$ gives $R_g = K\Delta c^n$ where $1/K = 1/k_d + 1/k_r$, and when $n = 2$, $R_g = k_d \left[(1 + k_d/(2k_r\Delta c)) - [(1 - k_d/(2k_r\Delta c)^2) - 1]^{1/2}\right] \Delta c.$, but there is no general solution for all n's (*cf* Garside and Mullin, 1968).

Particles in a crystalliser are kept in suspension by a stirrer or a pump. Lim *et al* (1999) study the effect of the attrition when crystals encounter with high speed impellers inside these apparatuses. When the impact energy exceeds the crystal strength, crystals fracture which gives rise to a particle size distribution as a result. Impact can occur at faces, edges or corners of crystals, but in the attrition model only the contact of a crystal corner with another flat, much harder object, for instance the steel impeller, was considered. Furthermore, crystals are assumed to have the shape of a cube or other polyhedra, but in the model the crystal faces forming the contacting corner are replaced by a cone having an included angle of 120°. In the vicinity of impellers the flow is turbulent and crystals travel in a random manner. Repeated attritions reduce the crystal size, assuming no competing effect of crystallisation. They assume a normal volume shape factor distribution and proceed to simulate according to a procedure recaptured here as Algorithm 5.1. The input to the algorithm is the impact energy W_p; the outputs are $w_f(J)$ and $c(J)$. The hardness of a solid is its resistance to local plastic deformation. The contact pressure of a plastically deformed cone is assumed to be the same as the Vickers hardness H_v. An isotropic material has two independent elastic constants, namely the shear modulus μ and the Poisson's ratio ν. All the other constants can be determined from these two quantities. Here α is the volume shape factor, Γ the fracture resistance, K_r the efficiency of stress field created by the impact between the crystal and the impeller, κ stress field parameter, $\kappa = 5$, μ_v quasi-isotropic shear modulus, N the total number of fragments, W_p impact energy, H dynamic hardness, $L_{\min}$ and $L_{\max}$ respectively the minimum and maximum size of a fragment, a characteristic size of the plastic zone, r the distance from the peak of the cone, $r_{\max}$ the maximum distance from the peak of the cone to the newly created surface and J the particle size classes. The values of H_v, μ_v and W_c used were that of magnesium sulphate heptahydrate and potash alum; r_1 and r_2 are random numbers.

Algorithm 5.1 *Fracture by attrition, Lim et al (1999).*

```
for $i = 1$ to 50 do
  $\alpha \leftarrow r_1$;
  $\Gamma/K_r \leftarrow \kappa W_c^{1/3} H^{5/3}/(5.2\mu)$
  $N \leftarrow 7 \times 10^{-4} W_p H^5 K_r^3/(\alpha\mu^3\Gamma^3)$;
  for $j = 1$ to $N$ do
    $L_{\min} \leftarrow 32\mu\Gamma/(3K_r H^2)$;
    $L_{\max} \leftarrow r_{\max}/2$;
    calculate $r_{\max}$;
    $r \leftarrow \exp[(13\log a - \log r_2)/13]$;
    $L \leftarrow 3\mu\Gamma r^4/(W_p^{4/3} H^{2/3} Kr)$;
    find $J$ corresponding to $L$;
    $w \leftarrow w + L^3$;
    $w(J) \leftarrow w(J) + L^3$;
    $c(J) \leftarrow c(J) + 1$;
  endfor
```

```
endfor
$w_f(J) \leftarrow w(J)/w$;                                  □
```

§ 5.3 Fluid flow within networks

Flow of viscous fluids through networks of geometrical objects is an important topic in various disciplines of engineering. Happel (1959) studies two cases of the flow of viscous fluid relative to arrays of cylinders, one parallel while the other perpendicular to the cylinders. This is found in practice as the flow through a bundle of heat exchanger tubes.

The flow pattern within a void of a porous media follows the equations of flows with vorticity, in 2-d,

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}=\zeta,$$

and in the axisymmetric case,

$$\frac{\partial}{r\partial r}\frac{\partial\psi}{\sin\theta\partial r}+\frac{\partial}{r^3\partial\theta}\frac{\partial\psi}{\sin\theta\partial\theta}=\zeta \qquad (cf\ Rowe, 1965.)$$

Or equivalently to both equations,

$$\zeta=\frac{1}{r}\left(\frac{\partial(rv_\theta)}{\partial r}-\frac{\partial v_r}{\partial\theta}\right)$$

when $v_\theta=\partial\psi/\partial r$ and $v_r=-\partial\psi/(r\partial\theta)$ in 2-d, and $v_\theta=\partial\psi/(r\sin\theta\partial r)$ and $v_r=-\partial\psi/(r^2\sin\theta\partial\theta)$ in the axisymmetric case. The vorticity, ζ, can be used as boundary conditions, for instance $\zeta|_{r=0}=-\partial v_r/a\partial\theta$ or $\zeta|_{r=a}=\partial v_r/a\partial\theta$. Or it can be used within the hole, for example $\zeta=k\sin\theta$ which gives the solution $\psi=(c_1r^{-1}+c_3r+c_4r^2)\sin\theta$, or $\psi=(c_3r+c_4r^2)\sin\theta$ if $c_1=0$. This solution is a part of the general polynomial for the stream function,

$$\psi=(\cdots c_0r^{-2}+c_1r^{-1}+c_2+c_3r+c_4r^2+\cdots)\sin\theta.$$

§ 5.4 Material science

A chiral is a group of points, or geometrical figure, whose mirror image can not be brought to coincide with itself. Achiral is the antonym of chiral. A chiral object is an object which fails to be achiral. Chirallity is a purely geometrical property since all the operations involved, namely the plane symmetry of the reflection, the rotation, and the translation, are all isometries. Chiral is used in the studies of molecules and knots.

The Poisson's ratio, ν, is the ratio between the transverse contraction strain and the longitudinal extension strain, that is $\nu=-\varepsilon_t/\varepsilon_l$. The theoretical value of the Poisson's ratio can range from -1 to 0.5, but for normal materials it is generally positive. At $\nu=0.5$ the bulk modulus is much greater than the shear modulus and the material is incompressible, while at $\nu=-1$ the opposite is true and the material is very tough and highly compressible. The bulk modulus B, and the shear modulus G are related to each other by the equation $B=2G(1+\nu)/(1-2\nu)$. For rubber this value is 0.5, for aluminium 0.33, while for cork it is approximately zero. Materials with a negative Poisson's ratio have been found whose structure is re-entrant (Lakes, 1987). Applications of such materials include robust shock absorbing material, fasteners, and stoppers of the wine bottles.

Chemical reactions can be thought of as phase changes in percolation. For example the extraction curves for Sb shown in Tiyapan (2003, KNT8(iii), § E.1). These curves represent an *s*-shape starting from one phase, represented by 0% Sb, to another at the maximum per cent extraction where it saturates. All the per cent extraction graphs shown in Tiyapan (2003, KNT8(iii), § E.1) show them with high slope at $t=0$. This is only for the convenience of drawing, since it is difficult to draw curves with an *s* shape smoothly on the computer. Also, the data obtained from the experiments do not extend to the time immediately following $t=0$. Extracting the solution using pipette takes some time to do, and therefore it has not been possible to prove experimentally whether the extraction yields start off with zero slope at $t=0$ or not, even though one might conjecture that this is likely to be the case if one considers the extraction yield in the leach solution as a developing phase in the continuum of the solution.

Particle sizes come in a variety of definitions. Svarovsky (1977) divides them into three groups, namely definitions by equivalent sphere diameters, by equivalent circle diameters and by statistical diameters. These are listed together in Table 5.1.

diameter	*criterion*
volume diameter	equivalent volume of sphere
surface diameter	equivalent surface of sphere
surface volume diameter	surface to volume of sphere
drag diameter	resistance to motion of sphere in the same fluid at the same velocity
free-falling diameter	free-falling speed of sphere, same fluid and particle density
Stoke's diameter	free-falling speed of sphere if Stoke's Law is used ($Re < 0.2$)
sieve diameter	diameter of sphere passing through the same square aperture
projected area diameter	projected area of a circle, the particle resting in a stable position
projected area diameter	projected area of a circle if the particle is randomly oriented
perimeter diameter	perimeter of the outline of a circle
Feret's diameter	diatance between two tangents on opposite sides of the particle
Martin's diameter	length of the line which bisects the image of the particle
shear diameter	particle width obtained using an image shearing eyepiece
maximum chord diameter	maximum length of a line limited by the contour of the particle

Table 5.1 *Particle size definitions.*

Particle size distribution comes in four types, distribution by number $f_n(x)$, by length $f_l(x)$, by surface $f_s(x)$ and by mass or volume $f_m(x)$. Conversions among them are done by $f_l(x) = k_1 x f_n(x)$, $f_s(x) = k_2 x^2 f_n(x)$ or $f_m(x) = k_3 x^3 f_n(x)$. Conversion is only possible if we know the shape factor's dependence on particle size, because k_i often contain a shape factor. The distribution frequency is by definition $\int_0^\infty f(x)\mathrm{d}x = 1$.

$$\begin{tikzcd}
(m, \mathrm{d}f(x)/\mathrm{d}x, Q) \arrow[r, "\text{feed}"] & \text{separator} \arrow[r, "\text{overflow}"] \arrow[d, "\text{underflow}"] & (m_f, \mathrm{d}f_f/\mathrm{d}x, O) \\
& (m_c, \mathrm{d}f_c/\mathrm{d}x, U) &
\end{tikzcd}$$

Figure 5.1 *Schematic diagram of a separator.*

Small particles tend to flocculate with one another. This makes it impossible to do experiments with very fine powder. Therefore any experiment which claims too fine a size as a control parameter, for instance in the micro metre range, must bear in mind that particles of such small size ranges tend to cluster into hard agglomerates bound by strong chemical bonds, for example from a previous chemical or thermal treatment, or into soft agglomerates by van der Waals attraction or by capillary forces. In ceramic making, this results in nonuniform packing during forming which leads to large voids or flaws after thermal processing. As one expert put it, you can not have powders in the size range of microns because they will disappear instantly into the air. One way to deal with them is to mix them into slurry.

Powder is also subject to impurities from the earliest stage of its life, that is during the comminution, which can be done by various methods, for example mortar and pestle, ball milling, jaw crushers or crushing rollers. The impurity comes from abrasion of the grinding media, which can be effectively avoided in the case of the jet mill, where particles are driven by air streams in opposing directions to collide with one another among themselves. Powder can be characterised by its size distribution, surface area, shape, composition and crystal structure.

Particle size distribution can be determined by means of Stoke's law, screening and microscopy. Stoke's law states that the steady state velocity of a spherical particle travelling through a fluid is $v_r = 2(\rho_p - \rho_f)ar^2/9\eta$, where a can be due to gravity, in which case $a = 9.8$ m^2/s, or alternately due to centrifugal force, $a = \omega^2 x$. The time required for a spherical particle of of radius r to travel a depth x is $t_r = x/v_r$. No particles of radius larger than r will be found at depths less than x at time t_r. The fraction extracted mass at short time intervals in an experiment using a graduated cylinder is constant, $\pi R^2 \int_{-k}^{0} f_o \mathrm{d}x = m$.

The equivalent spherical radius is the radius of a solid sphere that has the same steady state velocity in the slurry as the particle. The volume distribution function is $f_v(r) = \mathrm{d}V/(V_\mathrm{T}\mathrm{d}r)$. The number of particles in the powder having radius r is $N(r) = 3V_\mathrm{T} f_v(r)/4\pi r^3$, the mode of the distribution is the peak and the mean radius is $\int_0^\infty rN(r)\mathrm{d}r / \int_0^\infty N(r)\mathrm{d}r$.

When finding particle size distribution using screening methods, sieves are arranged in a stack with their apertures decreasing from top to bottom. Mass fractions retained on each of the screens is $f_i = m_i/m_\mathrm{T}$, which can then be converted to $N(r)$.

Microscopy methods use a variety of microscopes, for instance optical, SEM or TEM, depending on the magnification required. Measuring the dimensions and determining the shapes of the particles can be done manually with a ruler, or by using an image analyser.

Powder surface area is determined by gas adsorption methods. The specific surface area, *i.e.* surface area per unit mass, can be found by $s = N\sigma V_m/V_o$, where N is Avogadro's number, σ_0 the effective cross sectional area of the adsorbate molecule, V_m the volume per gram of a monolayer at STP and V_0 the STP number, 2.24×10^4 cm^3/mole.

Monolayer adsorption is used when molecules strongly interact with a surface to in a uniform monolayer. The Langmuir equation is $\theta = kP/(1+kP)$, where $\theta = V/V_m$ is the fraction of surface covered and K a constant. Then $P/V = 1/(KV_m) + P/V_m$ and the Langmuir isoterms are plots between V and P/P_0, P_0 being the saturated vapour pressure. For multilayer adsorption, $x/\left[V(1-x)\right] = 1/(cV_m) + x(c-1)/cV_m$, where $x = P/P_0$.

Permeametry measures the flow of gas under a pressure head through a packed bed of particles. The Darcy's law of laminar flow through porous medium is $u = (1/A)(\mathrm{d}V/\mathrm{d}t) = K\Delta P/\ell$. The Poiseuille equation for a fluid flowing through a tube of radius r is $Q = r^4\Delta P/(8\eta\ell)$, where η is the viscosity. It is also written as $h/L = 32\eta v/(gd^2)$, where h/L is the headloss per unit length, η the kinematic viscosity (m^{2}s^{-1}), $\eta = \mu/\rho_w$, ρ_w mass density of the water (kg$\cdot$m^{-3}), μ absolute viscosity of water (kg(ms)$^{-1}$) and d the diameter of the particles. The hydraulic radius is the ratio between the wetted area and wetted perimeter, $r = (\pi d^2/4)/(\pi d) = d/4$. Therefore, $h/L = 2\eta v/(gr^2)$. The number of particles is $n = V_\mathrm{T}/V_i$, and the total wetted surface area of solids is $A_\mathrm{T} = nA_i$. Therefore $A_\mathrm{T} = (1-\varepsilon)(\pi d^2)/(\pi d^3/6) = 6(1-\varepsilon)/d$. Then, replacing the factor 6 with the shape factor s, $r = \varepsilon d/[s(1-\varepsilon)]$. Because $v_pA_p = v_fA_f/R$, where p and f means pore and filter surface respectively, R is a constant which accounts for friction loss, and also $J = 1/R$, therefore we have $h/L = 2\eta v_p/(gr^2) = J\eta(1-\varepsilon)^2 v_f s^2/(g\varepsilon^3 d^2)$.

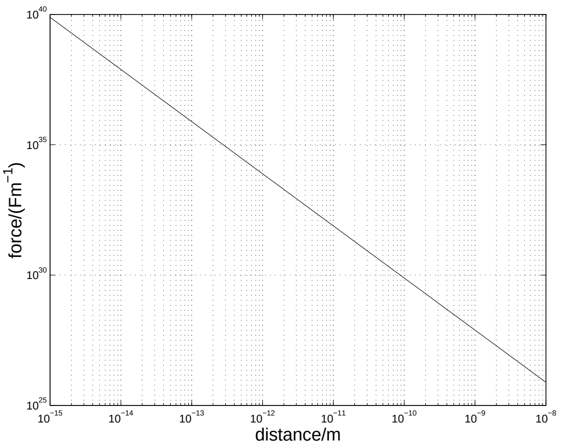
Kozeny used the ratio $V/A = d/4$ for a cylinder and the porosity $p = V_v/(V_v + V_s)$ and gavethe equivalent cylindrical diameter for a packed bed as $d_e = 4pV_s/\left[A(1-p)\right]$. Then the average velocity through the channels is $u_p = \mathrm{d}v_p/\mathrm{d}t = \left[(V_s^2\Delta P)/(2A^2\eta l)\right](p/(1-p))^2$, where l is the channel length. The Carman-Kozeny equation accounts for the tortuosity of the channels in the bed, $s^2 = 2.5p^3\Delta P/(2\rho\eta l v(1-p)^2)$, where $V_s = m/p$, the specific area $s = A/m$ and ρ the solid density of the powder. The determination of the phases and composition of a powder is done by x-ray powder diffraction, thermogravimetric analysis and differential thermal analysis.

Wet methods are preferred when preparing material before firing in order to control agglomeration during forming, for otherwise when the particles are agglomerated the narrowest particle size distribution becomes of little or no help. In wet forming, solid particles are suspended within a liquid the chemistry of which can be adjusted to make them mutually repulsive or mutually attractive, for respectively deflocculation and flocculation. Stabilising makes particles mutually repulsive either by adsorbing polymer chains on to the particles in steric stabilisation or by putting charged ions or polar molecules on the particle surface in electrostatic stabilisation. In steric stabilisation, one end of the long polymer chains, which is hydro- or lyophobic, is adsorbed on the particle while the remaining end, which is hydro- or lyophilic, extends in to the liquid.

Electrostatic stabilisation results in diffusive double layer built around particles. Zeta potential measures the repulsive potential between particles travelling in a medium. It defines the electrical potential in the double layer at the surface of shear slippage when the particle is forced to travel through the fluid when an electric field is applied. The composition and thickness of the double layer can be changed by adjusting the pH of the suspension. The point where the pH gives a zero zeta potential is called the isoelectric point. It is a point where spontaneous agglomeration occurs. The electrophoretic mobility $\nu_e = u(E)/E = \zeta\epsilon_r\epsilon_0/(f_h\eta)$, where u is a steady state velocity, f_h a constant determined by the dispersing medium and the particle size and $1 \le f_h \le 3/2$.

§ 5.5 Forces between particles

Essential to the simulation of stochastic models of particles is the consideration of forces acting on each particle (*cf* Schumacher, 1996). Since in theory the forces acting between two particles extend their effects to infinity, one has to make approximations. The degree of justification to these approximations depends on how the force in question vary with distance.



In the case of the Coulomb force where the potential energy is $u(r) = q_1q_2/(4\pi\epsilon_0\epsilon_r r)$ and the force $f(r) = -\mathrm{d}u(r)/\mathrm{d}r$ is

$$f(r) = q_1q_2/(4\pi\epsilon_0\epsilon_r r^2),$$

the justification is high as is seen in Figure 5.2 where the force-distance is a decreasing straight line on the log-log graph, q_1 and q_2 are protons, and r the distance in metre.

Figure 5.2 Coulomb force *vs* distance in log-log scale..

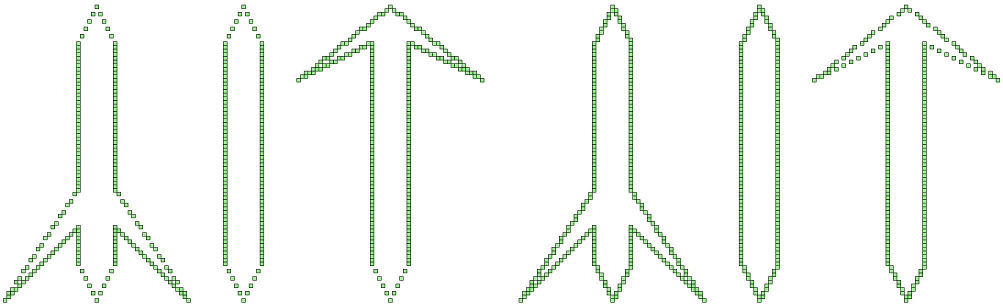
The effects of the Coulomb force are prominent when sizes of the particles are small. In quantum mechanics where the scale is atomic, for instance, the problem becomes one in which there are a large number of particles interacting with one another, *i.e.* an n-particle problem. In theory the Hamiltonian operator can be applied and then the problem solved numerically. But in general this is not possible due to the too many particles involved in the calculation. Theoretical solution is possible by the various methods of approximation, for example the Hartree method (*cf* Brown, 1972) where the best wave function is found in terms of the one electron function, *i.e.* orbitals, Φ, or a Hatree-Fock approximation which reduces the number of equations to $n/2$.

§ 5.6 Arbitrarily shaped particles

An arbitrarily shaped particle in a divided space will necessarily have a harder time travelling around compared with a sphere even if, or rather especially when the space is a mathematical one. Small particles are roughly spherical, or so the lore of science says, but when one's technology has led one down to the realm of dimensions in the order of those small particles, for instance the nano technology, then shape does matter a great deal and the universal spherical assumption can no longer suffice.

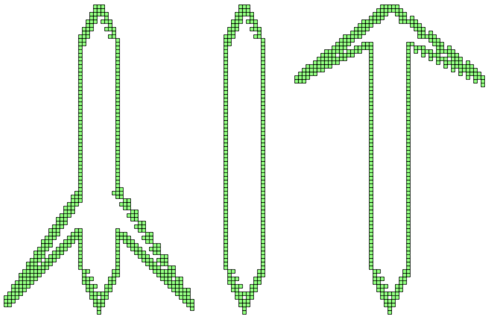
This is only one of the reasons which justify the investigation into the cases where the ratio of surface area to volume is not a minimum. When physics and motion are involved, this kind of study can incorporate both the continuous trajectory calculation as well as the discrete raster grids of the digital visualisation technology. The best way to hold such data is as matrix maps or masks.

The best and quickest way to define the boundary of an irregularly shaped object while allowing ease of investigation of its interaction with the surrounding is as raster outlines. To do this, I first investigated such outlines computed from the equation of straight line segments which make up the boundary of the object. Hereafter object boundaries are defined without loss of generality as straight line segments that link all the vertices together with no gaps. The first investigation is unsuccessful as it violates this definition by leaving many gaps, as can be seen in Figure 5.3 (a). The second algorithm calculates the grids from the coordinate axis along the direction of which the slope of the edge is minimum. But this too still leaves gaps along the boundaries.



(a) (b)

The object shape is defined by a set of landmark vertices or corners. This must be listed in intended order, otherwise it would be impossible to deduce the order intended unless one assumed that the object is convex. Figure 5.3 is an example of irregular object shapes, the name *Kit* in Anglo-Saxon's runic script (*cf* Freeborn, 1992) created by three different algorithms.



(c)

Figure 5.3 *Kit* the existential object.

There are three different programmes in § A.24. Only the third one, `tioa3.m` gives a satisfactory result which leave no holes in the surface of the objects. In the second programme, `tioa2.m`, `v` is a data set which contains the number of vertices, the vertices, slope, map matrix and the object's dimension. In `tiao3.m`, `v` is somewhat different. It contains the number of vertices, the vertices in real dimension, the grid dimension of the box containing the object, the vertices in grid dimension and the matrix map or image of the object. In order to exploit possible parallelisation, the last programme ignores the warnings when there is a division by zero as this means that all the grid positions of the whole line can be operated upon together as a vector quantity.

The third programme operates as Algorithm 5.2 does. It utilises the same idea as that used by a child or an artist alike when they draw or paint. Drawing and painting are both one-dimensional process which seeks to produce two-dimensional results. Each pencil- or brush stroke travels along a path the direction of which has one dimension. Unless he uses a very thick brush, there will always be a possibility of gaps forming between stroke lines, which can be annoying because however small they may be one needs to use a great deal of paint in order just to cover them up. This problem can be overcome by painting along two directions, for instance perpendicular to each other especially when the person concerned is a child. At a first glance, or to a novice, this may seem a wasteful practice, especially with dear types of ink. But a little practice and experience will show that this proves in most cases to be more economic than painting along only one direction. And since it can be readily seen that painting in two directions perhaps already uses twice the amount of ink required to cover the paper, it follows that doing so in one direction only would use a great deal more ink than this.

Algorithm 5.2 *Particles or objects with arbitrary shape.*

```
$(i, j)$ ← find grid coordinates for $(x, y)$;
find all vectors linking vertices;
$\{p_i\}$ discretise these vectors into sets of points;
for all intervals between consecutive $p_i$'s do
    draw along $y$-direction to fill gaps;
    draw along $x$-direction to fill gaps;
endfor
```

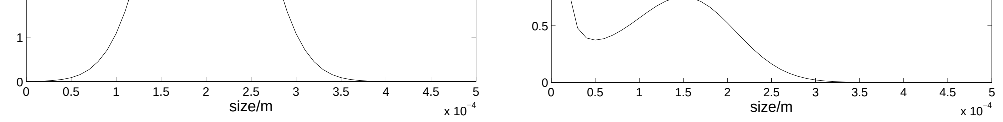
□

The problem of texture quantification is in the difficulty in expressing such quantity by a single parameter. Properties such as coarseness, smoothness, heterogeneity and regularity are finger prints of particles, and are the result of physical and chemical processes. By using fractal geometry, it is possible to describe quantities like texture. A fractal is a set whose metric properties can only be consistently illustrated with a dimension $D > T$, where T is the standard topological dimension. We can express this dimension as $D = T + (1 - H)$, where H is the codimension. Curves can have their roughness described. By giving a fractal number between one and two the space filling ability of the curves is established (*cf* Van Put *et al*, 1994). The first method defines the fractal dimension of a function in terms of the Fourier power spectrum $P(\omega) \approx \omega^{-(2H+1)}$; H and D are determined by doing a linear regression on the log-log plot of the observed power spectrum as a function of frequency. Another method defines fractal dimension in terms of how the variance of interpixel differences changes with distance; D is estimated from a log-log plot, a variogram, of the variance of increments versus increments $\sigma^2(x) \approx x^{2H}$.

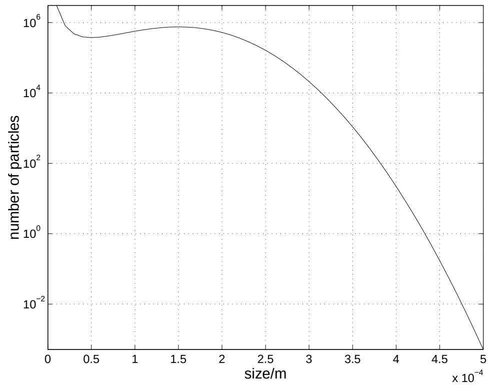
Techniques used for measuring sizes of particles include sieving, microscopic analysis, electronic particle counters, laser diffraction analysis, permeability-, sedimentation and elutriation methods. Usually arbitrarily shaped particles are characterised by a variety of methods. For a single particle this usually means transforming its property to the corresponding value of a sphere. A particle may be represented by a sphere which has the same volume, surface area, surface area per unit volume, or the area projected perpendicular to the flow direction. It may be compared with a sphere which has the same settling velocity in the same fluid, or a sphere which will just pass through a square aperture of the same size. Or its area projected on to the ground, when it is resting in the position of maximum stability, may be compared with that of a sphere in the same situation. To be consistent in one's choice is more important than which choice one chooses from. The sphericity of a particle is defined as $\Psi = A_V^s / A_p$, where A_V^s is the surface area of sphere which has the same volume as the particle and A_p surface area of the particle itself (*cf* Coulson *et al*, 1991).

§ 5.7 Non Poisson number distributions of particles

Let us look at an aggregate of spherical particles whose average diameter is 200 ± 50 μm, density $\rho = 3.7$, and the total weight one gramme. Assuming the gravitational constant to be 9.81 $\mathrm{kg \cdot ms^{-2}}$.



(a) (b)



The overall weight is $W = n \cdot w$, where $w = \rho V g$ is the weight of each particle. The particles being spherical their individual volume is thus $V = (4/3)\pi r^3$ and their distributions by weight and by number are shown in Figure 5.4. Figure 5.4 shows (a) distributions by weight, (b) distribution by number in normal scale and (c) with log scale in the y-axis.

Figure 5.4 Distributions by weight and by number.

In particle size classification the separation size, D_{50}, is the size where there is an equal chance of particles being reported to the fine or coarse fraction. The grade efficiency curve is normally plotted between the weight fraction to coarse product and the normalised particle size D/D_{50}. The choice of classification equipments depends on many factors, for instance the size of the particles in question and their electrical or magnetical property. For fine particles, Treasure (1965) discusses several types of classifier, namely the solid-bowl centrifuge, the Hosokawa micro-separator and the Head, Wrightson air classifier all three of which have $D_{50} \propto \sqrt{q}/\omega$, the hydrocyclone where $D_{50} \propto 1/q^{0.5}$ and $q \propto p^{0.5}$, and the Alpine Mikroplex classifier where $D_{50} \propto u/(w\sqrt{V})$ where u and w are respectively the radial and tangential velocities and V is the volume air flow rate, and $D_{50} \propto 1/w$ in practice.

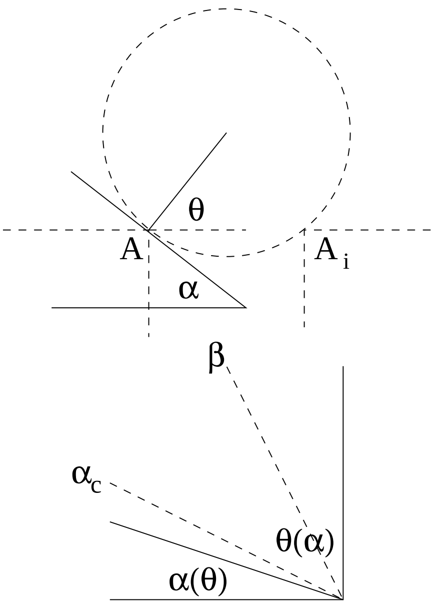
The distribution of bubble size in gas fluidised beds had been represented by various kinds of distribution, for example the log normal, Gamma and χ^2 distributions. Rowe and Yacono (1975) preferred the gamma distribution of the volume, that is $\Gamma(V|m,n) = [1/(n^m\Gamma(m))]\, V^{m-1}\exp(V/n)$, in its normalised form $\Gamma(v) = \left[m^m \exp(-mV)V^{m-1}\right]/\Gamma(m)$.

Sieve testing is affected to blinding materials, that is parts of materials under test which lodge themselves in the apertures of the sieve. To account for this Rose and English (1973) give a general rule of thumb which says that a particle for which $\theta < \tan^{-1}\gamma$ will blind the sieve. Here θ is the angle that the line of reactive force exerted by the material on the particle, which is normal to the contact surface, makes with the horizontal plane. Then they give the empirical value of γ to be approximately 20°, which results in $\theta < 0.3358$ radians as being the criterion for blinding to occur. For a spherical particle this gives the ratio $x/r = \cos\theta > 0.9441$, where r is the particle's radius and x half the minimum distance between two normal reactions which act on the particle. Therefore the radius of particles which blind the sieve must be such that $r < 1.0592x$, which corresponds to the aperture of 1.0592 times greater than the aperture if we assume that the aperture is $2x$ through out, or equivalently that the opening has its corresponding faces vertical.

It would be of a theoretical interest to prove this equation $\theta < \tan^{-1}\gamma$ given by them. Also, the angle of friction, γ, is not a constant but varies for different pairs of materials in contact with each other, as well as changes the value when we change the medium surrounding them to a different type, or even when the moisture content changes for that matter. All of this goes to say that the value $1.1d_{\min}$ extensively used in chemical engineering literature is probably nothing more than an engineering rule of thumb based on two conjectures, one in the limiting value of θ, the other in the value of γ. But it gives a simple and convenient criterion for use when doing computer simulations, which are nothing but means to visualise the *conceptualised* physics anyway. Other values of $\theta_{\max}$ should give the same qualitative result. We may, for instance, choose the coefficient of our $d_{\max}$ to be 1.02, corresponding to $\theta > 1/\cos 10°$, instead of 1.06 which corresponds to the $\theta > 1/\cos 20°$ used above.

Since no explanation of the formula regarding θ given by Rose and English has been given elsewhere, I arrive at my own derivation as follows. First, let us recall from our sixth form physics how friction can be described by either the coefficient of friction or the angle of friction, both of which are merely different sides of the same coin. The angle of friction is also known as the angle of frictional resistance, the internal angle of friction and the angle of shearing resistance. For clean sand it is approximately the angle of repose. It reduces with moisture content and is zero for saturated clay. For solid it is approximately the angle of inclination of the surface of one material at the point when a block of another material, placed on top of the former, starts to slide. This is precisely the method by which the coefficient of friction is determined. There are two different coefficients of friction for each pair of material, namely the static- and the kinetic coefficients of friction, the former having a slightly higher value than the latter, and both are defined as being the ratio between the limiting friction and the normal reaction, $\mu = f_k/F_N$. This gives rise to the formula for the friction being $f_k = \mu F_N$, which is equal to $mg\sin\theta$ at the point when the object of the second material slips, θ being the inclination of the plane made by the first material. Therefore we have $\mu = \tan\theta$, where μ is the coefficient-, and θ the angle of friction.

Coming back to our formula in question, $\theta < \tan^{-1}\gamma$. Figure 5.5 shows a spherical object sitting on top of an aperture in a sieve. From Figure 5.5, $\alpha + \theta = \pi/2$. Without loss of generality, assume $\gamma < \pi/4$. Let $\mu = \tan\alpha_c$ be the coefficient of friction at some critical value α_c. Furthermore assume that α_c is small, so that we can approximate μ by the angle of friction γ. Then we have $\alpha_c = \tan^{-1}\mu = \tan^{-1}\gamma$. It follows that the particle will block the pore whenever $\alpha < \alpha_c$, whereas it will pass through the latter when $\alpha = \pi/4$.



Because we know from experiments that some of the particles does lodge against, and blind the pore, there must exist some $\alpha_c < \beta < \pi/4$ such that this occurs. For the reason that the number of particles that blind is empirically small we know that β is small. Since we already have the relationship among α, θ and $\pi/4$, namely $\alpha + \theta = \pi/4$, and since α_c is also small from our assumption, then in order not to unnecessarily introduce another parameter into what necessarily already contains some error due to all the approximations so far made, we let $\beta = \alpha_c$. Now we may say that the particles will blind if and only if $90° - \alpha_c = 90° - \beta < \alpha < 90°$. For other remaining values of α, *viz.* both $\alpha_c \leq \alpha \leq 90°$ and $\alpha < \alpha_c$, in other words $0 \leq \alpha \leq 90° - \alpha_c$, then, particles must block. But $\theta = 90° - \alpha$, therefore the particles pass through when $\theta = 0$, blind when $0 < \theta < \alpha_c$ and block when $\alpha_c \leq \theta \leq 90°$. So the blinding particles have $\theta < \alpha_c = \tan^{-1}\gamma$ and the approximation is explained though not *q.e.d.*'ed.

Figure 5.5 Spherical object on a sieve.

Let A_a be the available area free from blinding material at some instant, A the physical area of the sieve cloth, G the area of cloth blinded by a unit mass of blinding material, and w_0 and w the mass of blinding material on the sieve respectively at $t = 0$ and at some $t > 0$. Then $A_a = A - G(w_0 - w)$ and $\mathrm{d}w/\mathrm{d}t = -k_1 w A_a$, where k_1 is the diffusion coefficient, and then $\mathrm{d}w/\left[w(b+w)\right] = -a\mathrm{d}t$, where $a = -k_1 G$ and $b = (A/G) - w_0$. Assume $b \neq 0$. At $t = 0$, $w = w_0$. It follows that

$$w = \frac{b w_0 \exp(-abt)}{b + w_0(1 - \exp(-abt))}. \qquad \mathbf{(21)}_{\mathrm{v}}$$

Furthermore, let W be the total weight of material on the sieve, W_∞ the residue on the sieve for a theoretically infinite time of sieving, and K a constant. Then, $\mathrm{d}W/\mathrm{d}t = -K(W - W_\infty)A_a$, and it follows that $\mathrm{d}W/(W - W_\infty) = -KG(b+w)\mathrm{d}t = -KGb\mathrm{d}t - KGb\exp(-abt)\mathrm{d}t/(q - \exp(-abt))$, where $Q = (b/w_0) + 1$. Again assume $b \neq 0$, and $W = W_0$ at $t = 0$. Then,

$$\ln\left[(W_0 - W_\infty)(W - W_0)\right] = K\left\{Gbt + \left(\frac{G}{a}\right)\ln\frac{((b/w_0)+1) - \exp(-abt)}{(b/w_0)}\right\}, \qquad \mathbf{(22)}_{\mathrm{v}}$$

and then $W = W_\infty + (W_0 - W_\infty)\delta$, where $\lambda = b/w_0 = (A/Gw_0) - 1$ and

$$\delta = \left[\frac{\exp(a\lambda w_0 t)(\lambda+1)}{\lambda} - \frac{1}{\lambda}\right]^{-KG/a}. \qquad \mathbf{(23)}_{\mathrm{v}}$$

When the total amount of blinding material is sufficient to bind the sieve completely, $A/Gw_0 < 1$ and so $-1 \leq \lambda < 0$, when it is exactly enough to completely blind the sieve, $\lambda = 0$, otherwise $A/Gw_0 > 1$ and $0 < \lambda < \infty$, Gw_0 being the area of sieve cloth which will be blinded by all the blinding material present. At $t \to \infty$, $\delta = 0$ when $0\lambda < \infty$ and $\delta = (-\lambda)^{KG/a}$, a positive fraction, when $-1 \leq \lambda \leq 0$.

6

§ 6. Filtering membranes

Filters sometimes surprisingly foul up by colloids containing only particles of size orders smaller than their internal pore size. Electrostatic forces are believed to play a vital role in bringing the particles together in this case, and clump them in such a way that they clot the openings in these membranes. It is suggested in this thesis that we should incorporate the effects of these forces into a setting of continuum percolation, and then put this in a percolation of network.

The problem of continuum percolation of polygon has been first suggested to Tiyapan in 2002. Works had already been done on percolation of circles and spheres in space, but no results from a similar thing for polygons have ever been published. In such case the problem becomes more complicated since a polygon may not only be placed somewhere but also be tilted. In the present work Tiyapan solved the problem and wrote an algorithm for the case of n-gons percolating in continuum. But instead of using it to look at manufactured Zeolite's structure, he sees it as directly explaining the case of fouling in a filtering membrane already mentioned.

Tanemura *et al* (1983) give a good algorithm which is both powerful and simple for constructing the three-dimensional Voronoi tessellation. The circumcentre of a DT being a vertex of the VT and the circumsphere of a DT being empty, this second condition of which is known as the *contiguity condition*, they arrive at an algorithm which works for both nondegenerate and degenerate cases. Algorithm 6.1 summarises their algorithm. Associated with the atom i, let V_i be the polyhedron, v_i vertex atoms of V_i, and S_i set of atoms surrounding i, $S_i \supset v_i$. Here $H_i(\alpha\beta|\gamma)$ is the half-space determined by $\{i, i_\alpha, i_\beta\}$ and does not contain i_γ, $S_i(\alpha\beta|\gamma) \subset S_i \subset H_i$.

Algorithm 6.1 *Voronoi construction in three dimensions, Tanemura et al (1983)*

```
find $i_1$, nearest, and thus contiguous, to $i$;
$C_i \leftarrow i_1$;
find $j = i_2$ s.t. $\{i, i_1, i_2\}$ has the minimum circumradius among all $\{i, i_1, j\}$, $j \in S_i$, $j \neq i_1, i_2$;
$C_i \leftarrow i_2 \cup C_i$;
find $j = i_3$ s.t. $\{i, i_1, i_2, i_3\}$ has the minimum circumradius among all $\{i, i_1, i_2, j\}$, $j \in S_i$,
  $j \neq i_1, i_2$;
$V_i \leftarrow \{i, i_1, i_2, i_3\}$;
$C_i \leftarrow i_3 \cup C_i$;
clear $\{m_\alpha\}$ and $\{l_{\alpha\beta}\}$;
$\alpha \leftarrow 1$;
for all $i_\alpha \in C_i$ do
  if $m_\alpha = 1$ then
    $\alpha \leftarrow \alpha + 1$;
  else
    find $\{i, i_\alpha, i_\beta, i_\gamma\} \in T_i$, which has $\{i, i_\alpha\}$ in common and where $l_{\alpha\beta} = 1$;
    $S_i(\alpha\beta|\gamma) \leftarrow S_i \cap H_i(\alpha\beta|\gamma)$;
    describe a circumsphere to each quartette $\{i, i_\alpha, i_\beta, j\}$, where $j \in S_i(\alpha\beta|\gamma)$ and $m_j \neq 1$;
    find $j_{\min}$ for which the centre of the circumsphere of $\{i, i_\alpha, i_\beta, j_{\min}\}$ has the minimum
      signed $z$-coordinate value among all circumspheres obtained, $z$-axis being normal to
      $\{i, i_\alpha, i\beta\}$ and lies towards the side of $H_i(\alpha\beta|\gamma)$;
    $T_i \leftarrow \{i, i_\alpha, i\beta, i_\delta\} \cup T_i$;
    if $i_\delta \notin C_i$ do
      $C_i \leftarrow i_\delta \cup C$; $l_{\alpha\beta}++$; $l_{\alpha\delta}++$; $l_{\beta\delta}++$; $l_{\beta\alpha}++$; $l_{\delta\alpha}++$; $l_{\delta\beta}++$;
    endif
    if all $\{l_\alpha\}$ are equal to 2, $m_\alpha \leftarrow 1$;
    if all $\{l_\beta\}$ and/or $\{l_\delta\}$ are equal to 2, $m_\beta \leftarrow 1$ and/or $m_\delta \leftarrow 1$;
  endif
endfor
find geometrical quantities of $\Pi_i$;                                        □
```

Algorithm 6.1 is justified by proofs of theorems which confirm that atoms nearest to each other are contiguous to each other, a triangle with the minimum circumradius is a face of a DT, which means that all its three atoms are also mutually contiguous, likewise a tetrahedron with the minimum circumradius is a DT, all its four atoms are mutually contiguous, two DT's sharing a plane have their fourth vertices at a signed minimum distance on either side of the plane they share. Some interesting geometrical lemmas help towards the proof of these theorems.

Jackson (1994) studies porous media and represent them using Voronoi models. His interests are in the synthetic membranes, which can be symmetric or asymmetric, homogeneous or heterogeneous structure, neutral or charged, and passive or active transport. The resistance to mass transfer is mainly in the top layer if it exists. This is a dense layer 0.1–0.5 μm thick which lies on top of a porous sublayer 50–150 μm thick. Regular tessellations in two dimensions come in 11 distinct tilings called Archimedean tilings. Tessellations by random lines in 2-d have the equation of the lines $x\cos\theta + y\sin\theta - h = 0$, where $0 \le \theta \le \pi$ and $-\infty < h < \infty$, which is analogous to tessellations of random planes in 3-d with the equation of a plane $x\sin\theta\cos\phi + y\sin\theta\sin\phi + z\cos\theta = 0$, where $0 \le \phi \le \pi$. The definition he gives of the *Voronoi polygon* is $\bigcap_{i\neq j}^{N} H(x_i, x_j)$, where the half-plane $H(x_i, x_j)$ is $\{x \in E^2 | d(x, x_i) \le d(x, x_j); i \neq j\}$. The bisector is $B(x_i, x_j) = \{x \in E^2 | d(x, x_i) = d(x, x_j)\}$. There are $n_c + n_b - 2$ triangles in the corresponding Delaunay triangulation if in the Voronoi tessellation there are n_c interior- and n_b boundary polygons. Scanning electron microscopy is explained and several SEM pictures of cellulose nitrate are given. AVS is used to do the pore perimeter calculation by connecting the modules `read any image` with `image measure` and *image viewer*, the latter two of which are in turn connected together. To calculate the pore area connect `read any image` with `sketch`, `histogram`, and `image viewer`. Then connect `sketch` with `histogram`, `sketch` with `image viewer`, and `histogram` with `print field`. A 3-d process is a homogeneous Poisson point process if the number of points in any region of volume V has a Poisson distribution with parameter λV and the random variables corresponding to the number of points in disjoint regions are independent among one another. Among the characteristics calculated are, in our terminology, the perimeter of polyhedron $P_c = \sum_{i=1}^{n_c^e} e_c^i$, the surface area $A_c = \sum_{i=1}^{n_c^f} \sum_{j=1}^{n_c^{t,i}} A_t^{c,i,j}$, and the volume is $V_c = \sum_{i=1}^{n_c^f} \sum_{j=1}^{n_t^{c,i}} V_t^{c,i,j}$. His numerical results given (see also, Jackson *et al*, 1999) are $n_f^e = 5.203390$, $n_c^e = 39.191489$, $n_c^v = 26.127660$, $n_c^f = 15.063830$, $P_c = 17.358196b$, $A_c = 5.910997b^2$, b^3 being the mean cell volume; the coordination number is of course 4. The distribution of n_c^v shows the distinct Voronoi characteristic of containing exclusively of even numbers, whereas that of n_c^e the equally distinct characteristic of being exclusively in multiples of three. This has been explained as odd number implying an impossible fractional number of edges, there being 3/2 edges as there are vertices. For the cross section data he found $n_c^e = 5.831$ and $P_c = 3.563s$, s^2 being the mean polygon area. In membranes, if the diameter of a particle is less than that of a pore then the particle will entrain the pore unless it should encounter other fouling particles, but if the particle is bigger than the pore then it will be retained. He assumes fouling particles to be spherical. Ascribed to each face is the diameter or size of its largest inscribed circle. Algorithm 6.2 is used for this purpose; d is the largest inscribed circle diameter sought for.

Algorithm 6.2 *Find the largest inscribed circle*

```
$d \leftarrow 0$
for all combinations of three edges of the face
  find the corresponding triangle
  find the largest inscribed circle of this triangle
  if the centroid lies within the face then
    if the perimeter lies within the face then
      if the diameter $> d$ then
        $d \leftarrow$ the diameter
      endif
    endif
  endif
endfor
```
□

In carrying out his simulation, the mean particle diameter $\bar{d}_p$ is calculated from the mean pore diameter $\bar{d}_v$ by $\bar{d}_p = k\bar{d}_v$ where k is a constant. The corresponding standard deviation of particle sizes is $\sigma_p = \sigma_v(\bar{d}_p/\bar{d}_v)$. The distribution of particle diameters is dependent on both k and another dimensionless parameter $m = (\sigma_p/\bar{d}_p)/(\sigma_v/\bar{d}_v)$. Then the central limit theorem gives the frequency distribution and thus the diameter of the i^{th} particle as $d_i = \bar{d}_p + \sigma_p\left(\left(\sum_{j=1}^{q} r_j - q/2\right)/\sqrt{q/12}\right)$, where $0 \le r_j \le 1$ are random variables and $q \ge 12$. Interaction between particles and membranes starts from Algorithm 6.3 which has been adapted from Jackson (*ibid.*).

Algorithm 6.3 *Interaction between particles and surface pores.*

```
while another particle exists do
  $i \leftarrow i + 1$;
  $p_i \leftarrow$ particle ;
  find particle entry coordinates;
  $c_i \leftarrow$ of the nearest Voronoi cell;
  $f_i \leftarrow$ the nearest face of $c_i$;
  if the face is fouled then
    for all the neighbouring faces $f_j$
      if $f_j$ is not fouled then
        $\{f\} \leftarrow f_j$;
      endif
    endfor
  else
    $\{f\} \leftarrow f_i$;
  endif
  if $\{f\}$ is not empty then
    $j \leftarrow 0$
    while particle travels and $j < |\{f\}|$ do
      $j \leftarrow j + 1$;
      if $d_p > d_i$ then
        if the centre of $p_i$ lies outside the largest circumscribed circle of $f_i$ then
          $f_i \leftarrow$ the neigbouring face $f_i$ of the adjacent cell;
        endif
        if $d_p > 1.1d_i$
          particle cakes or blocks surface;
        else
          particle blinds surface face;
        endif
      else
        particle enters membrane;
      endif
    endwhile
  else
    particle cakes or blocks surface;
  endif
endwhile                                                        □
```

Backflushing the membrane may clean blocking- but not blinding particles. Once inside the membrane, each particle independently and randomly walks until it can get out of the membrane or can go no further and thereby necessarily fouls a cell. The interpretation shown here as Algorithm 6.4 is an adaptation of that given by Jackson (1994).

Algorithm 6.4 *Interaction between each internal particle and the pores.*

```
for each internal particle $p_i$
  $\{c\}$ pores to enter;
  $j \leftarrow 1$
  while $j \leq |\{c\}|$ and $p_i$ has not entered a polyhedron do
    find pore to enter;
    if the polyhedron is fouled then
      particle enters the polyhedron;
      if there is no space in the polyhedron then
        particle blocks or blinds pore;
      endif
    endif
    $i \leftarrow i + 1$
  endwhile
endfor                                                          □
```

On the cake, the particles drop, roll, or nest among one another. The cake formation is described as Algorithm 6.5 which closely follows the description which he gave.

Algorithm 6.5 *Particle cake formation, cf Jackson (1994)*

```
while next particle exists do
   $p_i \leftarrow$ next particle;
   if route not directly leads to surface face then
      $p_i$ and $p_1$ collide;
      find rolling path;
      if $p_i$ does not roll off then
         $p_i$ and $p_2$ collide;
         if $p_i$ does not roll off then
            $p_i$ and $p_3$ collide;
            $p_i$ comes to rest;
         endif
      endif
   endif
endwhile                                                        □
```

Fifty particles is the empirical rule of thumb for the number of neighbouring particles to consider at the surface and cake level. Good care has been taken by his programme to ensure that no two particles within this set overlap. All of his particles are gentle for they bounce not but often roll, for these no hard billiard balls but mathematical particles with assumptions. In this way a spherical particle would first vertically drops, then touches another particle and starts rolling off the latter along its surface. Once the two particles are abreast with each other they part. A dropping particle can either roll off one- or simultaneously two particles at a time. Rolling simultaneously off three particles is not physically possible. For if we picture i meets 1, rolls off 1, meets 2, starts to roll simultaneously off 1 and 2, and then meet 3. At this point it can either choose to give up 1 to roll simultaneously off only 2 and 3 or, it nests on the cradle of 1, 2 and 3, and thereby stops. There are three phases or layers associated with this membrane packing model. These are the cake-, surface- and internal layers. For the internal layer the thickness is limited by the membrane thickness, for the surface layer by the maximum diameter of the dropping particles, and for the cake layer by their amount and sizes. The flux decline across the membrane model is calculated by Algorithm 6.6. Here C_n is the particle concentration, P_0 initial pressure, P_L final pressure, J_m initial volumetric flux rate, that is to say, clean solvent flux for membrane, d diameter of the circular membrane, n the number of particles to be dropped on to the membrane in this simulation, L the depth of the packing retion, $\bar{d}_p$ the mean spherical particle diameter, μ the solvent viscosity, ϵ_v the free volume, that is to say, the ratio between the volume of voids and volume of bed, v initial velocity of the fluid, P_s pressure between the cake and the surface fouling layer, R_1 the pressure-flux relationship for the particles in contact with the membrane, R the pressure-flux relationship of the membrane,

Algorithm 6.6 *Flux decline across the membrane model.*

```
define $\Delta t$, $C_n$, $P_0$;
$v \leftarrow (4/\pi) J_m / d^2$;
$t \leftarrow 0$;
while simulation should last do
   $t \leftarrow t + \Delta t$;
   $n \leftarrow \mathrm{f}(\Delta t, t, J_m)$;
   find the packing density, dimensions and number of particles of the cake;
   find the packing density, dimensions and number of blocking- and blinding particles
      of the surface fouling layer;
   find the packing density, dimensions and number of blocking- and blinding particles
      of the internal fouling layer;
   $P_c \leftarrow$ the pressure drop across the cake calculated from $v = \frac{(P_0 - P_L)}{L} \frac{\bar{d}_p^2}{150\mu} \frac{\epsilon_v^3}{(1-\epsilon_v)^2}$
   $P_s \leftarrow P_0 - P_c$
   $P_l \leftarrow$ the pressure drop for the surface layer;
   $P_i \leftarrow$ the pressure drop for the internal fouling particles;
   $R_1 \leftarrow \frac{P_l + P_i}{J_m}$;
   $R_2 \leftarrow R_1 + R$;
```

```
        $J_m \leftarrow P_s/R_2$;
    endwhile                                                    □
```

Flux is the rate of flow by weight. The number of particles per second is obtained by multiplying[†] the concentration by the volumetric rate. And if we divide this by the area of the membrane we get the number of particles per second per unit area. The random walk of particles during removal by backflushing is shown here as Algorithm 6.7.

Algorithm 6.7 *The random walk of particles during removal by backflushing*

```
remove cake and all surface blocking particles;
while next particle exists and backflushing do
    backflushing ← false;
    if not blocking then
        remain blinding;
        if there are more particles in cell then
            blocking ← true;
        else
            no more particle;
            backflushing ← true;
        endif
    else
        blocking ← true;
    endif
    if blocking then
        do
            if cell inlet pore unfouled or fouled and the particle passes through adjacent pore then
                move into the next adjacent cell;
            else
                remain blocking;
                backflushing ← true;
            endif
        while the particle is still within the membrane enddo
        particle has left membrane;
        backflushing ← true;
    endif
endwhile                                                        □
```

Filters are sometimes made of non-woven materials, in which case they are modelled as tessellation by random straight lines. Wilkinson *et al* (1986) studies this type of filter by modelling it as intersecting random rods. The position and orientation of a rod is defined by one point, an angle $0 \leq \theta_i$ 180° and the diameter of the fibre d_i. The free area is $\varepsilon_A = (A - \sum A_s)/A$, where the total fibre cross sectional area is $\sum A_s = \sum_m l_i d_i - \sum_m \sum_n \delta_{ij} d_i d_j \operatorname{cosec} \theta_{ij}$, where l_i is the length of the fibre, δ_{ij} is 1 if lines i and j intersects and 0 otherwise, θ_{ij} the angle of intersection of the fibres. Their model worked well for small fibre diameters compared to those of the particles.

Algorithm 6.8 rewrites their method in an algorithmic form. Here $0 < r_j < 1$ is a random number with uniform distribution, d_c the critical diameter, that of a circle which just fits the inside an irregular polygon.

Algorithm 6.8 *Non-woven fibre simulation, cf Wilkinson et al, 1986.*

```
for all particles $i$ do
    $d_i \leftarrow \bar{d} + \sigma(\sum_n r_k - n/2)/\sqrt{R/12}$;
    choose the particle position $0 \leq x_i \leq X$ and $0 \leq y_i \leq Y$;
    for each layer of the filter $j$ do
        $d_j^c \leftarrow$ the largest circle which fits inside the polygon;
        if $d_k > d_j^c$ then
            the particle is retained at this position;
        endif
    endfor
```

[†] It is possibly a typing error in Jackson (1994) when he says *dividing* instead of *multiplying* here.

```
endfor
```

□

The fibre here is in the form of fibre matt, which stacks one upon another in layers. The equation for the volume ratio suspension to inlet of a fibre matt, $V_n/V_0 = \exp(-\xi n)$ where ξ is the mean capture efficiency of the filter, is similar to that of the concentration ratio in deep bed filtration, $C/C_0 = \exp(-Kx)$. The mean capture efficiency is related to the layer efficiencies by $\xi = 1 - [(1-\xi_1)(1-\xi_2)\cdots(1-\xi_n)]^{1/n}$, and is dependent on $\bar{d}$, σ, ε_A, $\bar{d}_f$, σ_f, where the subscript f means *filament*.

Jafferali (1995) studies Voronoi tessellation and applies it to microfiltration. The programme which he used to create the Voronoi structure has the algorithm of Algorithm 6.9.

Algorithm 6.9 *Construction of 3-d Voronoi network (cf Jafferali, 1995)*

```
find set $S$ which contains $m$ nuclei nearest to the nucleus point $p$;
find the nucleus $q$ which is nearest to $p$;
find a nucleus $r \in S, r \neq q$, such that the triangle $pqr$ has the minimum circumradius,
  that is one facet of the Delaunay triangulation;
do
  find $s \in S, s \neq r \neq q$ such that the circumradius of the tetrahedron $pqrs$ is minimised,
    that is a Delaunay tetrahedron
  find $m, n, o \in S$ such that $m \neq n \neq o \neq q \neq r \neq s$ and the tetrahedra $pqrmj$, $pqsn$
    and $prso$ all have minimum circumradii, these are also Delaunay tetrahedra
until each triplet $pp_ip_j$ is in two Delaunay tetrahedra, that is every face of every
  Delaunay tetrahedron containing $p$ as a vertex is closed enddo
```

To find the centre of the circumscribed circle of the triangle ABC, let Π_A, Π_B and Π_C be the planes perpendicular respectively to the sides opposite to A, B and C. Respectively let n_A, n_B and n_C be the vectors normal to Π_A, Π_B and Π_C, and a, b and c the position vectors of A, B and C. Then Π_A, Π_B and Π_C will intersect at a point the position vector of whom is r which is obtained by solving the three simultaneous equations $(r - n_i) \cdot a = 0$ where $i = A$, B, C, $n_B = n_C \times n_A$, $n_A = b - c$, $n_C = b - a$, $d = 0.5(a+b)$, $e = 0.5(b+c)$. It follows that $r_x = [m_1 - (r_x a_y + r_z a_z)]/a_x$, $r_y = m_6 + m_7 r_z$ and $r_z = (m_4 - m_6)/(m_7 - m_5)$ where $m_1 = r \cdot a = a \cdot n_B$, $m_2 = r \cdot d = d \cdot n_C$, $m_3 = r \cdot e = e \cdot n_A, m_4 = (m_2 a_x - m_1 d_x)/(d_y a_x - a_y d_x)$, $m_5 = (a_z d_x - d_z a_x)/(d_y a_x - a_y d_x)$, $m_6 = (m_3 a_x - m_1 e_x)/(e_y a_x - a_y e_x)$, and $m_7 = (a_z e_x - e_z a_x)/(e_y a_x - a_y e_x)$.. The centre p of the circumscribed sphere is obtained by solving the four simultaneous equations the first one of which is $(p_x - A_x)^2 + (p_y - A_y)^2 + (p_z - A_z)^2 = r^2$ while the other three have B, C and D in place of A. Solving these we get $p_x = (m_2 - m_4)/(m_3 - m_1)$, $p_y = m_1 x + m_2$, $p_z = [2p_x(c_x - d_x) + 2p_y(c_y - d_y) + e_2 - e_1]/[2(d_z - c_z)]$ where $m_1 = [(a_x - b_x)(d_z - c_z) + (d_x - c_x)(b_z - a_z)]/[(c_y - d_y)(b_z - a_z) + (b_y - a_y)(d_z - c_z)]$, $m_2 = [(e_4 - e_3)(b_z - a_z) - (e_1 - e_2)(d_z - c_z)]/[2(c_y - d_y)(b_z - a_z) + 2(b_y - a_y)(d_z - c_z)]$, $m_3 = [(a_x - d_x)(b_z - c_z) + (b_x - c_x)(d_z - a_z)]/[(c_y - b_y)(d_z - a_z) + 2(d_y - a_y)(b_z - c_z)]$, $m_4 = [(e_2 - e_3)(d_z - a_z) - (e_1 - e_4)(b_z - c_z)]/[2(c_y - b_y)(d_z - a_z) + 2(d_y - a_y)(b_z - c_z)]$, $e_1 = a_x^2 + a_y^2 + a_z^2$, $e_2 = b_x^2 + b_y^2 + b_z^2$, $e_3 = c_x^2 + c_y^2 + c_z^2$, and $e_4 = d_x^2 + d_y^2 + d_z^2$. Statistics obtained from his simulation include n_c, n_c^f, n_c^v, n_c^e, perimeter of polyhedron, area of face, and volume of polyhedron. The number of faces and the number of edges are found during face construction the method of which is Algorithm 6.10.

Algorithm 6.10 *Find faces of a polyhedron, cf Jafferali (1995).*

```
find the set of all the $n$ vertices of the polyhedron, $\{v\}$;
for all combinations of three vertices $v_i$, $v_j$ and $v_k$ of $\{v\}$ do
  find the plane $\Pi$ which contains $v_i$, $v_j$ and $v_k$;
  if all the remaining vertices lie on one side of $\Pi$ then
    $v_i$, $v_j$, $v_k$ together with all the other vertices which lie on $\Pi$ form a face;
  endif
enddo
```

□

To find out whether all the vertices lie on one side relative to a plane, find whether the sign of the distance from each of these points to the plane changes. This distance is $d = \overrightarrow{pa} \cdot n = (a - p) \cdot n$ where n is the unit normal vector, $n = \overrightarrow{pa} \times \overrightarrow{bc}/|\overrightarrow{ba} \times \overrightarrow{bc}| = [(a - b) \times (c - b)]/|(a - b) \times (c - b)|$, and a, b and c form a plane while p is the point in question. To arrange the vertices of a face in cyclic order, first pick one point among them and then find the angle made by the lines from it to every two combination of the remaining vertices, by $\theta = \cos^{-1}[(\overrightarrow{v_p v_i} \cdot \overrightarrow{v_p v_j})/|\overrightarrow{v_p v_i}| \cdot |\overrightarrow{v_p v_j}|]$. Two such lines which maximises the angle are both edges, and the angle from either one of them to each of the remaining vertices increases as we tread the edges from one vertex to another in succession.

Nuclei points of the modified point process where $d \geq d_{\min}$ are generated by Algorithm 6.11.

Algorithm 6.11 *Nuclei points of the modified point process, Jafferali (1995).*

```
$n$; $d_{\min}$;
find $p_1$;
$i \leftarrow 1$;
while $i < n$ do
   $i \leftarrow i = 1$;
   find $p_i$;
   for $j = 2$ to $(i - 1)$ do
      find $d(p_i, p_j)$;
      if $d(p_i, p_j) < d_{\min}$ then
         $i \leftarrow i - 1$;
      endif
   endfor
endwhile
```
□

The statistics that he found for these modified Voronoi structures are shown here again in Table 6.1.

$d_{\min}$	n_c	$\min(n_f)$	$\max(n_f)$	$\min(n_v)$	$\max(n_v)$
0.0	226	8	23	12	42
0.2	218	8	24	12	44
0.4	215	8	23	12	44
0.6	226	8	24	12	44
0.8	229	9	21	14	38
1.0	214	10	20	16	36
1.2	204	10	20	16	36

Table 6.1 *Statistics of the modified Voronoi cells, cf Jafferali (1995).*

His asymmetric Voronoi tessellations have cell volume increasing with the z-coordinate position of the cell. The procedure for finding the $d_{\min}$ between two cells in such structure is described as Algorithm 6.12. Here the volume ratio is $\delta = V_0/V_1$, and thus $\delta = y_0/y + 1$, $d_{\min} = mz_i + c$ where $m = (y - y_1)/(z - z_1)$.

Algorithm 6.12 *Asymmetric Voronoi tessellation, cf Jafferali (1995).*

```
$\delta$;
$y_1 \leftarrow 1$;
$y_0 \leftarrow \delta$;
$c \leftarrow \delta^{1/3}$;
$m \leftarrow (1 - c)$;
$d_{\min} \leftarrow k[(1 - c)z + c]$;
```
□

The statistics that he found are given for reference as Tables 6.2 and 6.3. Real numbers have been rounded to leave at most four decimal points to make it easier to read (*cf* Jafferali, 1995).

	point distribution		
	Poisson	*modified*	*asymmetric*
n_c	1276	1171	1069
$\sum_i n_i^v$	33340	29946	27749
n_c^v	26.1285 ± 6.4207	25.5730 ± 3.5573	25.9579 ± 3.5301
$\sum_i n_i^e$	50072	44945	41643
n_c^e	39.2414 ± 9.6235	38.3817 ± 5.3317	38.9551 ± 5.2740
$\sum_i n_i^f$	19284	17341	16032
n_c^f	15.1129 ± 3.2114	14.8087 ± 1.7821	14.9972 ± 1.7514
$\sum_i V_i$	$0.6073b^3$	$0.6582b^3$	$0.5912b^3$
V_c	$4.76 \times 10^{-4}b^3 \pm 2 \times 10^{-4}$	$5.62 \times 10^{-4}b^3 \pm 6.8 \times 10^{-4}$	$5.53 \times 10^{-4}b^3 \pm 6.0 \times 10^{-4}$
$\sum_i p_i$	$641.3870b$	$664.1398b$	$554.01b$
p_c	$1.0053b \pm 0.2290$	$1.1343b \pm 0.1058$	$1.0365b \pm 0.3255$
$\sum_i A_i$	$40.9930b^2$	$44.2189b^2$	$36.4954b^2$
A_c	$0.0321b^2$	$0.0378b^2$	$0.0341b^2$

Table 6.2 *Comparative Voronoi statistics, Jafferali (1995).*

	distribution					
	Poisson		*modified*		*asymmetrical*	
	min	max	min	max	min	max
V	$8.8 \times 10^{-5}b^3$	$1.4 \times 10^{-3}b^3$	$3.9 \times 10^{-4}b^3$	$8.4 \times 10^{-4}b^3$	$9.9 \times 10^{-5}b^3$	$3.7 \times 10^{-3}b^3$
n_c^f	7	28	10	21	10	21
n_c^e	15	78	24	57	24	57
n_c^v	10	48	12	34	12	34

Table 6.3 *More comparative Voronoi statistics, Jafferali (1995).*

The Voronoi domain created has ragged boundaries due to polyhedra protruding. The method used in his thesis is to slice the domain by a horizontal plane and then redefine those vertices and edges on that plane and at the same time reject everything above it. This is more similar to a carpenter filing away at a block of wood than a stone mason choosing his stones. The volume of voids was updated by using AVS which does this by counting the number of pixels contained within a given area. Statistics of an asymmetric Voronoi structure were given. His study concerns with the simulation of dead end filtration and cake formation. The particle size distribution is quantified in terms of the ratio of the mean particle diameter to the mean inlet pore diameter, α, and that of the standard deviation of the particle diameters to the mean particle diameter, β. The first parameter indicates the relative size between the particles and pores while the second one that amongst particles. Thus $\alpha > 1$ would mean that the particles are bigger than the inlet pores, while $\beta = 0$ that particles are mono dispersed. The interaction between the particles and cake is essentially the same as that used earlier by Jackson (1994). The evaluation of pore properties described is equivalent to Algorithm 6.13.

Algorithm 6.13 *Pore property evaluation, cf Jafferali (1995).*

```
identify edge pores;
identify inlet- and outlet pores;
for all pores do
   find the largest inscribed sphere;
   for all faces of the pore do
      find the largest inscribed circle;
   endfor
endfor                                                        □
```

Identifying edge pores amounts to first finding neighbouring polyhedra, and then finding for all faces whether each of them belongs to one and only one polyhedron. Finding neighbouring polyhedra amounts to finding for all polyhedra pairs whether each of them possess no less than three common vertices.

Particles pass through a face into a pore if they could, but since they are spherical whereas the pores and faces are polyhedral and polygonal, the largest inscibed circle of the facet is computed. If the particles are irregularly shaped, they may fit through a polygon when challenging from a certain direction but not another. However, the assumption of spherically shaped particles provides most authors on the subject with an approximation both satisfactory and convenient, because then there is no need whatever to rotate them. Jafferali (1995) applies an algorithm equivalent to our Algorithm 6.14 to find the inscribed circle for all faces. Essentially this involves computing an inscribed circle of a triangle arisen from every one of the possible combinations of three sides of the face. Then the largest one of such circles which lies inside the face is the inscribed circle of the face. Here a, b and c are the positional vector of A, B and C respectively; n is a unit vector from A to B, q normal to the plane ABC, and m perpendicular to both; s_i are the sides opposite (x_i, y_i). From 3-d each face is transformed into 2-d in such a manner that $A = (x_1, y_1) = (0,0)$, $B = (x_2, y_2) = (0, y_2)$ and $C = (x_3, y_3)$. From $c - a = x_3 m + y_3 n$, x_3 and y_3 are obtained; z – for *Zentrum* – is the centre (x_z, y_z) of the face in 3-d or its equivalent transformed centre $(\tilde{x}_z, \tilde{x}_z)$; ξ is the nucleus of our polyhedron while ξ_i its neighbouring nuclei. Then the k^{th} combination gives the largest inscribed circle the radius of which is r_k.

Algorithm 6.14 *Inscribed circle of faces, cf Jafferali (1995).*

```
for each face do
  for all the ${}^pC_3$ combinations $j$ of sides of the face do
    $R \leftarrow 0$;
    find $A$, $B$ and $C$, the vertices of the circle formed by them;
    $n \leftarrow (b-a)/|b-a|$;
    $m \leftarrow$ solve $m \cdot n = 0$, $m \cdot N = 0$, $|m| = 1$ and $|n| = 1$;
    $q \leftarrow (b-a) \times (c-a)$;
    $\{x_1, y_1, x_2\} \leftarrow 0$;
    $y_2 \leftarrow |b-a|$;
    $y_3 \leftarrow [(x_3 n_y + y_3 n_y) m_x - (x_3 m_x + n_x y_3) m_y] / (n_y m_x - n_x m_y)$;
    $x_3 \leftarrow [(x_3 m_x + n_x y_3) - y_3 n_x] / m_x$;
    $p \leftarrow \sum_{i=1}^{3} s_i$;
    $S \leftarrow p/2$;
    $r_j \leftarrow \left[S \prod_{i=1}^{3} (S - s_i)\right]^{1/2} / S$;
    $\tilde{x}_z \leftarrow (s_1 x_3)/p = (s_1 x_3 + s_2 x_1 + s_3 x_2)/p$;
    $\tilde{x}_z \leftarrow (s_1 y_3 + s_3 y_2)/p = (s_1 y_3 + s_2 y_1 + s_3 y_2)/p$;
    $x_z \leftarrow \tilde{x}_z m_x + \tilde{x}_z n_x$;
    $y_z \leftarrow \tilde{x}_z m_y + \tilde{x}_z n_y$;
    $z_z \leftarrow \mathrm{ABC}(x_z, y_z)$;
    $d \leftarrow |z - \xi|$;
    $\zeta \leftarrow 1$;
    for all $\xi_i$ do
      $d_i \leftarrow |z - \xi_i|$;
      if $d_i < d$ then
      $\zeta \leftarrow 0$;
      endif
    endfor
    if $\zeta = 1$ then
      if $r > R$ then
        $R \leftarrow r_j$;
        $k \leftarrow j$;
      endif
    endif
  endfor
endfor                                                          □
```

Algorithm 6.15 finds the inscribed sphere of each polyhedron. A tetrahedron is formed from each combination of four planes of polygonal faces. The largest of all the inscribed spheres of these tetrahedra is the inscribed sphere of the polyhedron. The intersection of bisecting planes $\cap_{ij=i1}^{ik} \Pi_{ij}$ is the facet in $(3 - k \bmod (3))$ dimensions which is equidistant from Π_i and Π_j, $j = 1$ to k. Finding $\Pi = ax + by + cz + d = 0$ amounts to solving $A \cdot (B \times C)$ for a, b, c and d. Finding intersection of planes amounts to solving their equations simultaneously. Here Π_i is the plane containing the i^{th} face, Π_{ij} the plane bisecting Π_i and Π_j.

Algorithm 6.15 *Inscribed sphere of a polyhedron, cf Jafferali (1995).*

```
for each polyhedron do
```

```
  $R \leftarrow 0$;
  for all the ${}^{n_f}\mathrm{C}_4$ combinations $j$ of the polyhedron do
    for $i = 1$ to 4 do
      $A \leftarrow v_i^2 - v_i^1$;
      $B \leftarrow v_i^3 - v_i^1$;
      $C \leftarrow v_i^4 - v_i^1$;
      $\Pi_i(a_i, b_i, c_i, d_i) \leftarrow$ solve $A \cdot (B \times C) = 0$;
    endfor
    find $\Pi_{1i}$; $i = 2, 3, 4$;
    $p_i \leftarrow \cap_{i=2}^{4} \Pi_{1i}$;
    $r_j \leftarrow \mathrm{d}(p_i, \Pi_1)$;
    if $r_j > R$ then
      $R \leftarrow r_j$;
      $k \leftarrow j$;
    endif
  endfor
endfor                                                            □
```

Particles move down vertically in discrete time steps towards the membrane. Upon reaching the latter, the i^{th} particle is greeted by an entrance the aperture of which is circular with radius r_c. If $r_i < r_c$ it penetrates into the pore, if $r_c \leq r_i \leq 1.1 r_c$ it blinds and if $1.1 r_c < r_i$ blocks the entrance. Blinded particles are not removed by reversing the flow. They become a part of the membrane. Reaching the membrane is by no mean their destiny, and the particles only begin their long and tortuous pelerinage hereafter by embarking on a random walk in the direction towards the centre of the earth. Each particle enters a pore via the face furthest away from the latter, and it leaves via the face closest to it. The position of a facet is that of its inscribed circle. If the lowest facet is not viable, the particle first repositions itself precisely at the centre of the inscribed sphere, and then either moves out from some facet lower than itself or, when all possibilities of travelling having been exhausted, become a residence of that pore. The repositioning part above may seem like a gross approximation, but there is no reasons why this should not make a sound assumption if we consider the fact that a real particle is never spherical in the first place. But if the particle blinds or blocks a facet, then the calculation used by Jafferali (*ibid.*) becomes more accurate. That is $\alpha = \xi + \lambda(z - \xi)$, where $\lambda = |d|/(|d| + |a|)$, $|a| = (r_\alpha^2 + r_z^2)^{1/2}$ and $d = |\xi - z| - |a|$. To summarise, a particle is in a perpetual search for a lowest facet, which of course is closest to the centre of the earth. This kind of study is important because it lets us know which part of the membrane is prone fouling by the particles.

Non-woven fibres are simply stacked layers of material. Extra holes produced by needles which are used to increase their strength are difficult to model and are normally assumed away before the simulation (*cf* Chan, 1990). Layers are considered as flat when the fibres which make them are considered to be flexible. Capturing of particles must take into account the separation between the layers as well as the usual aperture perpendicular to the flow direction. This is an example of man-made stacked 2-d layers. Examples in nature are numerous, including the honeycomb and cell growth in tissues.

In his simulation, Chan (1990) represents random lines with $(y - y_i) = \tan\theta_i (x - x_i)$, where $0 \leq \theta_i \leq \pi$, a method of interior randomness. Other possible methods include the μ randomness where the line is represented by $x \sin\alpha_i + y \sin\alpha_i = d_i$, where d_i is the distance of the line from a pole point p_i which is usually taken as the centre of the area, and surface randomness where the position on the boundary of the area is chosen.

Flow channels within a medium follows Poiseuille's equation for flow within capillaries, $Q = \Delta P \pi r^4 / 8 \mu \Delta x$, where Δx is the length of the capillary. Darcy's law relates the flow of a fluid through a porous medium to the overall pressure drop, $\Delta P = \mu L u_0 / K$, where u_0 is the superficial fluid velocity, L the depth of the medium and K the permeability coefficient. The permeability coefficient for packed media, $K = \varepsilon^3 / \left[K' S^2 (1 - \varepsilon)^2\right]$, where ε is the voidage of the medium, S the specific surface and K' the Kozeny constant. Particles can blind during both the filtration and the backflushing stages. The backflushing efficiency is the ratio between the volume of the particle flushed away and the total volume of particles captured in the media before backflushing.

§ 6.1 Separation processes

The largest aperture in the screen is the determining factor by which the selection is made, in other words it defines the size of the cut. This is a more logical argument than the one by which it is the mean aperture that defines the cut size (*cf* Rose and English, 1973). This largest aperture determines the smallest size of the particles retained on the screen. But in practice much smaller particles than these will remain on the screen for various reasons the most important one of which is blindings caused by particles with sizes $d_{\min} < d < 1.1 d_{\min}$. The factor 1.1 was first used by Rose and English (*ibid.*), and arises from the angle of friction γ being approximately 20°.

On a sieve at each instant, the area that is free from blinding material is $A = A_s - G(m_0 - m)$ where A_s is the sieve area, G the area blinded by a unit mass of blinding material, m and m_0 the mass of respectively the blinding- and initial blinding material on the sieve. Also, $\mathrm{d}m/\mathrm{d}t = -kmA = -km\left[A_s - G(m_0 - m)\right] = -\alpha m(\beta + m)$, where $\alpha = kG$ and $\beta = [(A/G) - m_0]$.

In surface straining filtration the particles are larger than the pore size, in depth straining filtration both the pore and the particle sizes are commensurate to each other, and in adsorptive filtration the particle size is smaller than the pore diameter. Adsorptive filters can be made to have higher filtration efficiency, higher capacity and higher flow rate than depth straining filters (Raistrick, 1986).

Examples of solid-liquid separations normally found in Chemical Engineering are filtration, sedimentation, flocculation, centrifugation, electro-osmotic and electro phoretic dewatering and hydrocyclonic separation. Poole and Doyle (1965) listed some of the aspects in filtration which they thought need further investigation: the effect of rapid pressure increases on the approach to equilibrium porosity in cakes, for instance in rotary filtrations; the relation between drag forces on particles and particle arrangements and shapes; migration of fines within cakes and media; the effect of changing flow paths during washing on porosity.

§ 6.2 Dead-end filtration

Because membranes are porous media, the flux across them follows Darcy's Law, $j = \Delta p/(\mu R)$, where $j = \mathrm{d}V/(A\mathrm{d}t)$, R is the hydraulic resistance, $R = R_m + R_c$ where R_m and R_c are membrane and cake resistances respectively. Assume $R_m = 0$, then $R = R_c = \alpha m/A$, where α is the specific resistance of the cake, m the mass of the filter cake and A the septum area.

§ 6.3 The centre of gravity

The formula for the centre of gravity of objects in general is $c_g = \int x w(x)\mathrm{d}x / \int w(x)\mathrm{d}x$. For a real object the theoretical procedure is complicated, since there is always the possibility that the density is not uniform, but the practical procedure is simple and straight forward, that is by simply hanging the object by strings in various positions and then find the intersection between the lines of string, provided of course that this is possible with the object.

The centre of gravity is important whenever there is a gravitational interaction with an object. Both the translational and rotational motions of objects through space are relative to this point. The c.g. of a triangle is positioned at one third its height whereas that of a half circular disk is $4r/3\pi$ from the straight boundary line. A c.g. always lies on the lines of symmetry when these exist. Other names for c.g. include geocentre, centroid and barycentre.

The centroid of a triangle lies at the intersection of its median. That of a tetrahedron lies at the intersection of all the lines joining the vertices with the centroids of their respective opposite faces. Its coordinates are the mean coordinates of the four vertices.

To find the centroid of a polygon, first tessellate it into triangles and find the centroid of each one of them. Then we have another system of point masses located at the centroids of these triangles, with the mass proportional to their respective areas.

The centroid of a rod is at its mid point, and the weight of a rod is proportional to its length. This can be helpful when we want to find the centroid of a network of rods or tubes.

The centroid of a quadrilateral is the intersection between its two bimedians. This is also the mid point of the lines joining the mid points of the two diagonals.

The medians of a triangle which has its vertices at (a_1, b_1), (a_2, b_2) and (a_3, b_3) are the lines connecting these three vertices to the mid points opposite to them, the first one to (c_1, d_1) where $c_1 = (a_2 + a_3)/2$ and $d_1 = (b_2 + b_3)/2$, and similarly for the other two. Their intersection is obtained by solving their equations, $(d_1 - b_1)/(c_1 - a_1) = (y - b_1)/(x - a_1)$, and so on. This gives the solution as being the average value of the vertices, $[(\sum_3 a_i)/3, (\sum_3 b_i)/3]$.

Having mentioned the centroid, it is natural to add the other two points related to it. On the Euler line also lie the orthocentre, where altitudes intersect, and the circumcentre, where perpendicular bisectors intersect.

Let a_{ij} be $(a_j - a_i)$. Then for the orthocentre we need to solve any two of the three equations of the altitudes. For example, $-a_{23}/b_{23} = (y-b_1)/(x-a_1)$ and $(y-b_2)/(x-a_2) = -a_{13}/b_{13}$ can be simultaneously solved to give $x = [a_1(a_2b_{12} + a_3b_{31}) - (a_2a_3 + b_{21}b_{31})b_{32}]/(a_3b_{21} + a_1b_{32} + a_2b_{13})$ and $y = -[a_{32}(a_1a_2 - a_2a_3 + b_2b_{31}) + a_{31}(a_1a_{23} + b_1b_{23})]/(a_3b_{21} + a_1b_{32} + a_2b_{13})$.

Similarly the circumcentre can be solved from two equations, for instance $(y-b_1)/(x-a_1) = m_1$ and $(y-b_2)/(x-a_2) = m_2$ where $m_1 = -a_{23}/b_{23}$ and $m_2 = -a_{13}/b_{13}$, to give $x = b_{32}(a_2a_3 + b_2b_3)/(a_3b_2 - a_2b_3)$ and $y = a_{32}(a_2a_3 + b_2b_3)/(a_2b_3 - a_3b_2)$.

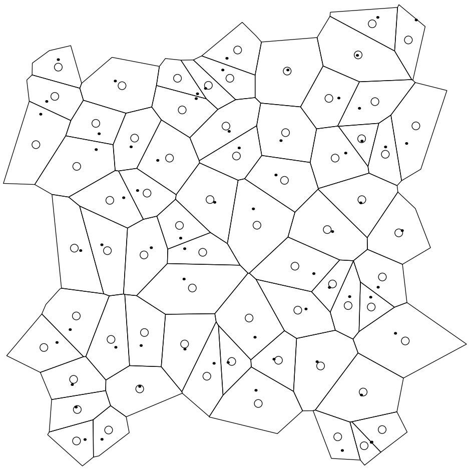


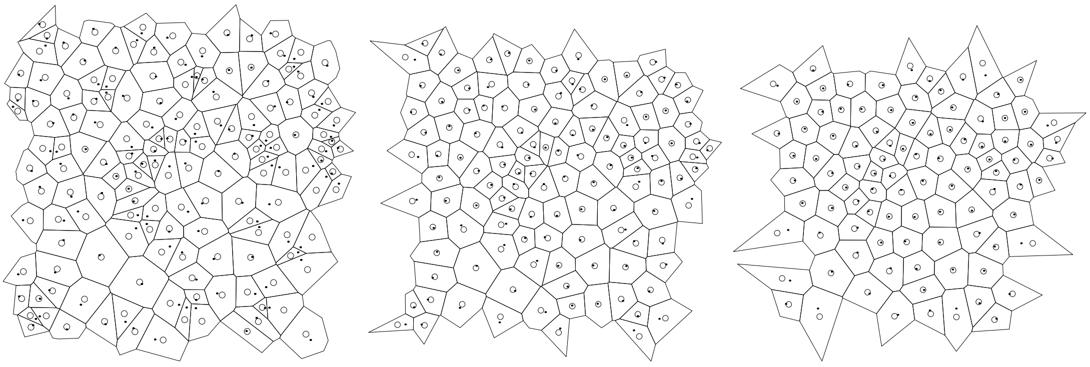
Figure 6.1 shows a 2-d Voronoi tessellation with its nuclei and centroids, which are represented respectively as dots and circles. Ironically, the centroid represents the position of a cell better than its nuclei, because it positions itself in the most balanced situation relative to the cell, whereas the nuclei does so with regard to its neighbours. The centroids are noticeably more evenly distributed than our Poisson nuclei.

Figure 6.1 Centroid of the Voronoi tessellation.

Because the centre of gravity is a property which has more to do with the individual cells than with their neighbours, there may be real situations in nature where the c.g. of individual cells in a network has a role to play. For example, it is usually assumed in a mathematical Voronoi tessellation that all the nuclei stay fixed and do not move. But in a real situation this would imply that there are some nuclei which lie very close to the walls. As no such nuclei would be stable, they more likely move away, and in that case the most reasonable prediction is that they move towards the centre of gravity of the cell.

Because they are stochastic in nature, computational VT could in theory take up any shape however awkward. Jafferali (1995), for instance, imposes constraints on the algorithm which creates VT's, namely in the course of the generation no new nuclei can assume a position too close to those of the existing ones. Thus there is an excluded volume within the network which develops throughout the generation process, and the Poisson point process does not cover the whole space. Again, this excluded volume is relative to the neighbours of the cells rather than to themselves.

I suggest that an evolution centred around the centroids of a network can replace such constraint regarding the minimum allowable distance between two nuclei. And that this would be a more natural for a network to do because the nature of the procedure which makes it look inwards to itself, as opposed to being totally governed by its environment. An attractive feature of the movement of the nuclei towards c.g.'s is that the cells in effect takes into account not only themselves but also their surrounding, because, if nothing else, it was the latter who shaped their appearance in the first place. Therefore the nuclei would start to grow from a purely random position according to a Poisson point process. Then it would move towards the centroid of the cell, and as it does so the definition of its boundaries is also changed. The centroid is a stable position, as can be seen in Figure 6.2. Similar to the processes of covering, $\mathrm{C}^{\cdot}(\mathcal{V})$ introduced in § 2.2 on page 45, and dual Voronoi, $\mathcal{V}^n(\cdot)$ in § 3.10 on page 97, this centroid process can be recursive, that is $\mathcal{G}^n(\cdot)$. However, unlike the other two mentioned, this process very quickly becomes stable; Figure 6.2 (b) is much different from (a), whereas it is very similar to (c). The difference is in the size distribution as well as in the location of nuclei. The nonuniformity in the shape of a half-circle which can be seen in Figure 6.2 (a) is propagated to both Figure's 6.2 (b) and (c), but it has become much fainter than in the original.



(a) (b) (c)

Figure 6.2 *Nuclei evolution towards a centroid; (a) a Voronoi tesssellation based on the Poisson point process, $\mathcal{V}$, (b) the VT around its c.g.'s, that is $\mathcal{G}(\mathcal{V})$, and (c) the same process applied again to give the second order $\mathcal{G}^2(\mathcal{V})$.*

The programme in § A.22 finds the centroid of a Voronoi network and $\mathcal{G}^n(\cdot)$.

There are three main steps in finding the centroid of a polygon. For each triangle of the triangulation of the polygon, first find the coordinates of its centroid, then its area. And then the centroid of the polygon is the weight average of the centroids of all the triangles. The first step involves finding the equation of two medians, which can be done by finding the mid point of an edge and then the slope to it from the opposite vertex. The second step can be carried out using the Heron's formula which finds the area by means of edge lengths.

Without loss of generality, let the polygon has n vertices at (a_i, b_i), where i increases from 1 to n in the clockwise direction. Assuming that all triangles which triangulate our polygon share one vertex at (a_1, b_1), so that they all are $\triangle_{ijk}$, where $i = 1$, $k = j + 1$ and the three vertices are in the clockwise direction (a_i, b_i), (a_j, b_j) and (a_k, b_k). There are three medians in each triangle, corresponding to the three vertices, but we only need two in order to solve for the centroid. Any two of them will give the same result, but in order to fix the algorithm let us choose those medians going out from the vertices i and j. Then, since there is no loop in the procedure, we can save the space by describing it here in linear steps as $c_i \leftarrow (a_j + a_k)/2$, $d_i \leftarrow (b_j + b_k)/2$, $m_i \leftarrow (d_i - b_i)/(c_i - a_i)$, and the first median has the equation $(y_1 - b_i)/(x_1 - a_i) = m_i$. Likewise the second equation can be obtained as $(y_1 - b_j)/(x_1 - a_j) = m_j$. Here the subscript of the centroid coordinates tells us in which triangle it belongs. As the result of the above, we obtain for the k^{th} triangle $x_k = (a_i + a_j + a_k)/3$ and $y_k = (b_i + b_j + b_k)/3$.

The second step is to find the area. For this, we first find the edge lengths $s_{k1} \leftarrow (a_{ij}^2 + b_{ij}^2)^{1/2}$, $s_{k2} \leftarrow (a_{jk}^2 + b_{jk}^2)^{1/2}$ and $s_{k3} \leftarrow (a_{ki}^2 + b_{ki}^2)^{1/2}$, then the half-perimeter $s_k \leftarrow (s_{k1} + s_{k2} + s_{k3})/2$, and then the area $A_k \leftarrow (s_k(s_k - s_{k1})(s_k - s_{k2})(s_k - s_{k3}))^{1/2}$.

The third step solves for the polygonal centroid (x, y) as $x \leftarrow (\sum_k x_k A_k)/\sum_k A_k$ and $y \leftarrow (\sum_k y_k A_k)/\sum_k A_k$. Notice that the first step is parallel to the second- but not the third one. Moreover, finding c's and d's are parallel, as well as finding the various s_{ki}'s. So we could vectorise these using parallel processing. But on Matlab all the matrix operations are already vectorised, so we only need to put all the components to be run in parallel in a single matrix and then do all the operations at once as a single operation on the matrix, provided that this is possible.

The general algebraic solution to the above algorithmic procedure is obtainable by solving all the equations involved. This gives for any polygon the following theorem.

Theorem 6.1.

$$x = \frac{\sum_\kappa (a_i + a_j + a_k)(a_i b_{jk} + a_j b_{ki} + a_k b_{ij})}{3 \sum_\kappa (a_i b_{jk} + a_j b_{ki} + a_k b_{ij})} \qquad \mathbf{(24)}_{\text{vi}}$$

and

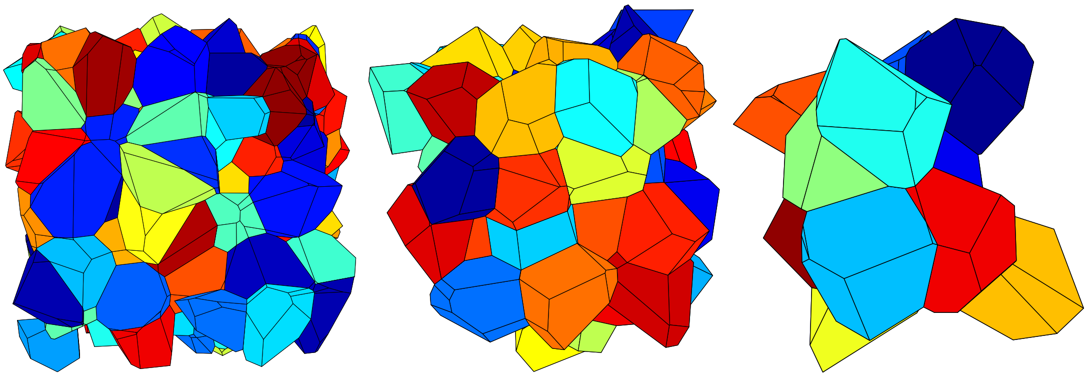
$$y = \frac{\sum_\kappa (b_i + b_j + b_k)(a_i b_{jk} + a_j b_{ki} + a_k b_{ij})}{3 \sum_\kappa (a_i b_{jk} + a_j b_{ki} + a_k b_{ij})}. \qquad \mathbf{(25)}_{\text{vi}}$$

Proof. Assuming that we accept the centroid of a triangle to be at the mean coordinates among its vertices, and the centroid of massive point bodies is their average coordinates weighted by their mass, then when $k = 1$ it is obvious by looking at the Equation's 24(vi) and 25(vi) that this theorem is true. Let us suppose that both these equations are valid for j triangles. We can present both of them in the form n/d where, necessarily, $n = \sum_\kappa x_i A_i$ and $d = \sum_\kappa A_i$. If we now add another triangle to this cluster, the new triangle will also be represented as a massive point and it will add the terms xA to the numerator and A to the denominator, making $n/d = (n+xA)/(d+A) = (\sum_{(J+1)} x_iA_i)/\sum_{(j+1)} A_i$. Now, x for a triangle $\triangle_{ijk}$ is $(a_i + a_j + a_k)/3$, and A calculated from Heron's formula above is $(a_ib_{jk} + a_jb_{ki} + a_kb_{ij})^{1/2}/2$, which yields Equation 24(vi) and 25(vi) for $\kappa + 1$. This proves both equations, and thus Theorem 6.1, by induction. □

These general formulae, Equation's 24(vi) and 25(vi), cover the triangle itself, when we consider the latter as a polygon, and the solution when $k = 1$ in this case reduces to (x, y) being simply the average coordinates of the three vertices, which is the same as that we have earlier mentioned.

Earlier we have seen the evolution of the nuclei towards centroids in two dimensions. Let us look at the same thing in three dimensions. For this purpose, the programme of § A.21, which is used in § 6.5 and explained by Algorithm 6.18 in page 183, is still inadequate, as its list of faces contains considerable amount of duplicates. This would have increased the resources required to do further work here. Therefore the programme has been adjusted to that in § A.23 for the described in the following.

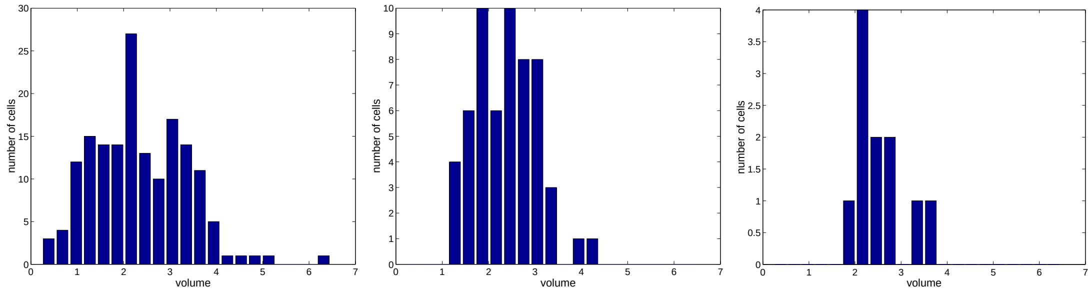
Similar to the idea shown in Figure 6.2, we draw in Figure 6.3 the VT's in three dimensions whose nuclei evolve towards the respective c.g.'s.



(a) (b) (c)

Figure 6.3 *Evolution of nuclei towards centroids in three dimensions, namely* (a) $\mathcal{V}^3$, (b) $\mathcal{G}(\mathcal{V}^3)$ *and* (c) $\mathcal{G}^2(\mathcal{V}^3)$.

This shifting in the nuclei positions preserves the mean of the size while cutting down its variance. The pictures in Figure 6.3 show how a 3-d VT transforms into its first- and then second order adjustment. If we plot the cell size distribution, we will come up with a picture similar to Figure 6.4. The appearance of the distribution where sizes are discretised into bins like this depends on the number of bins chosen.



(a) (b) (c)

Figure 6.4 *Change in size distribution as a result of a nuclei adjustment process,* (a) *the size distribution of a* $\mathcal{V}^3$, (b) $\mathcal{G}(\mathcal{V}^3)$ *and* (c) $\mathcal{G}^2(\mathcal{V}^3)$. *From a view point at 96° azimuth and 0° elevation.*

Figure 6.4 serves as a graphical presentation, but the statistics are the following. For our $\mathcal{V}^3$, $\mathcal{G}(\mathcal{V}^3)$ and $\mathcal{G}^2(\mathcal{V}^3)$ respectively the cell sizes are 0.9345 ± 0.4025, 0.9495 ± 0.2652 and 1.0315 ± 0.2086; the third central moment 0.0332, 0.0059 and 0.0061; the fourth central moment 0.0936, 0.0137 and 0.0036. These means are only meaningful when considered relatively with each other, since it depends on how one chooses the normalising basis. Here the basis chosen is the volume for the original Poisson point process, which is 1, divided by the original number of generators, which is 400. When we built this $\mathcal{V}^3$, 236 cells had been disposed of as they were thought to be those along the boundaries; when $\mathcal{G}(\mathcal{V}^3)$ a further 107 and finally when we built the $\mathcal{G}^2(\mathcal{V}^3)$ 46 more. This leaves us with 164, 57 and 11 cells remaining respectively in our $\mathcal{V}^3$, $\mathcal{G}(\mathcal{V}^3)$ and $\mathcal{G}^2(\mathcal{V}^3)$, all of which are shown in that order as Figure 6.3 (a), (b) and (c). Of course a great part of these are hidden behind others, so you can not possibly see them all there.

The sizes in Figure 6.3 are based on a volume magnification factor of 1,000, which is corresponds to the increase in size by a factor of three, but those in Figure 6.4 are based on a magnification 400, which corresponds to a factor of $(400)^{1/3}$ increase in the size.

As the new programme has considerably changed from the old one, it is listed again in § A.23 which has evolved from that which is listed in § A.22 on page 274. Wherever the sign © appears in this monograph it means *copyleft* not copyright.

§ 6.4 Molecular dynamics

The Lennard-Jones function can be written $u(r) = -A/r^6 + B/r^{12}$, where A and B determine respectively the attractive and repulsive parts. Let the range parameter $\sigma = (B/A)^{1/6}$ and the energy parameter $\varepsilon = A^2/4B$, and the equation becomes $u(r) = 4\varepsilon\left[(\sigma/r)^{12} - (\sigma/r)^6\right]$. In other words, σ is the diameter of one of the atoms and ε is the well depth, *i.e.* energy constant. The force here is not $F = -\mathrm{d}u/\mathrm{d}t = (24\varepsilon/r^2)\left[2(\sigma/r)^{12} - (\sigma/r)^6\right]$, but $F = -\mathrm{d}u/\mathrm{d}r$.

Another approach in molecular dynamics simulation is to use numerical methods on the Newton's equation of motion. The most widely used is perhaps the Verlet algorithm, shown here as Algorithm 6.16. The initial values are r_0 and r_1.

Algorithm 6.16 *Verlet algorithm, cf L. Verlet (1967)*

```
for $i = 1$ to $n$ do
    find $f_i$;
    $r_{i+1} \leftarrow 2r_i - r_{i-1} + f_i\Delta t^2/m + \mathrm{O}(\Delta t^4)$ and
    $v_i \leftarrow (r_{i+1} - r_{i-1})/(2\Delta t) + \mathrm{O}(\Delta t^2)$.
endfor                                                        □
```

There are variants and modifications of Algorithm 6.16, for example the Leapfrog Verlet algorithm which has three steps instead of two, that is $v_{n+1/2} = v_{n-1/2} + f_n/m\Delta t + O(\Delta t^3)$, $r_{n+1} = r_n + v_{n+1/2}\Delta t + O(\Delta t^4)$ and $v_n = (v_{n+1/2} + v_{n-1/2})/2 + O(\Delta t^2)$; or the velocity Verlet algorithm where $r_{n+1} = r_n + v_n\Delta t + f_n\Delta t^2/(2m) + O(\Delta t^3)$ and $v_{n+1} = v_n + \Delta t(f_{n+1} + f_n)/(2m) + O(\Delta t^3)$, or in the form normally used in practice in which $v_{n+1/2} = v_n + f_n\Delta t/(2m)$, $r_{n+1} = r_n + v_{n+1/2}\Delta t$ and $v_{n+1} = v_n + f_{n+1}\Delta t/(2m)$.

The van der Waals force is an attractive force acting between molecules. In the case of gases, each molecule consumes some space, so the dynamic volume is less than overall volume by an amount bn when b is a constant and n the number of molecules. Here b is $N_A v$, where v is the volume slightly larger than the volume of each molecule of gas and N_A the Avogadro number, $N_A = 6.022 \times 10^{-23}$ mol^{-1}. The pressure of gas we see is the pressure of gas detected, which is less than the real pressure by F/A, where $F = \sum F_i$ and $F_i = \sum f_j$. Here i runs from 1 to n and j from 1 to $(n-1)$ in the same set of molecules. The reduction of the total force comes from all molecules, but this reduction from each molecule is in turn affected by those molecules around it. Therefore $i \neq j$ and j runs from 1 to $(n-1)$ as mentioned above. Suppose that each molecule attracts another molecule by a force a. Then, in a given volume V, $F = \sum F_i = \sum_i \sum_j f_{ij} = \sum_i (n-1)a/V = n(n-1)a^2/V^2$. For very large n we can say that $(n-1) \approx n$, and therefore $(P + an^2/V^2)(V - bn) = nRT$.

The nature of this mutual attraction between molecules, which reduces the pressure in gas, comes to light in the case of solids. The distribution of charges around a neutral atom in solid fluctuates in the time scale $\tau < 10^{-16}$ s. This charge imbalance makes each atom behave as an electric dipole. This electric dipole has an electric field around it, which affects other atoms nearby and binds the two together.

The electric dipole moment is $p = qa$, where a is a vector from the negative to the positive charge. The electric field around a charge q_1 has a magnitude $E = q/(4\pi\varepsilon_0 r^2)$. It acts on a nearby charge q_2 with a force of magnitude $F = u = q_2 E = q_1 q_2/(4\pi\varepsilon_0 r^2)$, where u is the potential energy of the two charges. This force itself is the Coulomb force, and the electric potential around q_1 is u/q_2, that is $V = q_1/(4\pi\varepsilon_0 r)$.

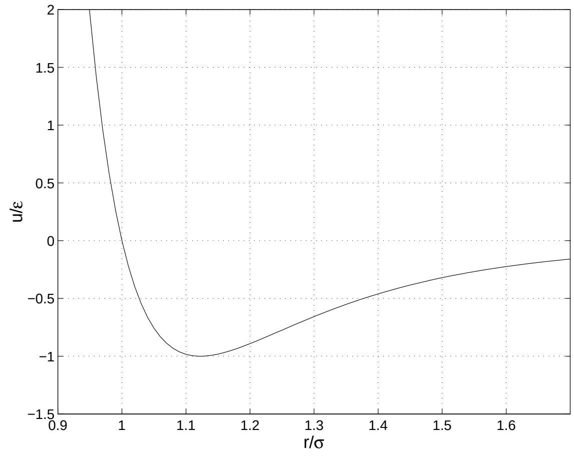
For an electric dipole, $V = (1/(4\pi\varepsilon_0))(q/r_1 - q/r_2) = q(r_2 - r_1)/(4\pi\varepsilon_0 r_1 r_2)$. Since $a \ll r$, a being of the order of the atomic size, $r_1 r_2 \approx r^2$. Let θ be the angle between a and r. Then, also since

$a \ll r$, we have $r_2 - r_1 \approx a\cos\theta$. Therefore we now have $V \approx qa\cos\theta/(4\pi\varepsilon_0 r^2) = p\cos\theta/4\pi\varepsilon_0 r^2$ and consequently $E_r = -\partial V/\partial r = 2p\cos\theta/(4\pi\varepsilon_0 r^3)$ and $E_\theta = -\partial V/(r\partial\theta) = p\sin\theta/(4\pi\varepsilon_0 r^3)$.

We neglect E_θ for long-range interaction, and simplify E_r to $E_r = a/r^3$. We shall hereafter use the terms *dipole* and *atom* interchangeably. If this electric field is produced by atom p_1, it could induce in another atom $p_2 = \alpha E_r$, where α is the molecular polarisability of the second atom. This second atom will have an energy of interaction with the first one, $u = -p_2 E_r = -\alpha E_r^2 = -\alpha a^2/r^6$. This force is attractive.

The repulsive force is more complicated, and is generally thought to arise from the Pauli exclusion principle and the coulombic repulsion of the electrons in the outer orbit. It is generally assumed to be either $u \approx A/r^{12}$ or $u \approx a\exp(-r/\rho)$, where ρ is a range parameter (*cf* de Podesta, 2002).

The Lennard-Jones potential is often written as $u = -4\varepsilon\left[(\sigma/r)^6 - (\sigma/r)^{12}\right]$, where σ is a range parameter that indicates the approximate size of an atom and ε an energy parameter that indicates the strength of the interaction between atoms. In other words, it is the minimum value of u.



The Lennard-Jones potential between two atoms has the minimum value $u = -\varepsilon$ at $r = 1.1225\sigma$. Figure 6.5 is a plot $y = -4(1/x^6 - 1/x^{12})$, where $y = u/\varepsilon$ and $x = r/\sigma$. The force which the potential curve of Figure 6.5 acts on another particle is shown in Figure 6.6. The force produced by a dipole is calculated from $F = -\mathrm{d}u/\mathrm{d}r = 24(\varepsilon/\sigma)[(\sigma/r)^7 - 2(\sigma/r)^{13}]$. But in our units $y = u/\varepsilon$ and $x = r/\sigma$ above the equation becomes $y = 24(1/x^7 - 2/x^{13})$, which is plotted in Figure 6.6.

Figure 6.5 Lennard-Jones potential.

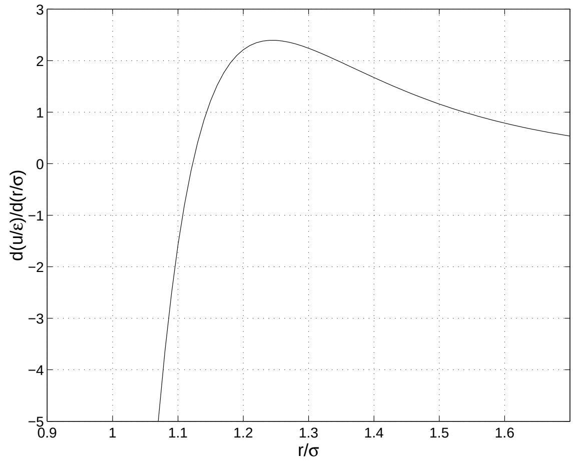


Figure 6.6 shows that the force is zero at the distance $r = 1.1225\sigma$ away from the particle. Closer than this point the repulsive force increases rapidly towards infinity. Further away from this point, however, the attractive force increases to a maximum and then gradually dies down. This maximum force occurs when $\mathrm{d}^2y/\mathrm{d}x^2 = 0$, where $y = u/\varepsilon$ and $x = r/\sigma$, that is at $r = 1.2445\sigma$.

Figure 6.6 Force corresponding to the Lennard-Jones potential.

In gases, the molecules travel past one another so fast that they hardly notice the wells of negative energy surrounding other molecules, let alone stop and rest there. But even here the attractive force produced by these wells is probably what give rise to the term an^2/V^2 in the van der Waals equation. Solid molecules, in contrast, hardly have kinetic energy and therefore prefer to sit in such wells of their neighbours, which is the reason why solids hold and neither flow as liquid does nor disperse like gas.

The most cohesive structures of solids are crystals, where the Lennard-Jones potential culminates in a minimum cohesive energy of the lattice. This cohesive energy is $U = N_A u_i/2$. Here u_i is the summation over all pair potential energies of Lennard-Jones type, $u_i = \sum_{i \neq j}^{N_A} u(r_{ij})$. Let $r_{ij} = \alpha_{ij} r_0$, where r_0 is the nearest neighbour distance. Then $U = -2\varepsilon N_A\left[(\sigma/r_0)A_6 - (\sigma/r_0)A_{12}\right]$, with the lattice sums being $A_6 = \sum(1/\alpha_{ij}^6)$ and $A_{12} = \sum(1/\alpha_{ij}^{12})$. These lattice sums are calculated from the network structure in terms of infinite series that converge very quickly.

To find the optimum value of σ/r_0 we shall let $b = \sigma/r_0$ and write differentiate $\mathrm{d}U/\mathrm{d}b = -2\varepsilon N_A(6b^5A_6 - 12b^{11}A_{12}) = 0$. As a result, $r_0 = \sigma(2A_{12}/A_6)^{1/6}$, the cohesive energy per mole $U = -(A_6^2/2A_{12})N_A\varepsilon$ and the cohesive energy per molecule $u = U/N_A = -(A_6^2/2A_{12})\varepsilon$.

§ 6.5 Problem definition and algorithms

Walls in a 3-d Voronoi tessellation are isomorphic to bonds that link between its cells. Therefore we can redefine the problem of a particle passing through the hole in a wall, into the cell chamber and out through a hole in another wall, and so on, as that of the passage of a mathematical particle through a tube that links between two cells, into a cell and out through one of the other tubes. In reality the solid parts which make up the partitions have thickness, therefore the holes through the walls will have a nonzero thickness and the volume of the cell chamber is smaller than that of the otherwise mathematical Voronoi cell.

Each vertex of a Voronoi polygon has three walls of the latter attached to it. The centre of gravity† of these three faces is calculated and linked together. This truncates the coigns and produces for each polygon its dual self. The next step is to link together the midpoints of all those edges of the polygon that have a vertex in common. The edges of this last polygon bound and define the void volume of the cell chamber.

Next, the cross section of the hole within each wall is taken to be the area on the original wall which is bound on all sides by the second order covering lattice of that wall.

The filtering membrane in this case is assumed to be isomorphic and homogeneous. The size of the particles is taken to be reasonably smaller than the void in general, so that the attraction between particles and the adsorption to the wall play a more prominent role than the physical blockages by individual particles. The particles are all assumed to be of the same size.

Algorithm 6.17 is an algorithm to do filtration that is being developed. Here N contains the neighbourhood information

Algorithm 6.17 *Filtration in Voronoi tessellation*

```
find $(v_a, c_a) \leftarrow$ Voronoi tessellation;
find $d_l \leftarrow$ Delaunay triangulation;
find $N_c \leftarrow$ cell neighbours;
$c \leftarrow c_a$ which lie completely within $[0,1]^3$;
$v \leftarrow v_a$ which belong to some $c$;
```

□

My work on filtering membranes results in another Voronoi algorithm and programme which is different from the programmes listed in §'s A.3 and A.4. Because it is also written anew from scratch, this new algorithm has nothing to do with the previous two as regarding the data structure and the logic of its method. It is given in § A.21.

The current convention for variables is this. A single alphabet or entity means a list, for example `c` is the cell list and `va` is the list of all vertices original created. Another example is b, a structure for bonds which contains the list of the cells of bonds, the number of vertices of the face represented by each bond, the list and the map of these vertices, the ordered list of the vertices of each face and another list similar to this but cyclic, and the number of cells connected to each bond. Similar to the structure of `b` is that of `bdr`, or b_d in Algorithm 6.18, which is a list of the bonds along the border, each of which is connected to only one cell.

Two entities xy makes x of y, for instance `vc` is vertices of cells, which is a structure that contains the number of vertices of each cell and the list and the map of all these vertices. I try to preserve the space by creating a short but easy to understand naming convention. Also, the lines are put together, which means that the number of lines of the codes is more than what appears here, some of the lines having six logical lines or more to them. But the structure of the programme still remains intact, therefore it should be possible without much difficulty to compare the programme in § A.21 with Algorithm 6.18 which explains it.

Three entities xy^2 or y_x^2 are $y \times y$ matrix of x, where the latter relates two entities of the former, for example `bcc` is the bonds mapped on to a cell matrix. In the algorithm, v_n and c_n are respectively the `vin` and `cin` in the programme, the `vn` and `cn` there being verbosely described in the algorithm as the *number of* v and c.

Algorithm 6.18 *Voronoi data structure for the study of membrane filters.*

---

† aka centre of mass, centroid.

```
$(v_a, v_c^a) \leftarrow$ find a 3-d Voronoi tessellation;
$v_n \leftarrow$ index of $0 < v_a < 1$;
$c_n \leftarrow$ index of $c$ all the $v$'s of which are in $v_n$;
$v \leftarrow v_a \in v_n$;
$v_c \leftarrow v_c^a \in c_n$;
$t_a \leftarrow$ find a 3-d Delaunay tessellation;
for all $t_i \in t_a$ which are connected to two cells do
  $t \leftarrow t_i$;
else
  $b_d \leftarrow t_i$;
end
$b \leftarrow c_b^2 \leftarrow t$;
$b \leftarrow b_d$;
find Delaunay triangulation in $(3 - 1)$ dimensions for all faces;
order the vertices of each face;                                   □
```

§ 6.6 Simplified algorithm for filtration

The first algorithm we shall now develop is a reasonably simplified one. But it turns out to my surprise that this simple algorithm links us to the continuum percolation of hard spheres or even of particles with irregular shapes. This is because of the assumptions that we shall make.

One such assumption is that the variance of the filter cell size distribution is small. For this, Jafferali (1995) would probably have applied his favourite constraint code on a Voronoi tessellation. The approach which I developed uses the operator $\mathcal{G}(\cdot)$ on a $\mathcal{V}$ (*cf* § A.23). It is more logical and therefore is what we shall use for now.

One favourite constraint that has been related to a Voronoi Tessellation in application is that which limits the minimum distance between any pair of the nuclei. The reason given for this is usually that in nature cells are of similar size and shape. This implies that the nuclei never get together closer than than a certain distance, which can be found empirically by physically making some observations and measurements. The cause of this is often put to the hydrostatic pressure inside the cells, which pushing at the cell wall against the pressure outside at the same time pushes the nucleus back, away from them. For this reason a rule of thumb is created as a criterion used in the generation of Voronoi tessellations for the study of various kinds of structure.

Centre of gravity is a very important point for each object. The stability of a double-decker is determined not by its height but by the location of the c.g. All forces acting on an object with no angular velocity can be represented by an imaginary force passing through the c.g. In martial arts when it comes to a face-to-face combat you want to keep your eyes on the c.g. of your opponent. The limbs and anything may fly all over the place but wherever his centre of gravity be ultimately that is his move. Though in this last case one has to keep in mind the body is nonrigid, and each of its components has a different momentum.

In doing the simulation, a Voronoi tessellation is first created using Poisson points as nuclei. Then for each cell the centre of gravity is found. And then another Voronoi tessellation is created using all these centres of gravity calculated as nuclei. Repeating this procedure, the centre of gravity function $\mathcal{G}(\cdot)$, n times eventually gives us the Voronoi tessellation $\mathcal{V}(\mathcal{G}^n(\mathbf{x}))$ we want. Here $\mathbf{x}$ is the set of all the nuclei.

The use of centre of gravity this way is new and has not been found in literature. However it is simple while at the same time provides a logical starting point, a reasonable basis to work upon which can satisfactorily replace the arbitrary limiting distance criteria in use.

The second assumption is that the particle size is reasonably smaller than the size of the smallest void of the original system. This is the assumption which seems to link us to the percolation of spheres in continuum, the space considered being that of each void.

Also, we assume that the particles are attracted towards one another and towards the walls by various interactions which may include the van der Waals force and the force from electrostatic interactions.

Let θ be the angle that the line from the c.g. of a each cell to the mid point of a face makes with the horizontal plane. Then another possible assumption is that all particles prefer a trajectory which goes through a face which has the maximum θ.

The third assumption is that the volume of each membrane pore is 80 per cent the volume of the corresponding Voronoi pore. The fourth assumption, blockages due to the clustering caused by the attrition forces between particles can occur within pores. This excludes cake formation and blockage at entrances to, or exits from the pore.

The last assumption above is similar to assuming that clustering due to attrition can occur in stagnation regions. Since the flow of the suspension reaching the membrane is strong, *i.e.* the pressure high, there can be no clustering there. The flow from one pore into another is also faster than the flow within one. This is amount to assuming that the effective diameter of the hole within each face of a pore is considerably smaller than its diameter.

Upon reaching the membrane, each particle in the troop seeks out the bond closest to it, and begins its journey through the membrane. Within the membrane, it always follow the bond which has the greatest gradient available. Therefore the top bonds should map to the bottom ones one to one. But because the possible blockages during the course of operation, this may not be so and the mapping is instead one to many.

First we consider the case in two dimension, so volumes becomes areas and the volumetric flow rate the distance travelled. In our discrete time, if v is the flow velocity, n the number of particles, ρ_v the total volume ratio of the particles and r their radius, then we have $n\pi r^2 = v\Delta t$. Assuming that $v = 1\ \mathrm{ms}^{-1}$ and $r = 100\ \mu\mathrm{m}$, then $n = 10^3\rho_v/\pi$ or approximately $318\rho_v$ particles for each time step.

Next we will study the suspension in a square box in order to find out ρ_v. Because we shall assume that the particles flowing through the pores percolate as though there is no flow, we will consider here the suspension that is simply contained within our box without moving about. I feel that the assumption that particles could percolate that way in pores is justified since the flow is in steady state, protected from the turbulence outside by all the solid structures making up the walls of the membrane.

The percolation programme first generates random positions of the particles, then moves apart those which are too close together so that in the end they only touch each other. Particles which are separated by a distance less than $1.4r$ move towards each other until they touch. Lastly, touching particles never separate.

We know that the total volume ratio of the particles has the upper limit of 0.9069, because that is the density of the closest packing of circles on a plane.

Packing circles on the plane becomes densest when the circles are arranged as a hexagonal lattice with the packing density of $\pi/2\sqrt{3}$. Packing of spheres is similarly at its highest density if the arrangement is that of the face-centred cubic lattice with the packing density $\pi/3\sqrt{2}$.

§ 6.7 Filtering problem when physical blockage is prominent

Assuming particles to be spherical, the size of particles to be constant, and that this size is compatible with the size of the holes in the walls and the voids so that physical blocking is responsible for most of the blockages in the membrane. As in § 6.5, the hole in each wall is taken to be the polygon which results from recursively finding a covering polygon for the face twice.

Redefine the problem of particles' passage through walls and voids as that of particles travelling along edges of the dual lattice of the Voronoi structure, *i.e.* the Delaunay triangulation. Each of these edges corresponds to a face in the original physical lattice. To each edge is thus ascribed the details of the cross section of the hole through that face, namely the shape and the dimension, as well as the gradient it makes with the horizontal plane. Upon reaching a vertex, the particle ball will choose the next path, *i.e.* bond in the dual lattice, which has the maximum gradient to pass through. However, if this bond is too small or if it is blocked, the particle with choose from among the remaining paths the one which has the maximum gradient, and so forth. If the size of the particle is approximately that of the hole it tries to pass through, within two per cent of the latter, say, then the particle will blind the passage at that point. The difference between blinding and blocking is in the degree of tightness that the particle sits in the hole, which reflects in the degree of difficulty to remove it by backflushing. This degree is not constant but a function of the relative size between the two parties involved, *i.e.* the particle and the hole.

§ 6.8 Percolative filtering with very small particles

At this point a new filtering algorithm is considered and investigated. We introduce an assumption that the particles are all of the same size which is very small compared with the size of the pores. So there is neither cake formation nor blinding by a single particle. Assume that the solid particles suspended in a fluid medium, being dragged downwards under their own weight.

Assume that the van der Waals force plays a significant part and, similarly to § 6.4 on page 181, that these interactions between particles are governed by the Lennard-Jones equation. Assuming that each particle is spherical and acts as a dipole with the electric dipole $p = 10e$, where e is the electric charge on a proton, $e = 1.602 \times 10^{-19}$ C. Then (*cf* de Podesta, 1996)

$$E_r = \frac{p}{4\pi\varepsilon_0 r^3}, \qquad \mathbf{(26)}_{\mathrm{vi}}$$

and the energy of interaction between two particles becomes

$$u_r = -\alpha E_r^2 = \frac{-\alpha p^2}{(4\pi)^2\varepsilon_0^2 r^6}, \qquad \mathbf{(27)}_{\mathrm{vi}}$$

where α is the molecular polarisability of one particle under the influence of the other. If we assume that the repulsive force acts in such a way that the repulsive energy is $u = c/r^{12}$, where c is a constant, the Lennard-Jones potential becomes Equation 28(vi).

$$u_r = -\frac{\alpha p^2}{(4\pi)^2\varepsilon_0^2 r^6} + \frac{c}{r^{12}}. \qquad \mathbf{(28)}_{\mathrm{vi}}$$

Comparing Equation 28(vi) to the form

$$u_r = -\frac{A}{r^6} + \frac{B}{r^{12}}, \qquad \mathbf{(29)}_{\mathrm{vi}}$$

we have $A = \alpha p^2/((4\pi)^2\varepsilon_0^2)$ and $B = c$. Or if we compare it to the form

$$u_r = -4\varepsilon\left[(\frac{\sigma}{r})^6 - (\frac{\sigma}{r})^{12}\right], \qquad \mathbf{(30)}_{\mathrm{vi}}$$

then we have $\sigma = (B/A)^{1/6} = (16c\pi^2\varepsilon_0^2/(\alpha p^2))^{1/6}$ and $\varepsilon = (A^2/4B) = \alpha^2 p^4/(4c(4\pi)^4\varepsilon_0^4)$. Notice that here σ and ε are simply parameters, not the Stephan-Boltmann constant and the dielectric constant, σ is a range parameter and approximates the size of an atom while ε is an energy parameter which shows the strength of the interaction between particles. At $r = \sigma$ the value of u_r is zero, whereas u_r has the minimum value of $-\varepsilon$.

The force between two particles is Equation 31(vi).

$$F = -\frac{\mathrm{d}u_r}{\mathrm{d}r} = -\frac{6\alpha p^2}{(4\pi)^2\varepsilon_0^2 r^7} + \frac{12c}{r^{13}}. \qquad \mathbf{(31)}_{\mathrm{vi}}$$

At the equilibrium separation, r_0, $F = 0$ and therefore Equation 31(vi) yields the minimum distance in Equation 32(vi),

$$r_0 = \left(\frac{32c\pi^2\varepsilon_0^2}{\alpha p^2}\right)^{\frac{1}{6}}. \qquad \mathbf{(32)}_{\mathrm{vi}}$$

The equation for the minimum energy is obtained by substituting r_0 from Equation 32(vi) into Equation 28(vi), which gives

$$u_r = -\frac{\alpha p^2}{(4\pi)^2\varepsilon_0^2 r^6} + \frac{c}{r^{12}} \qquad \mathbf{(33)}_{\mathrm{vi}}$$

$$= \frac{1}{r_0^6}\left(-\frac{\alpha p^2}{(4\pi)^2\varepsilon_0^2} + \frac{c}{r_0^6}\right) \qquad \mathbf{(34)}_{\mathrm{vi}}$$

$$= \frac{1}{r_0^6}\left(-\frac{\alpha p^2}{(4\pi)^2\varepsilon_0^2} + \frac{c\alpha p^2}{32c\pi^2\varepsilon_0^2}\right) \qquad \mathbf{(35)}_{\mathrm{vi}}$$

$$= -\frac{\alpha p^2}{32\pi^2\varepsilon_0^2 r_0^6}. \qquad \mathbf{(36)}_{\mathrm{vi}}$$

Returning to Equation 31(vi), the attractive force has the maximum value at the point where $\mathrm{d}F/\mathrm{d}r = 0$. Differentiating F in Equation 31(vi) with respect to r, we arrive at Equation 37(vi).

$$\frac{\mathrm{d}F}{\mathrm{d}r} = \frac{42\alpha p^2}{(4\pi)^2\varepsilon_0^2 r^8} - \frac{156c}{r^{14}} \qquad \mathbf{(37)}_{\mathrm{vi}}$$

From this the distance where this maximum force occurs is given by Equation 38(vi),

$$r_m^6 = \frac{156c(4\pi)^2\varepsilon_0^2}{42\alpha p^2} = \frac{416c\pi^2\varepsilon_0^2}{7\alpha p^2}. \qquad \mathbf{(38)}_{\mathrm{vi}}$$

Then by putting Equation 38(vi) into Equation 31(vi) we have this maximum force in Equation 41(vi).

$$F_m = -\frac{6\alpha p^2}{(4\pi)^2\varepsilon_0^2 r^7} + \frac{12c}{r^{13}} \tag{39$_{\text{vi}}$}$$

$$= \frac{1}{r^7}\left(\frac{-6\alpha p^2}{(4\pi)^2\varepsilon_0^2} + \frac{12c(42)\alpha p^2}{156c(4\pi)^2\varepsilon_0^2}\right) \tag{40$_{\text{vi}}$}$$

$$= -\frac{9\alpha p^2}{52 r^7 \pi^2 \varepsilon_0^2}. \tag{41$_{\text{vi}}$}$$

Let each of the spherical particles has a radius $r = 1$ μm and the density $\rho = 3{,}000$ kg $\cdot$ m^{-3} (compare the density of silicon, which is 2,329 kg $\cdot$ m^{-3}). Then the mass of the particle is $m = \rho V = 3000 \times (4/3)\pi \times (10^{-6})^3 = 1.26 \times 10^{-14}$ kg and the weight is $mg = 1.26 \times 10^{-14} \times 9.8 = 1.23 \times 10^{-13}$ N. We may now find the attractive distance, the maximum distance whereby two particles will come together under the van der Waals force. Figure 6.7 shows the capturing of one particle by another when one particle is fixed in space and another particle has zero velocity but is free to move. Recall that the capture occurs when the accelaration due to the weight of the particle g equals the acceleration a due to f.

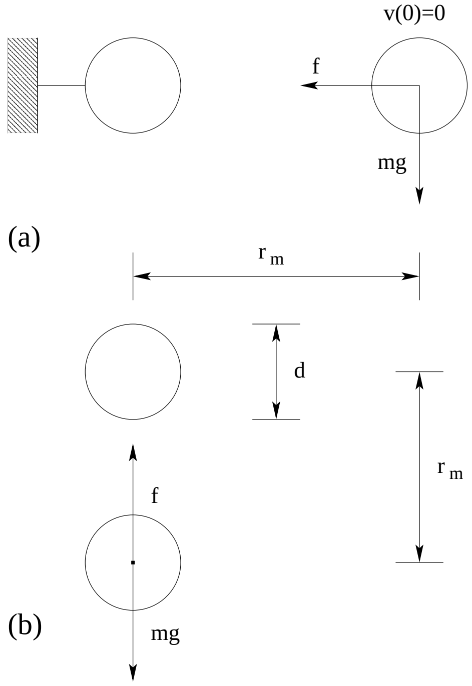


Figure 6.7 Force balance with one particle fixed.

If we suppose that our particle have the same polarisability as that of a benzene in its gaseous state, C_6H_6, *i.e.* $\alpha = 11.61 \times 10^{-40}$, then in this case $\alpha = 11.61 \times 10^{-40}$ F^{-1}m^4 (*cf* de Podesta, 1996) and Equation 41(vi) gives us the separating distance between the two particles which gives the maximum attractive van der Waals force, which is 444 μm. Giving our particle other values of α, with the value of α for methanol gas CH_3OH, *i.e.* $\alpha = 3.860 \times 10^{-40}$ F^{-1}m^4, we have this distance $r = 379$ μm, and with $\alpha = 1.647 \times 10^{-40}$ F^{-1}m^4, that of water vapour, the distance becomes $r = 336$ μm.

These values are rather large compared with the radius of the particles, therefore we shall opt instead to a bigger size of particles, when $r = 5$ μm. With the radius of five microns, the particle has a mass of 1.571×10^{-12} kg and its weight becomes 1.64×10^{-11} N. Then if we adopt the value of α for water vapour, $\alpha = 1.647 \times 10^{-40}$ F^{-1}m^4, then from Equation 41(vi) the distance where the force is maximum becomes $r = 167$ μm.

But this is not everything. So far we have only considered what two particles will do when one of them is fixed and the other one has no initial velocity. There are two other things that can happen, the particles may both be moving down beside each other under gravity and the path of the particle which is effected by their mutual attraction. Even when two particles do not come together, their path can be deviated by the van der Waals force. But here for simplicity we shall neglect this and assume that particles either come and stay together or they experience no mutual force whatever. Only if they come together will their final path and velocity be affected, and these depend on their combined momentum.

When particles move relative to each other, their captive velocity depends on their relative velocity. They tend to join each other more easily if their paths are along side each other and goes in the same direction. In the extreme case where both particles move in the same direction with zero relative velocity, they would come together across a vast gap in between indeed, had there been no frictional force due to viscosity that acts to drag them.

In the present study, the friction due to the fluid in the medium is neglected, together with the relative velocity between the particles, for the purpose of deciding whether particles will come together. It is expected that on average particles coming to within the capturing radius of each other will come together. This radius is larger than the radius of maximum force in Equation 41(vi) above. In other words, we expect our particles to behave like solid spheres with a well defined boundary for their sphere of influence and a much simplified capturing mechanism.

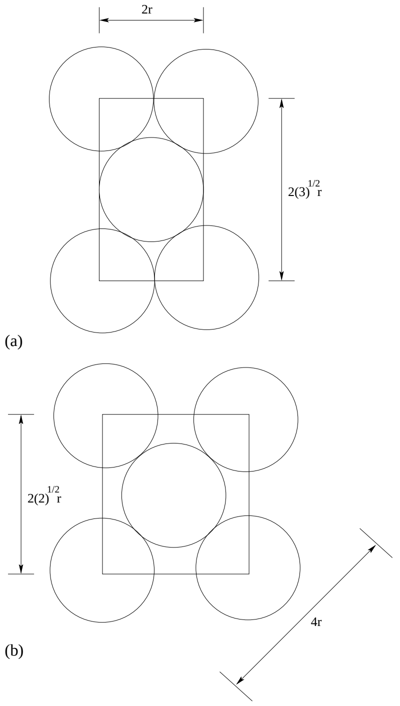
We shall define this radius of captivity, r_c, to be one third that of the radius of maximum force r_m. From Figure 6.5 this is the point where $r_c = 1.5\sigma$, and because r_m is here 1.2445 we have $r_c/r_m \approx 1.2$. For Figure 6.5, this is the same as saying that $r_c/r_0 = 1.34$.

One additional point is that, in real situation when particles form a cluster their collective value of the molecular polarisability will change, and this value will not be the same for the cluster as it is for the individual particles. But here we assume that they are the same, which means that the capturing radius of a cluster will be the same as that of the particles which form it.

It may worth mentioning here that the molecular polarisability is related to an optical property, *viz.* the refractive index n_r, by the relationship $\alpha = \varepsilon_0(n_r^2 - 1)/n$, where n is the number density of molecules. For gases $n = P/(k_B T)$, whereas for liquids $n = N_A\rho/m$, m being the mass.

Next we shall concern ourselves with clusters of particles thus formed. All clusters will be assumed to be a closest-conglomerate of particles which form them, with the densest packing density possible in three dimensions, that is $\pi/3\sqrt{2}$ of the face-centred cubic lattice. Numerically this is 0.7405.

Notice that the packing density in two dimensions can be higher than this. Packing circles on a plane is the densest of all with its packing density of $\pi/2\sqrt{3}$.



Shown in Figure 6.8 are the hexagonal lattice packing in two dimensions and the cubic close pack. In the cubic close packing spheres in every third layer lie vertically straight on top of one another. Each face of its cubic section looks like the second picture in Figure 6.8. Similar to the cubic close packing is the hexagonal close packing spheres in every alternate layer of which lie over one another. Both the cubic- and the hexagonal close packing have the same packing density which, in the case of the former, is calculated, from the second picture, as $\rho = \sum v_s / \sum v_c$, where v_s are the total volume of sphere segments in the unit cell which has the volume v_c. Here $v_c = (2\sqrt{2}r)^3$ and $\sum v_s = (8(1/8) + 6(1/2))4\pi r^3/3$. In 2 dimensions, the densest packing is calculated from the first picture.

Figure 6.8 The packing density calculation of the closest-packed densities.

The density of the packing is $\sum a_c/a_r$, where a_c are the areas of the circles and a_r is the area of a rectangle. These two areas are namely $a_c = 2(\pi r^2)$ and $a_r = 2r(2\sqrt{3}r)$, which give the density $\rho = \pi/2\sqrt{3}$.

If the particle size is very small there will be an effect from quantum mechanics. The de Broglie wavelength, $\lambda = h/p$, exists for all particles large and small. Here $p = mv$, $m = \rho V$ and $V = \pi d^3/6$ for a spherical particle. Figure 6.9 shows a graph between diameter and wavelength of particles.

We can see that the quantum-mechanical effect when the pore size is in the order of micrometre will become prominent if the size of particles challenging it is in the order of nanometre. When this is the case there will be not only the interference effect of the particle-wave passing through slits but also the effect of confined particles which produces standing waves of these wave-particles in each pore. These standing waves of the particle-wave have $\lambda_n = 2D/n$.

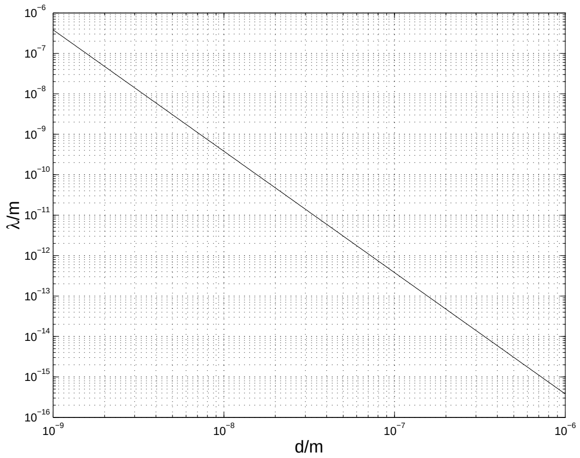


Figure 6.9 Quantum-mechanical effect in filtration.

§ 6.9 Percolation within percolation

Particles suspended in a fluid generally have their diameter d_p much smaller than the pore diameter d_v of a filtering membrane. Yet these small particles can cause blockages of the membranes due to attrition among themselves. Since in this case $d_p \ll d_v$, blockages due to blocking or blinding of the individual particles (*cf* Jackson, 1994; Jafferali, 1995) become out of question.

Having investigated both the percolations of networks and continuum, I suggest that the blockage of these smaller particles in membranes is due to a double percolation phenomena, one the percolation of the suspension continuum, the other the percolation of the centroidal Voronoi network. As a reminder of a centroidal Voronoi network, it is a Voronoi tessellation on generator points which are the centroids of a Voronoi network which either is generated from Poisson point generators or is another centroidal Voronoi network.

Because percolation is a study of the behaviour of two phases, and because in general $p_c \neq 1/2$, there are not only two but no less than three states or regions of behaviour to consider in each percolational investigation (*cf* Tiyapan, 1997, KNT5(ix); also in Tiyapan, 2003, KNT8(iii)). When $p_c < 0.5$ these three regions are $p < p_c$, $p_c \leq p \leq (1 - p_c)$ and $p > (1 - p_c)$, and when $p_c > 0.5$ they are $p < (1 - p_c)$, $(1 - p_c) \leq p \leq p_c$ and $p > p_c$. The case where $p_c = 0.5$ is assumed to be very rare in nature, and so can be neglected in the present study.

When a suspension becomes so concentrated that the average interparticle distance has become such that the attrition due to van der Waals force is prominent, it will solidify into a moisted bed of particles. According to the percolation theory, we may define the point where this spontaneous solidification occurs to be that point where there is a single cluster, under a mutual van der Waals force, which traverses the whole continuum in a certain well-conditioned direction, that is a direction which may represent the diameter of the network. Furthermore, let us call a critical concentration ρ_c the minimum concentration at which this infinite cluster appears.

Then we have for our suspension a continuum percolation with three regions of behaviour similar to those we have found in the case of network percolation. Furthermore we map the space of ρ_c on to that of $0 \leq p_c \leq 1$, where $p_c = 0$ means there are no particles suspended in the fluid, in other word $\rho = 0$, and $p_c = 1$ is where the suspended particles form a bed in the closest packed structure, that is $\rho = \rho_{\max}$. We also assume, without the loss of generality, that $p_c > 0.5$. Then we have the following as the three regions, $p < (1 - p_c)$, $(1 - p_c) \leq p \leq p_c$ and $p > p_c$.

We are interested in neither the cases $p < (1 - p_c)$ nor $p > p_c$, since the former implies too dilute a concentration for the attrition due to the van der Waals force to cause an infinite cluster, while the latter means that the suspension is so concentrated that they solidify instantaneously, simultaneously in all pores.

When $(1 - p_c) \leq p \leq p_c$, all pores has an equal probability of being solidified, and so the per cent total solidified cells now depends on the topology of the network, in this case Voronoi, and the probability where the critical phase change occurs becomes the critical probability of the network.

In the case of dead-end filtration the flow is in one direction, therefore the critical flux reduction which is the result of percolation of the network occurs at the point where the cross section of the network, not the network itself, percolates. This is because such percolation in the cross section in the plane perpendicular to the flow direction will cause a bottle neck in the flow and therefore determines the flux. This phenomenon is summarised in Figure 6.10.

Next we must define the critical probability of the overall system. Since the system involves two kinds of probability, that is continuum and network, and assuming p_{c_0} and p_{c_2} are two independent probabilities, then we have the overall probability is

$$p_c = p_0 p_{c_2}, \qquad \mathbf{(42)}_{\rm vi}$$

where $(1 - p_{c_0}) \le p_0 \le p_{c_0}$.

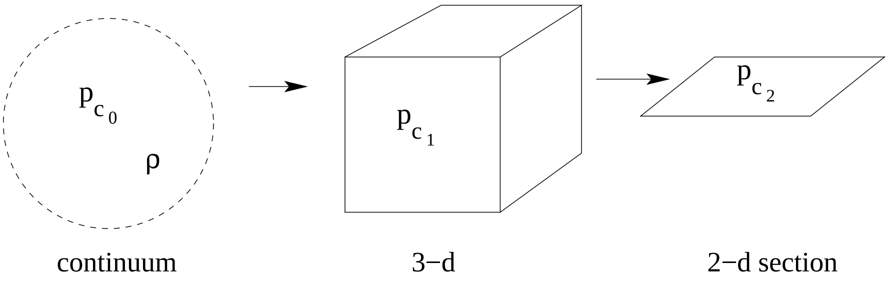


Figure 6.10 *Percolation within percolation;* $0 \le \rho_0 \le \rho_{\max}$, $0 \le p_0 \le 1$, $p_c = p_{c_2} \cdot p_0$.

If our membrane is homogeneous, the reduction in the area perpendicular to the flow becomes $\Delta A = A_v p_{c_2}$, where $V_v/V_t = A_v/A_t$, V_v and V_t are respectively the void volume and the total volume of the membrane, and similarly for the areas A_v and A_t.

§ 6.10 The first part, suspended particles

Because of the complex nature of the problem, there is no single algorithm but rather there is an algorithm for each job. The first task is to study the percolation of the spherical particles under the van der Waals force.

Continuing from the development in § 6.8, particles are spherical in shape with $r_p = 5$ μm and the capturing radius is $r_c = 167 \times 1.2 \approx 200$ μm. First we shall study particles within a cubic box of side length 2 mm. Particles start as a suspension with no obvious velocity. They stick together and to the walls.

When particles come together, they form a porous globule, having the densest packing density of the hexagonal or cubic close packing. When this happens, we discard the individual particles and consider instead the globular cluster which they formed. The cluster is porous, so its new radius is $r = (\pi/3\sqrt{2})(3\sum v_i/4\pi)^{1/3} = (\pi/3\sqrt{2})(3nv/4\pi)^{1/3} = \pi^{2/3}(nv)^{1/3}/(3^{2/3}2^{7/6})$.

But we do not know the rules by which these particles stick themselves together, whether they form a closest-packed globule or some other shapes. It is quite certain that whatever shape they are after, they may not retain it for long because there is a limited space within each pore that will put constraints on the way they grow. We shall call the growth of clusters into globules mentioned above *globular formation.*

Other clustering mechanisms possibly include what we shall call the *tetrahedra formation.* By this I mean that each one of our spherical particles attaches itself to three other particles, forming a tetrahedron whose side lengths are two times their radius. The next free particle may fit into any of the available attachment sites, *i.e.* the free triangular faces of an existing cluster. We may suppose that it always choose the closest one among such sites if this is available; if not, then the next closest one and so on.

These are only two among all the possibilities, namely the globular and tetrahedra formations. There could well be others, as well as a mixture of them. On the other hand, each material of which the particles are made may decide the particular cluster shape it prefers. Detailed analyses in thermodynamics and quantum mechanics are needed if we were to understand this cluster formation in continua under spatial constraints. Neither of these is within the scope and time constraint of the present work, though both of them merit a detailed investigation which I plan to carry out in the future.

Another problem arises when we come to consider percolation of our membrane. We may, for instance, say that it percolates when its structure in three dimensions percolates, or we may say that it percolates if there exists a cross section perpendicular to the flow which percolates in two dimensions. Since percolation of a cross section implies percolation of the structure but not vice versa, these two definitions of percolation due to suspension in membranes are not the same.

Choosing the percolation of sections as a criterion implies that we consider the superficial velocity of the flow whereas choosing the percolation in three dimensions as the criterion means that we focus on its interstitial velocity instead.

The algorithms for the study of percolation by tiny particles due to attrition in membranes which proposed here are Algorithm's 6.19 and 6.20. Algorithm 6.19 prepares the structure while 6.20 does the percolation simulation. Here both VT and $\mathcal{V}$ means the Voronoi tessellation. The appeal factor is the probability that a particle will choose to leave a cell via a certain bond. It is weight by the gradient of each bond, and is calculated over all bonds going in the downward direction from the cell. Transfer grids are square grids which help map the continuous plane at the top layer to bonds connected to it, that is to say, it maps a continuous Euclidean plane into discrete grids and from there on to bonds. In other words, $E^2 \to D^2 \to \{b\}$.

Algorithm 6.19 *Percolation by tiny particles due to attrition in membranes.*

```
generate a Voronoi tessellation in three dimensions;
transform the VT into a centroid VT;
find the cross section of its top layer;
$C \leftarrow C^2(\mathcal{V})$;
find transfer grids of $C$;
for every cell in VT do
    find the maximum chamber capacity of its cell;
    find appeal factors for all its bonds;
endfor                                                          □
```

Let the gradient of each bond be represented by an angle α that it makes with the horizontal plane. Then the gradient can be calculated from the coordinates of the two end points of each bond, providing that $z_2 > z_1$, from $\alpha = \tan^{-1}(z_{12}/((\Delta x)^2 + (\Delta y)^2)^{1/2})$, where the slope is downwards from p_2 to p_1, and as usual $z_{12} = z_2 - z_1$. Algorithm 6.20 describes the membrane percolation simulation proposed.

Algorithm 6.20 *Percolation by tiny particles due to attrition in membranes, percolation simulation.*

```
for each time step do
    for all arriving particles do
        find their random arrival position;
        round these positions to the precision of the grids;
        map positions on to bond numbers, using the grids;
    endfor
    for all particles do
        update distance travelled;
        update chamber crowding;
        find percolation of blocked chambers;
        if chambers percolate then
            terminate the simulation;
        endif
        endfor
    endfor                                                                  □
```

Here we concentrate on the interstitial flow velocity, therefore the percolation is supposed to occur when the chambers percolate in three dimensions. Coincidentally, this also makes the calculation easier. If we were to choose the percolation of sections as the deciding factor for percolation of the membrane, for instance, we would have needed to consider approximately $2n$ sections in total, where n is the number of chambers. With an equal probability for success for all the homogeneous sections, this would still leave us on average n sections to consider before we know that a sample percolates, if it does, though we would still need to test all the $2n$ sections in cases where it does not. This number $2n$ arises from the fact that to completely cover all combinations of grouping cells into sections we need to consider for each cell two sections for each existing cell, one touching its top while the other touches its bottom.

Sphere packing is a rich field of its own, within which the packing density is generally referred to as η, an efficiency, instead of the usual density symbol ρ. The rigid packings of spheres vary in density from the lowest in loose packing where $\eta \approx 0.06$ to the highest, which is shared by the cubic and the hexagonal closest packings, $\eta \approx 0.74$. A rigid packing is a packing in which all spheres touch at least four others, and the points by which each sphere touches its neighbours can neither be all in the same hemisphere nor all on an equator, *i.e.* a greatest circular section. So we can now limit the value of ρ that we shall use to be in accord with $0.06 < \eta < 0.74$. In this early stage we shall not use a Monte Carlo study to find the probable ρ, *i.e.* η, but will approximate it to be some value within the range mentioned. Since the most familiar sphere packing in human history must be that by which oranges are stacked at markets, especially open markets like the one at Bolton, which gives the efficiency of packing $\eta \approx 0.74$, we shall assume that this is the way the clusters arrange themselves.

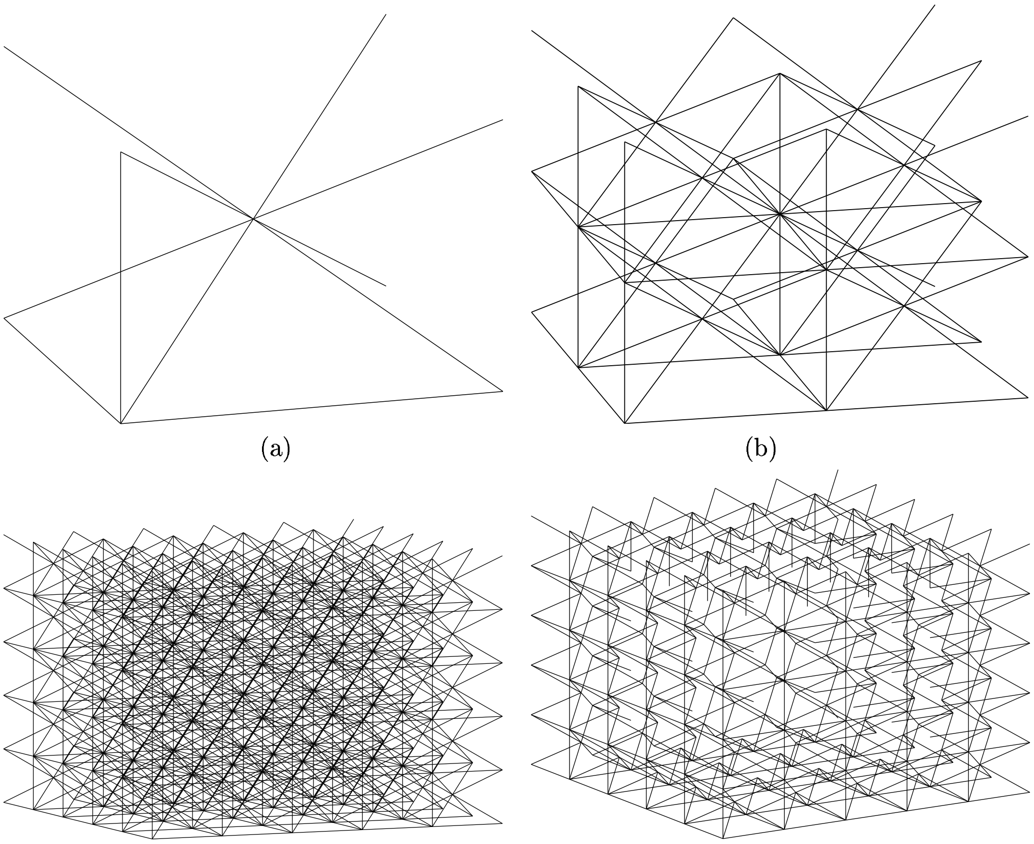
Notice also that piling oranges in a neat tetrahedral shape on a table and a packing them into a rectangular box both produce the same crystal structure, that is the face-centred lattice, the difference being only in their habits.

For all intents and purposes the percolation probability of spheres under the influence of the van der Waals force must be the same as p_c of a face-centred lattice. This is because the biggest cluster of both cases will have the same structure and their orientation will determine the orientation of the structure. We can do away with the orientation of other minor clusters precisely because they are much smaller, which justifies our grossing over their individual shapes and only concern ourselves about their statistics, that is to say, their number. I think that stacking oranges into a box is cubic close packing while a pile of oranges is hexagonal close packing, but this needs to be checked.

To find p_c of the close-packed cluster of spheres, one needs a programme similar to the one mentioned in § 6.10, but which would do the job for three dimensions instead of two. For this purpose, the programme for 2-d tilings mentioned in § 6.10 has been developed further to deal with regular lattices in three dimensions. At first I thought that there should be some other way to do this instead of having to develop another programme for a general lattice in three dimensions, since this is already the last week of the project and time is running out. But in the end I found it better to spend some time to systematically develop a programme for general cases than to opt for some adhoc approaches. As a result, a programme that creates regular lattices in three dimensions for the purpose of percolation study has been written and is listed in § A.30.

As the 2-d programme in § A.6 does for all 2-d regular lattices, this new programme can deal with all possible lattices in three dimensions. The difficulty is, however, in the meticulous nature of identifying all the vertices and links in each unit cell. In this respect, the cubic close packing is much simpler to do than the hexagonal close packing. Therefore we shall only do the first one while leaving out the second, which ideally could be used for the purpose of comparison.

Figure 6.11 shows the lattice generated by the programme and one which is used for finding the percolation threshold.


(a) (b)

(c) (d)

Figure 6.11 *Percolation of the cubic close-packed lattice;* (a) *a unit cell,* (b) *eight unit cells, one from each of the eight groups,* (c) *network of size* $5 \times 5 \times 5$ *unit cells, which is used in a simulation, and* (d) *the same network with only boundary edges drawn to make it easier to look at.*

At present the programme only finds p_v, p_e, x_v and x_e, not p_c, p_b, x_c and x_b. There may be altogether three types of cell and bond pairs, in comparison with two in the 2-d case, depending on whether the number of shared vertices required be 3, 2, or 1. For our purpose in the study of filtration, we only need to know p_v, which, when generated from a $5 \times 5 \times 5$ network as shown in Figure 6.11 (c), turns out to be 0.25. All the results from simulations are $p_v = 0.2501 \pm 0.0400$, $x_v = 8.0645$, $p_e = 0.1320 \pm 0.0209$, $x_e = 17.1709$, while $n_v = 341$ and $n_e = 1{,}375$.

This means that when the space will be blocked, *i.e.* percolates, when it is filled up to one quarter of its volume by suspended particles in the form of clusters of the highest packing density. Because the cubic close-packed spheres fill 0.74 of the space, this ratio translates into the real volume ratio of $0.74 \times 0.25 = 0.185$, that is 18.5 per cent by volume. If the fluid in our system is water, then $\rho = 1{,}000 \text{ kg} \cdot \text{m}^{-3}$ and the percentage by volume above is equivalent to a density of the suspended particles of $555 \text{ kg} \cdot \text{m}^{-3}$. Notice also in our simulation that the cluster shapes need not be convex.

In the light of the symmetry between particles and space, in other words between particles and anti-particles, which give rise to a symmetry and the three types operational regions that I originally proposed in a study of traffic congestion (*cf* § 7), the operational space of our filter may fall into three distinct regions when it is subjected to very small particles suspended in a fluid.

If we specify by ρ_v the ratio of the volume occupied by all the clusters to the total volume, and ρ the density in weight per volume, and if the three regions of operation are labelled I, II and III, then in the case of I, $0 \leq \rho_v < 0.25$, while for II, $0.25 \leq \rho_v < 0.75$ and for III, $0.75 \leq \rho_v \leq 1$. In other words, for I, II and III, we have respectively $0 \leq \rho < 555$, $555 \leq \rho < 1,665$ and $1,665 \leq \rho \leq 2,220$, where the unit of ρ is kilogram per cubic metre. Generalising this, the regions I, II and III correspond respectively to $0 \leq \rho < \rho_{c_1} = \rho_1$, $\rho_1 \leq \rho < \rho_{c_2} = \rho_2$ and $\rho_2 \leq \rho < \rho_c$, where ρ_c is determined by a physical constraint, namely the packing efficiency mentioned.

Qualitatively speaking, these three regions may correspond to the operational-, blocked and non-operational regions. If our system also contains other particles which are larger than the pores, then in Region I filters will operate normally until the effects of blocking or blinding by the large particles become prominent, as has been studied in literature (*cf* Jackson, 1994; Jafferali, 1995). In this case fouling of the filter is caused exclusively by the blocking or blinding of these larger particles, which result in the formation of cake, and unless $\rho = 0$ there will be some blockages of internal pores due to the blockage caused by small suspended particles forming cluster. This latter type of blockage, which is of our concern, is to be expected due to various reasons. Cluster formation may be caused by nonhomogeneity in the concentration of the suspension which raises the concentration in some region such that it exceeds ρ_1.

Additional reduction in the flux is to be expected from two reasons. Firstly blocking clusters may block some of the pores. And secondly, the suspended particles together with free clusters, *i.e.* those which are smaller than they could block pores, displace the volume of the liquid surrounding them and thereby reduce the flux. The first one of these will produce an effect similar to blinding described in literature (*cf* Jackson, 1994; Jafferali, 1995, *ibid.*), where no backflushing may recover the filters to their virginal state. On the other hand, the majority of those particles and clusters in the second scenario is expected to be easily removed when backflushed. Among these latter there could yet be some which adhere themselves to the walls, whose fixation defies backflushing. But these last ones are expected to be small in number, and thus can be neglected, because we shall assume that the combination of ρ and channelling results in the probability that pores are blocked being very close to either zero or one.

The channelling effect, or the occurrence of rivulets by some other authors, is the formation of preferred paths through a porous media through which liquid and the solids it contains pass. It is not yet clear what these rivulets would do to our system. If all the channels channel equally both the liquid and the solids, then the blockage along these path may be expected to rise above the average value of the whole structure if only because this becomes more probable statistically. But if there exist some channels which prefer channelling liquid to particles or vice versa, then the effect they produce will vary and become complicated. For example, channels which like to channel particles are more likely to find themselves blocked in the end by those particles which pass through them. On the other hand those channels which channel liquid better than solids will be less prone to blocking on average, but will leave other pores around them with the excess particles, and these latter will necessarily become blocked more often than usual. But, for our purpose here, we shall assume that it is solid particles that are being channelled. This should raise the possibility of blocking in some of the pores by certain amount. Channelling in general needs further investigation which will not be covered here.

Before going on to the next step of our study we should briefly look at the core idea that makes the programme in § A.30. There are eight types of unit blocks here, compared with the four types in the case of the programme in § 6.10. These correspond to the area drawn and labelled in Figure 6.12.

Figure 6.12 shows the eight areas defined by the eight types of unit blocks they contain. Area's from I to IV correspond to those previously defined for the 2-d programme. Although unit blocks in the various areas works differently, that is to say, they adopts different set of vertices from different sources, and create different edges, all of them follow the same four rules. These four rules in the mnemonic not cryptic forms which I use are, *take vertices from behind*, *make front vertices*, *draw edges behind* and *draw no front edges*. With these rules in mind, both programmes should become self-illuminating to such extent that no further explanation is needed. This set of rules does two things, namely organise vertices and then link them with bonds. The unit vectors in the three directions being orthogonal means that we will have a nice and square end product suitable for a percolation study.

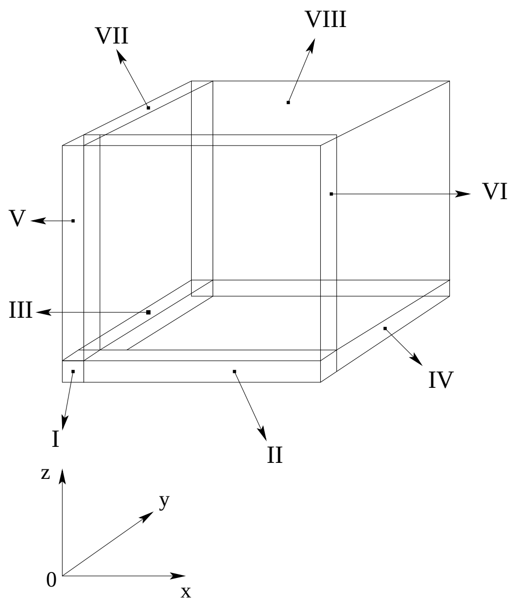


Figure 6.12 The eight areas defined by the eight types of unit blocks.

In fact it is wrong to say that all four rules work differently for each unit group. Only the two on vertices, *viz.* the first two, are different. The rest, *viz.* the last two which concern edges, are the same for all basic units. These are the only two crucial tasks with the discovery of which the writing of both programme becomes worthwhile.

The input data needs only contain details regarding units in Area II, III and V. Area I, being at the origin, is trivial, or should one rather say unique. Area IV can be derived from Area's II and III. Like wise VI is derived from II and V, and VII from III and V. Finally Area VIII turns out to be nothing but II, III and V combined.

So far we have only mentioned the situation where $0 \leq \rho < \rho_1$. In the second case, where $\rho_1 \leq \rho < \rho_2$, many more pores are blocked from small particles than in the first case. The concentration is already beyond the first critical point. But while the second critical point is still not reached, there would still be an infinite cluster of space – in this case the liquid – surrounding the particles. In other words the space still percolates. The presence of this infinite cluster, or continuum of the medium, means that the filter can still be in operation until it should be blocked or blinded by larger particles which individually can physically block the pores as found in existing literature earlier mentioned.

The third and last case, where $\rho_2 \leq \rho \leq \rho_c$, represents the extreme which can be easily comprehended. Here the solid particles occupy more space than the liquid does, as a result of which the combination is no longer a suspension but a slurry. Clay material produced by this slurry would block most of the pores within the structure and make filtration impossible. Backflushing will not be effective on filters which have undergone such fate.

§ 6.11 The second part, flow through the cells

Next we will investigate briefly the effect that channelling may have on the value of p_c. There are three cases considered here. The programme for this purpose is listed in § A.31. The programme creates a centroidal Voronoi tessellation in three dimensions. The first set of simulations works on a normal case of cell percolation, similar to that in § 4.3 but here the system is a centroidal Voronoi, which implies a constraint on cell sizes and distribution. The function `perd` in it is obtained by inputting the `Blocked` variable instead of generating it internally.

This section is written along with my developing the programmes, so the contents should be easier to follow than in other sections. This is not to say that in those other sections I had not kept records of things discovered. Every Ph.D. student starts off doing his project knowing that he should write along as he goes, and plans to nothing but that. But the truth is that even though we always write, but the way we write develops with our experience. Also, with the increase in the understanding of our problem, we no doubt would be able to write a better description of what we do and how we do it. This is unavoidable, and it is probably the reason why we should keep on working.

So much for an aside. From one hundred generators originally, Voronoi operator is applied twice. After the rims has been trimmed there are 280 centroidal Voronoi cells remaining, and this is the value of n_c. For x_c the value is 10.7929, while p_c from 2×5 simulations is 0.2314 ± 0.0602.

Next investigate the effect of channelling by assuming that the steepest gradient of all the bonds arriving at a cell decides how quick it percolates. Here cells are sorted according to their steepest gradient of incoming bonds. Working on the same network as previously, if the percolating order is such that the steeper the quicker, then $p_c = 0.2107$. But if on the other hand steeper incoming bond means slower percolation, then $p_c = 0.1429$. The critical probability is constant in this case since the order of percolation is predetermined by the orientation of bonds with respect to cells.

If instead of looking at only a single bond we take the signed summation of bonds entering and leaving a cell, then $p_c = 0.1321$ when the criterion is $\min(\sum b_i - \sum b_o)$, and $p_c = 0.1393$ when it is $\max(\sum b_i - \sum b_o)$, where b_i and b_o are respectively the incoming and outgoing bonds.

Our studies up to now tell us that if the suspension is homogeneous and there is a rivuleting effect, then the location of the blockages made by clusters of suspended particles among all pores of the structure is predetermined. This is in contrast with the blinding and blocking of large particles, where such location is random. In fact, even for these latter large particles, the location can only be random when the particles have a variety of sizes. It can never be wholly random, however, if this is not the case, since in the former case the randomness is introduced by the distribution of sizes which is random, but in the latter the blinding or blocking will be determined by the size of pore openings which is fixed by the geometry of each network. The randomness then can only be in the order not location of blockings.

The next study is a combination between continuum and network percolations. Here suspended particles are grouped into quanta, each channelling through a path or rivulet of interstitial distance with some interstitial velocity.

At the top, the layer of the structure where the incoming particles arrive, is a cross section all the cells of which are gridded to provide means to determine the path at the beginning of each quantum. The partitions in this layer is conveniently found by cutting some faces of the convex hull of each cell in the top layer by the plane $z = 0.9z_{\max}$, where $z_{\max}$ is the maximum z-coordinate of all the cells under consideration.

Given coordinates of two points, (x_1, y_1, z_1) and (x_2, y_2, z_2), and a plane equation $z - a = 0$, we may think of the plane equation as being one coordinate given, $z = a$, and find the coordinates of intersection between a line passing through the two points and the plane from the parametric equations for the line. Parametric equations are in fact interpolation done on each of the coordinates. In this case, which is useful when finding the intersection between an edge of a triangle and a plane perpendicular to some coordinate axis, the parametric equations are $x = x_1 + x_{12}t$, $y = y_1 + y_{12}t$ and $z = z_1 + z_{12}t$. From the plane equation $z = a$, therefore $t = (a - z_1)/z_{12} = (a - z_1)/(z_2 - z_1)$.

The programme being written finds the intersection of cells with the horizontal plane by finding the intersection of the faces of its convex hull with the same. Since every face of the convex hull is a triangle, the programme essentially finds intersection between edges of these triangles and the horizontal plane. The result obtained from an intermediate state during the course of development of the programme is shown in Figure 6.13. The partitions look incomplete because the picture is taken as a test while developing the programme as mentioned. A picture with the same degree of incompleteness as this one is not to be obtainable from the completed programme.

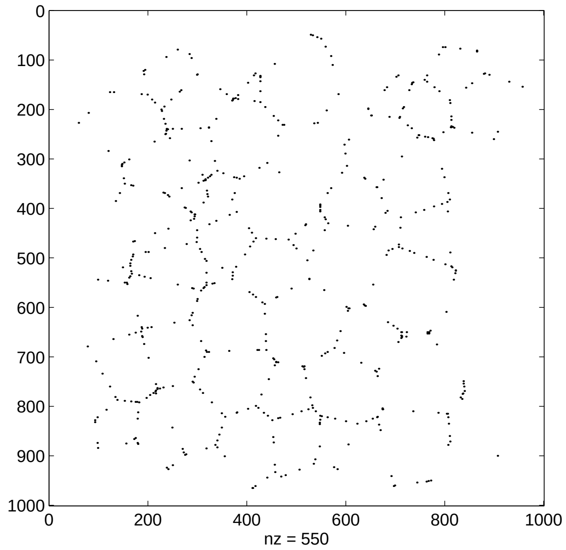
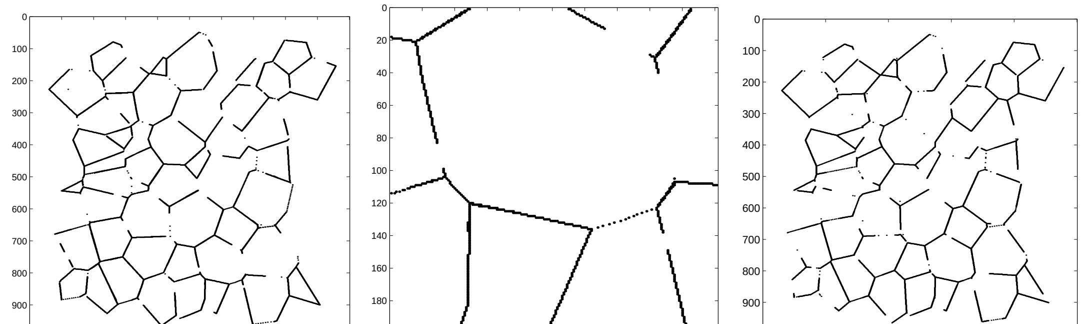


Figure 6.13 Intersection of convex hull faces and the horizontal plane.

For each face of the convex hull that intersects the plane, there will be two points of intersection arising from the two edges of the triangle intersecting it. Let (x_1, y_1) and (x_2, y_2) represent these two points. Then we may scan up in the y direction finding $x = x_1 + (y - y_1)x_{12}/y_{12}$ for each y along the way, and then scan in the x direction, this time finding instead $y = y_1 + (x - x_1)y_{12}/x_{12}$. Afterwards we could fill in the space by scanning along the x direction for all y positions.

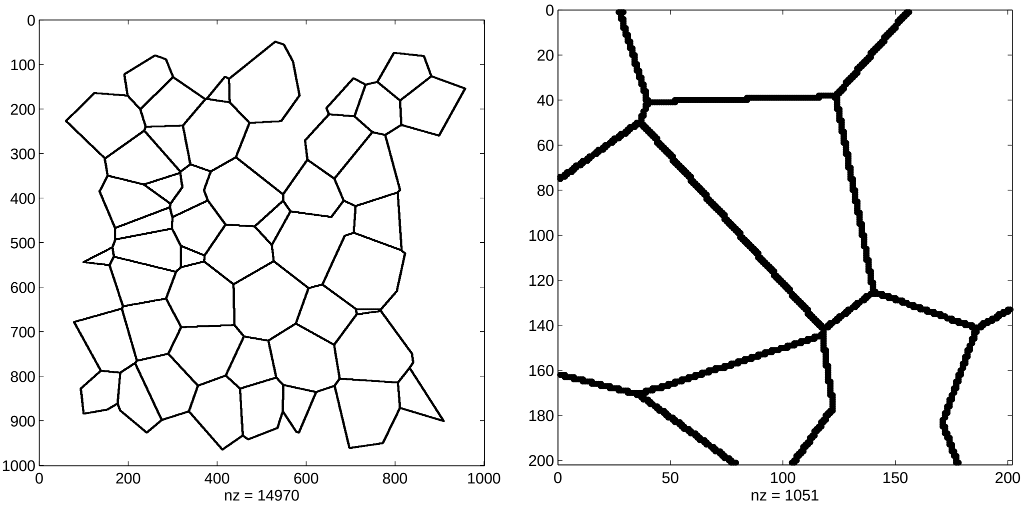
Figure 6.14 shows the progress of my programming the codes, step by step, trying to close all the partitions such that no gaps remain. Two problems have been discovered, namely those of rounding and precision. Before the correction the result looks like Figure 6.14 (a) and (b), and after rounding problem corrected Figure 6.14 (c).



(a) (b) (c)

Figure 6.14 *Correcting the effect of rounding, in other word discretisation.* (a) *The gaps resulted from rounding or discretisation,* (b) *a closed-up view of* (a) *and* (c) *partial remedy where roundings have been solved but with the degree of precision not yet raised.*

After having corrected the problem regarding precision, by increasing the number of steps when calculating x or y, the result still misses several walls, the cause of which is still unknown at present. This is shown in Figure 6.15.



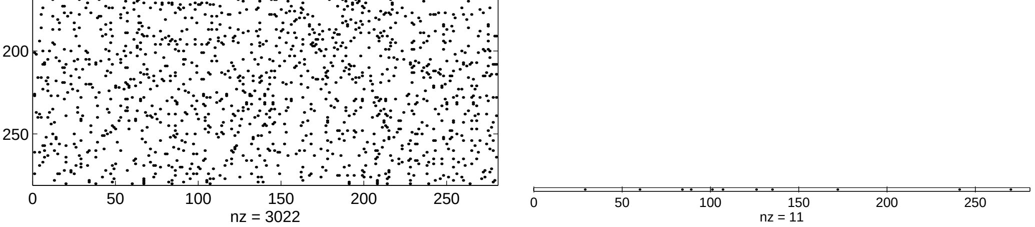
(a) (b)

Figure 6.15 *After having increased the precision to ten times the previous value,* (a) *and a closed-up view,* (b).

It seemed at first that there might be some triangular faces missing from the surface of the convex hull, which would have accounted for the missing boundary in the section. But after having tested the minimum number that an edge of each hull appears as the edges of all its triangular faces, and see that it's value is correctly two, this becomes out of question.

The figures, *viz.* Figure 6.13, 6.14 and 6.15, are produced from the `spy` command in Matlab, as a result of which the x-axis runs downwards while the y-axis runs to the right. This command looks at a matrix from above as we look at a map. In the present case our matrix is a full-, not sparse matrix. The number written at the bottom is the number of all its nonzero components, which is less than the number of times that we calculate them since we need to calculate some of the points more than once in order to increase the precision to eliminate gaps in other places.

The command `spy` is used more often with sparse matrices since these are often too large to list, and listing their members in pairs makes it difficult to visualise. As an example, Figure 6.16 (a) is what we get when we `spy` our neighbour matrix `necc`, while in Figure 6.16 (b) are all the neighbours that the 100th cell has. Cells which have few neighbours generally live along the border. For example the 110th cell has only three neighbours, and it is located not far from the lower x limit.



(a) (b)

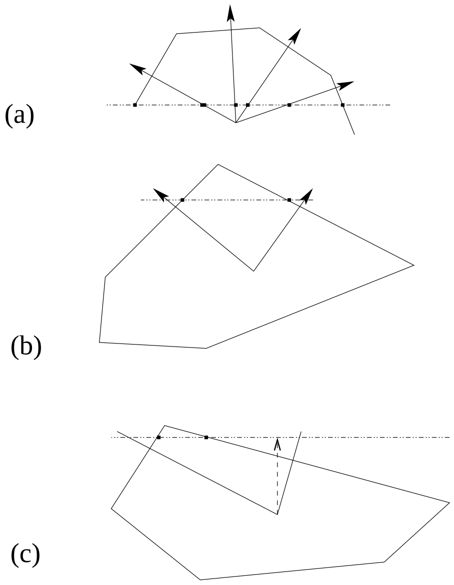
Figure 6.16 *The result when we* `spy` *our neighbour matrix. Here* (a) *shows the neighbours of every cells while* (b) *only shows those of the* 100th *cell.*

At this stage the neighbour matrix of the programme is double-checked, and find that it includes poorly defined neighbours, that is those which only have one vertex in common. Therefore the programme is first altered to make it only look for neighbours who share at least two vertices.

But this has shown no noticeable changes in the results that we have so far.

The next step is to colour the cells. Like a child painting a picture, there are so many ways one can paint or label the tiles of a tessellation. For example one could draw a vertical line first and then branch out either horizontally or diagonally. On the other hand, one could also draw the diagonals first. Yet another way is to expand radially, spiraling outwards. With parallel computing we could also divide the area into domains, paint each domain, and then merge the resulted areas together.

But here we opt for painting the cells by scanning horizontally, moving upwards in layers. Once past a wall, the programme moves on until the next wall is reached while gathering all the grids that are between the two walls into one group. It then colour the whole group by a colour picked up from one layer below it. If no colours exist, then it creates a new colour which in turn gradually propagates upwards this way until the upper wall is reached.



The intersection of bonds will not always coincide with the intersection of the cell, neither is the projection of a cell perpendicular to the plane the sectional plane of the cell. This is because of the three possible situations shown in Figure 6.17. In Figure 6.17 (a) the cell section contains several points, while both (b) and (c) contain none.

Figure 6.17 Sections of bonds and cells.

And here sadly the time runs out, so I will suffice myself to describing what I see should be done next. Up to now we have a three-dimensional network and its top section. We also have the list of all its bonds, which contains the connections and the draining angles sorted in a descending order.

Next we should find a mapping from each cell section to the corresponding nuclei. Then the times it takes to traverse each bond must be calculated. This time for each bond is then divided by half, one belonging to each of the two cells connected by the bond.

When it comes to bombarding the filter with our small particles, we can not keep track of millions of particles and therefore we should quantise them into units. These units or quanta can then be treated as individual particles. When a quantum enters a cell, it is assigned t, the time to reach the nucleus. Later time sees this t decreases in steps until it finally reaches its destination, the nucleus. Once there, it is assigned the next bond to go along, taking into account what bonds are available at the time and their comparative probabilities, which in turn depend on their gradient as mentioned. When this is decided, it is given t_1, the time it would take to reach the border that lies at mid point of the bond.

This goes on forever, apart from that at each time step we look to see whether the blockages in our filter has percolated. After updating the list of blocked cells, if we find that percolation has occurred then the simulation would end. Percolation occurs in each cell whenever its concentration has reached a certain value. This value we have found in § 6.10 to be 18.5 per cent by volume.

To calculate the flux decrease we find instead the decrease in the superficial area. This is calculated from the total volume of the void subtracted by the volume of all cells that had percolated, and then subtracted by the total volume of solid particles which are suspended inside the network. The area of the cross section is then the volume which remains divided by the thickness of the filter.

It is not a little to have to leave things unfinished after having started it off. But as one New Zealander poet says, ‘Alone we are born and die alone, yet see the red-gold cirrus over the snow mountains shines. Upon the up-land road ride easy, stranger. Surrender to the sky your heart of anger.’ And with this we go on to the next chapter.

7

§ 7. Percolation in traffic modelling

It does not take much imagination for an average person to see that traffic congestion is a percolative process. As I have lived and studied in Bangkok where, at that time, the traffic jams were renowned. Now I relate traffic jam to percolation of the network of streets. I remember once in 1990 it took me more than five hours on a bus to travel the distance of ten kilometers along the Sukhumvit Road in Bangkok. The newspaper of the following day reported that that traffic jam had been on the BBC news broadcast all over the world.

Soon after having decided to investigate the application of percolation to traffic, I gave a poster presentation at the Fourth Annual Conference of Thai Researchers in Japan in 1997, the proceedings of the event of which also contain a number of abstracts from some of the researches that I was currently working on (Tiyapan, 1997, KNT5(i) to KNT5(iv)). The poster presentation was considerably a success.

The paper listed in § E.20 of Tiyapan (2003, KNT8(iii)) is a reproduction of what I had submitted to the Journal of Statistical Physics in 1997. I tried but could not receive any reply for the paper I submitted. Afraid that the ideas in it could have been plagiarised, I decided to put it in the book above quoted.

In 2002 I did some more works on it. I have developed the theory further and done a number of simulations. I plan to carry on doing research along this line in the future, together with another area where percolation is used to explained economics transition (Tiyapan, 1997, KNT5(viii)).

In his paper submitted to the Journal of Statistical Physics Tiyapan (1997) introduces a new idea of considering the development of clusters in both phases at the same time. Applied to the context of traffic network, these phases are namely cars and spaces. Furthermore, because both phases reside in one and the same network, there is a symmetry which divides the probability space into three regions, symmetric with respect to $p = 0.5$. This helps divide the traffic condition into three regions as existing literature in traffic study at the time described, namely free flowing, congested and stand still. In particular, this idea explains the difference between the congested and the stand still states. There has been no reply from the journal.

In the 2002 simulation the networks are drawn which have their vertices as points where two or more roads meet one another. The proposed study is to compare the robustness of two traffical networks by comparing their percolation thresholds. When a new motorway is planned, for example a ring road around a city, the two networks, one with the ring road and the other one without, can be simulated to find their percolation thresholds and then these values compared.

Traffic status or condition when the percolation probability p_c of cars is more than 0.5. The traffic condition is generally described as free-flowing, congested, or stand-still. For cases where $p_c > 0.5$ as the one shown in Figure 7.1 the stand-still traffic corresponds to the situation where only cars have percolated but not the space, that is the places on the road available and accessible to the cars.

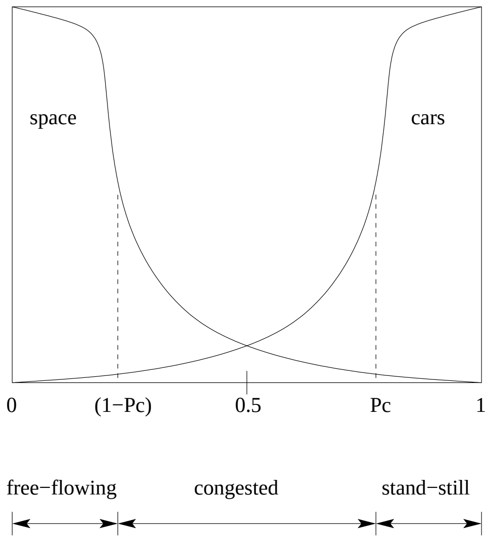
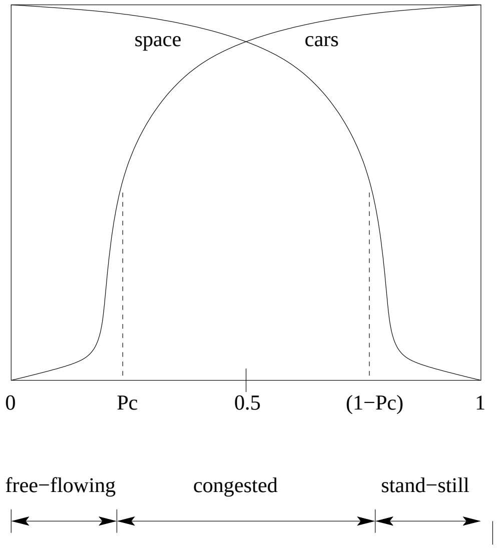


Figure 7.1 Traffic status, $p_c > 0.5$.

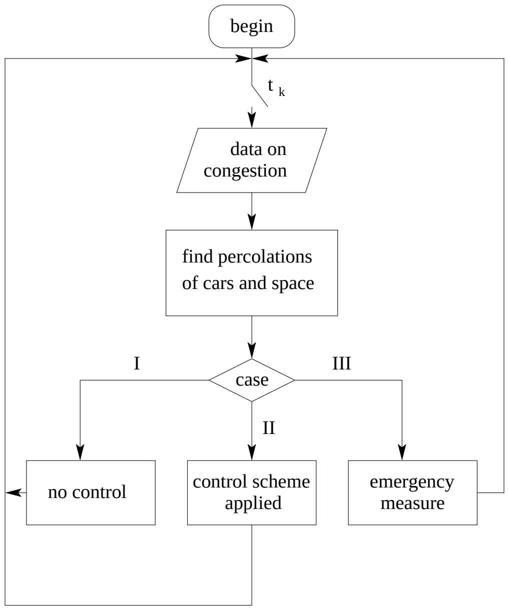
A free-flowing traffic is where only the space but not the cars has percolated, and a congested traffic is that when neither of the two has. Traffic status or condition when p_c is less than 0.5.



On the other hand, in the case where $p_c < 0.5$ shown in Figure 7.2 the critical probability of cars is now on the left hand side of the middle line instead of on the right hand side thereof in the previous case where $p_c > 0.5$. Definitions of the free-flowing and the stand-still statuses remain the same, but the congested traffic is now the traffic where both cars and space have percolated. An interesting question is whether or not there is a difference between the congested area in Figure 7.1 and the one in Figure 7.2. A *modelling algorithm* which studies the percolation of a traffical networks starts by finding all the vertices and edges forming a network from the road data. Then a blocking algorithm operates by randomly shutting off one edge after another until the network percolates when the critical probability of the network may be calculated.

Figure 7.2 Traffic status, $p_c < 0.5$.

A *control algorithm* for the real time traffic control, however, is as shown in Figure 7.4.



Let C means *cars have percolated* and S means *space has percolated*, then in Figure 7.4 the cases I, II, and III are respectively $\neg C \wedge S$, $(\neg C \wedge \neg S) \vee (C \wedge S)$, and $(C \wedge \neg S)$. Case I is the normal congestion, nothing to worry about. Examples of the control schemes used in Case II are overriding of the traffic lights manually by a traffic police at certain strategic points, temporary one-way systems, bird-eye view observation and feedback from helicopters, traffic control centre, distributed control centres together with traffic radio channel broadcast. And lastly the most important and critical Case III the emergency plan of which may include directing all cars away from congested clusters or if necessary out from the city, and directing all incoming traffic such that no more cars may enter the city until the emergency status ends.

Figure 7.4 Proposed traffic control in real time..

A congested cluster can be broken up by forming a one way flow channel cutting through it which leads cars away from the cluster. How far the channel needs to go before letting the cars on it circle and seep back into town depends on how badly congested the traffic is. Although it is a normal practice to lead cars along a long detour because this brings in more road surface and thus enlarges the network, identifying the percolating cluster and cutting it into two or more parts by guarded flow channels is much less common or even unheard of. The latter seems to be more important and will lead to a better and more effective control, namely the control of the percolating, in other words the biggest, cluster.

The percolation probability is important for the networks of traffic both inside cities and among them. It shows the degree of connectivity of the area being considered. Urban road networks have a general character which differ from one country to another, the simplest construction of which seems to be that of the square lattice. One can find this theme of the square lattice and its variations, similar to the dislocations and defects found in minerals, in America. Examples are Denver, Aspen, Durango, Pueblo, Salida and La Junta in Colorado; Boise, Pocatello and Twin Falls in Idaho; Butte, Bozeman, Coeur d'Alene, Kalispell in Montana; and Cheyenne, Laramie and Sheridan in Wyoming (*cf* Florence *et al*, 2001). The Great Junction in Colorado and the Great Falls in Montana are very close to being perfect square lattices. As more examples of these (*cf* Collins USA, 1999), in Arizona there are Phoenix, Yuma, Tucson; in California Bakersfield, Central San Francisco, Central Sandiego, Fresno, (Central) Los Angeles and vicinity, Modesto, Sacramento; in Colorado Fort Collins, Denver and vicinity, Greeley; in Florida Central Miami; in Georgia Central Atlanta; in Illinois Champaign and Urbana, (Central) Chicago and vicinity,, Quad Cities, Rockford; in Indiana Fort Wayne and Indianapolis; in Kansas Topeka and Wichita; in Louisiana Central New Orleans; in Maryland Central Baltimore; in Minnesota Central Minneapolis and Central St. Paul; in Missouri Central Kansas City; in Nebraska Lincoln; in Nevada Las Vegas; in New York Manhattan; in Oklahoma Lawton, Norman, Oklahoma City and Tulsa; in Pennsylvania Central Philadelphia; in South Dakota Sioux Falls; in Amarillo, Central Houston and Lubbock; in Utah Central Salt Lake City; in Washington Central Washington D. C. and Central Seattle; and in Wisconsin there is Central Milwaukee. One example in Canada is Toronto. The square lattices of these cities are sometimes cut through by motorways or interstate highways as is the case in Amarillo, Texas. Or they can be surrounded by a ring road or a county highway as is what happens with Lubock, also in Texas.

When the percolation probability is greater than 0.5, we have the interval $p_c \pm (p_c - 0.5)$ where neither the blocked nor the free roads percolate. If this interval is narrow, that is if $p_c - 0.5$ is small, then within this interval the condition of the traffic is very sensitive, and even a seemingly small change may lead to a standstill or instead to a free-flowing traffic. This is easily visualised, since in such situation there would be small islands of free-flowing roads within a large congested cluster, and vice versa small clusters of congested roads within an otherwise noncongested area. Despite their sizes, such small islands of anomaly in either of the phases are particularly important. Moreover, their importance increases the closer p_c is to 0.5. The same characteristic happens in politics where, in the case of coalition governments, a minority party which has relatively few representatives can become critically important and influential to the major party when the latter needs them in order to be able to govern (Ireland, 2002).

Plan to build a ring road around a city usually includes flyovers, overbridges, and tunnels in order to avoid having intersections. The design philosophy used in Europe is to have heavier traffic goes under a lighter one, which results in either the ring road going into a tunnel or having overbridges or viaducts over it. The philosophy used in Thailand which used to be for the heavier traffic go over the lighter traffic, which puts a more severe limit on the weight of trucks and is therefore not economic in the long run, but this has started to changed, if only to follow the practice of the west. Ring roads do not necessarily resemble a circle, as in the case of the circular speedway proposed around Saint Petersburg (Petersburg, 2001), which is in the shape of a cashew nut. It will link the arterial roads of the city to Helsinki, Kiev, Moscow, Murmansk, and Tallinn into a network. The route proposed is 155 kilometres in length, has 31 bridges, 16 overbridges, 55 viaducts, and will support the volume of traffic of 21 million tonnes. The implementation and contract work is looked after by a joint-stock company KAD Sankt-Peterburga, under the order of St. Petersburg and Leningradskaya Oblast.

One classical example of a ring road is the M25 motorway which forms a circumscribed ring around London. Another more recent example is the M60 orbital motorway around Manchester. In the case of M60, various sections of existing motorways have been put together and renumbered. The northwest quarter used to be M62, the southwest one M63, and parts of the remaining used to be M66. The motorway forms a complete ring around Manchester since 2000 with the opening of the final northeastern part which stretches from Denton to Prestwich. There is another smaller ring, an inner ring, which is formed by the A6010 and A576. Orbital motorways around cities have now become indispensable and are the hallmark of a city.

§ 7.1 Percolation of road networks

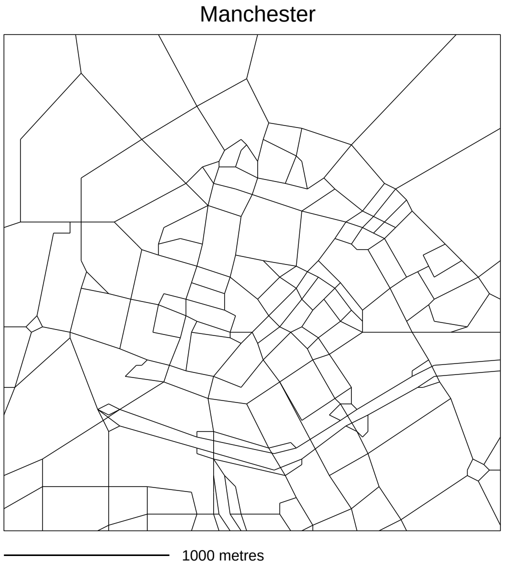
I began my study of the traffic networks in 1997 while in Japan. In the same year I submitted two papers,

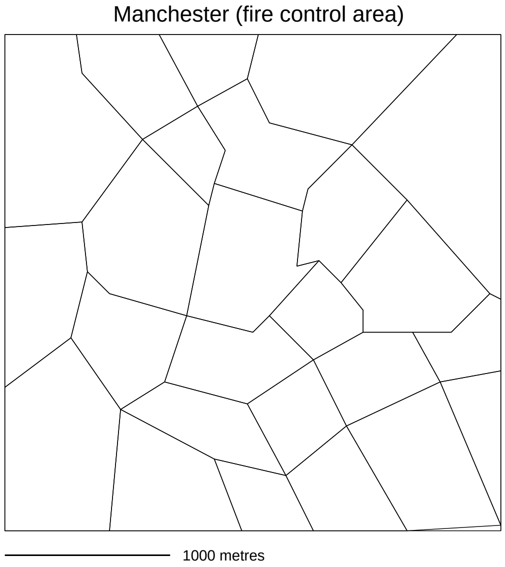
Tiyapan in 1997 (KNT5(ix)) has a novel idea of linking percolation to the management of street congestion. In his work submitted that year to the Journal of Statistical Physics, also in § E.19 and § E.20 of Tiyapan (2003, KNT8(iii)), electronically though the paper became lost and was never published, he explains how three phases could result from structures containing only two phases. Recently there have been some papers published in 2001 in the Journal of Statistical Physics which used this very idea that Tiyapan introduced in 1997. But what has passed is past and past cure, we need to think about it no further now. Everything comes from and goes to God, let that suffice! I have developed a programme and a procedure for finding the percolation thresholds of road networks. These programmes are listed in § A.27. They are different from the usual percolation programmes used for other kinds of networks including Voronoi. Because of the existence of flyovers and elevated express ways, we can not use the duality operator to transform a network of vertices and edges into one of cells and bonds. Having said that, the said transformation could become useful in the future in some other applications, in some other areas or even within the study of traffic network itself. But at present I have only one application in mind for the dual networks of roads, and that is related to fire prevention where such an application is by no mean obvious. Therefore, here the boundary is defined again to obtain the cells which represents the zones.

The codes also contain data of several towns and cities, namely Amsterdam, Brussels, Freiburg and Manchester. There are three main datasets. The first one contains a list of the coordinates of all the vertices. The second one is a list of edges, together with the numbers of the cells that each of them connects. The third one contains the coordinates of turns in each of the roads listed in the second dataset. This is in order for the graph to look like the actual roads it represents, instead of containing only straight lines, as would have been the case were the windings of the roads not to be taken into account. Also, these coordinates will make it possible to calculate the true length of each road. Even though we have no use for these lengths at the present stage, future developments may need them.

The procedures developed for gathering the data and processing them can be carried out by a single researcher, and require no sophisticated tools. Were these tools become available in the future, the former could be adjusted to accommodate them.

To my surprise, so much so that I at first thought that there was something wrong with the programme, the mean coordination number of the road networks of Manchester turns out to be exactly 3. The second simulation gives $x_v = 3.0513$, which is still very close to three.

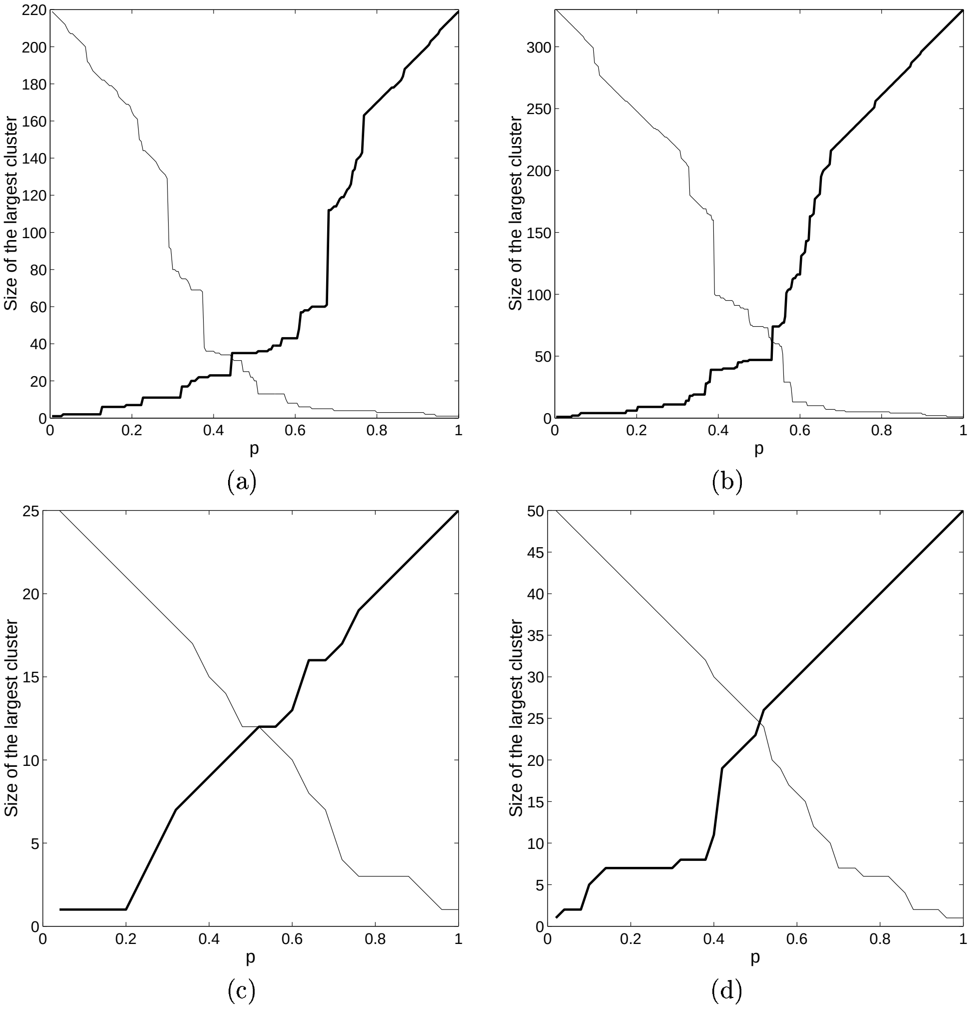




(a) (b)

Figure 7.5 *For Manchester in this picture,* $n_v = 220$, $n_e = 330$, $n_c = 25$, $n_b = 50$, $x_v = 3.0000$, $x_e = 4.7273$, $x_c = 4.0000$ *and* $x_b = 6.5200$.

Figure 7.6 shows the largest clusters plotted against p, the probability that each vertex, edge, cell or bond respectively for (a), (b), (c) or (d). Here p_v^s is the percolation probability of the space in the network of vertices, and similarly for p_e^s, p_c^s and p_b^s. Each plot represents one of the runs of simulation which, for networks of these sizes and variances, literally distributes the critical probabilities all over the place. But the plots of the largest cluster sizes always look very symmetrical. This seems to suggest that these sizes may represent the point of percolation better than the percolation probability.



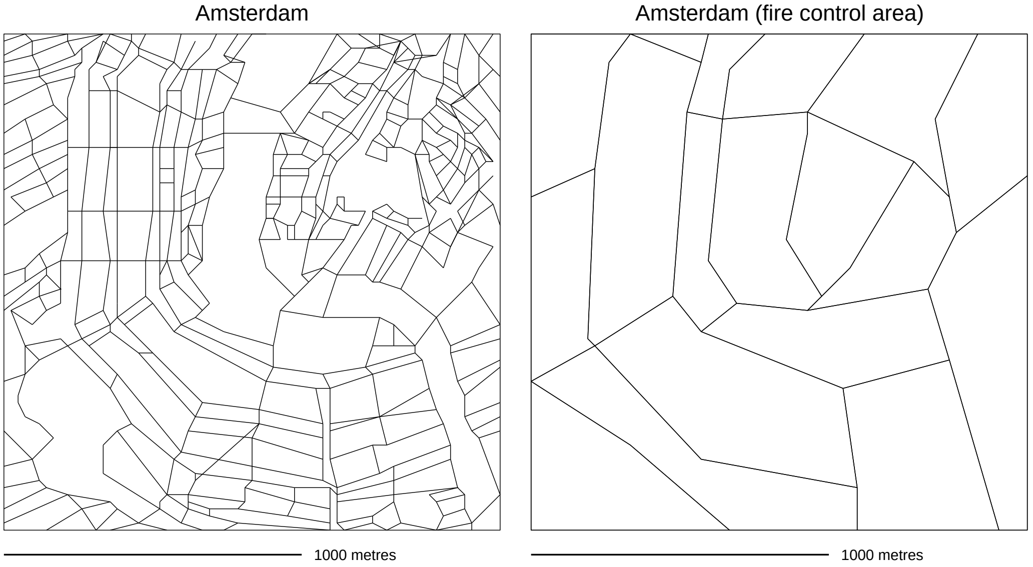
(a) (b)

(c) (d)

Figure 7.6 *Manchester. Plots of the largest clusters of,* (a) *vertices,* (b) *edges,* (c) *cells and* (d). *Here* $(p_v, p_v^s) = (0.5455,\ 0.7182)$, $(p_e, p_e^s) = (0.6152,\ 0.4939)$, $(p_c, p_c^s) = (0.6400,\ 0.6000)$ *and* $(p_b, p_b^s) = (0.1200, 0.5000)$ *respectively for the cases of* (a), (b), (c) *and* (d).

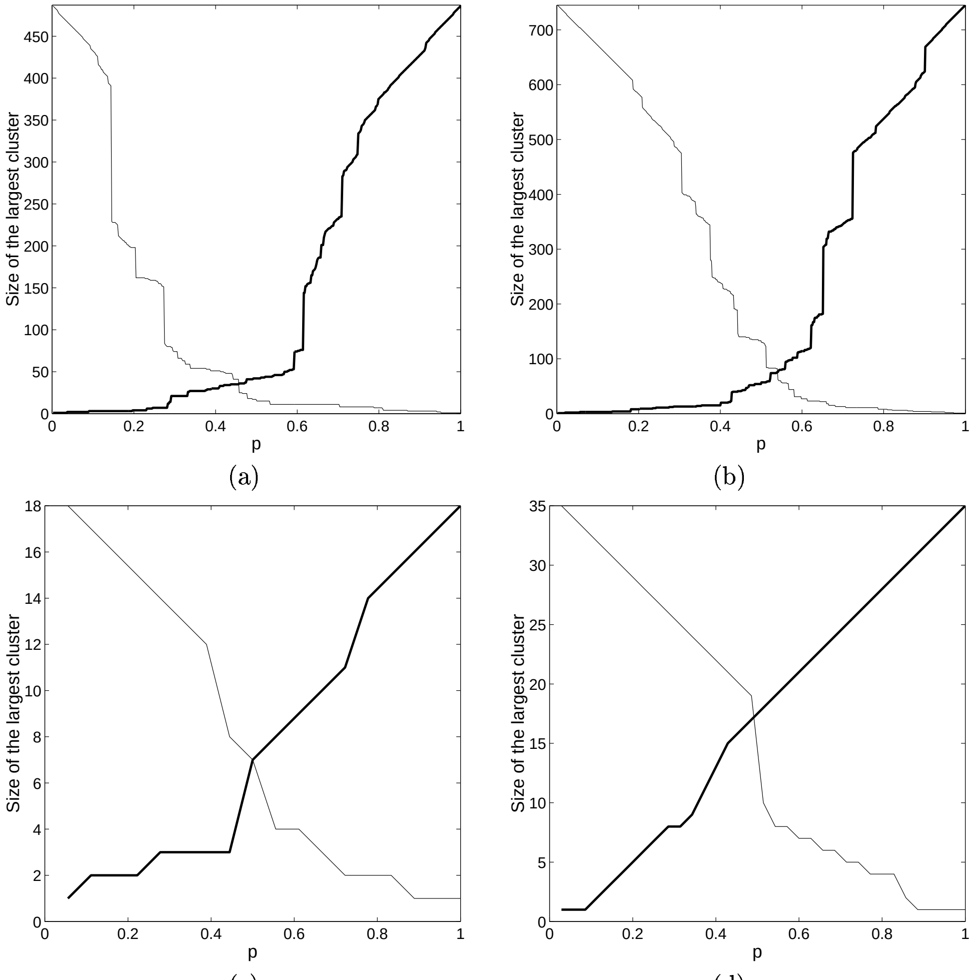
The mean percolation probabilities from (2×10) similar simulations as the two shown in each of the four cases of Figure 7.6 are $\bar{p}_v = 0.6723 \pm 0.0762$, $\bar{p}_e = \text{Er}[0.6083, 0.0838]$, $\bar{p}_c = \text{Er}[0.6240, 0.0921]$ and $\bar{p}_b = \text{Er}[0.3720, 0.1149]$.

Next consider Amsterdam in The Netherlands. The map for our purpose is shown in Figure 7.7. Areas shown in Figure 7.7 (b) are usually bound by main roads or the rims of the picture. There is no definite relations between vertices and edges on one hand, and cells and bonds on the other.



(a) (b)

Figure 7.7 *Amsterdam in this picture has* $n_v = 487$, $n_e = 745$, $n_c = 18$, $n_b = 35$, $x_v = 3.0513$, $x_e = 4.7007$, $x_c = 3.8889$ *and* $x_b = 6.8000$.

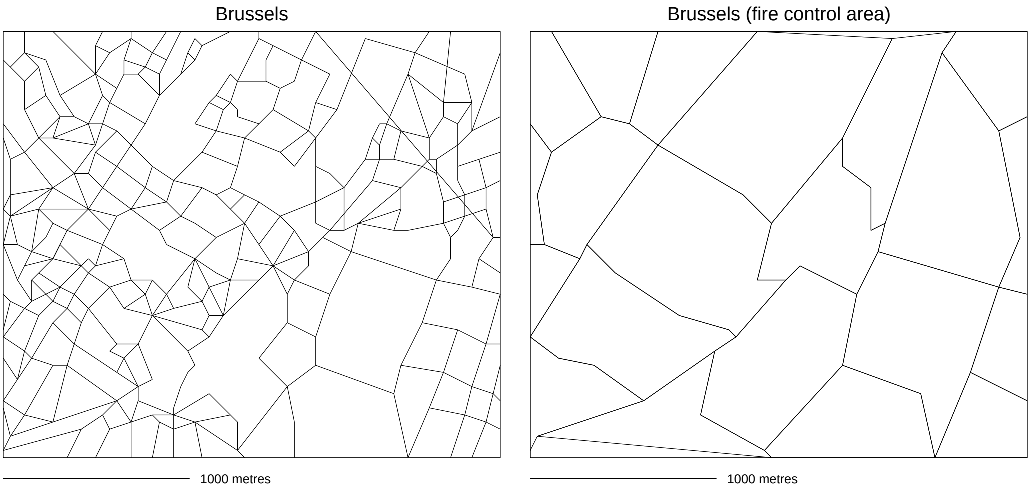


(a) (b) (c) (d)

Figure 7.6 *Amsterdam. Plots of the largest clusters of,* (a) *vertices,* (b) *edges,* (c) *cells and* (d). *Here* $(p_v, p_v^s) = (0.7187, 0.7146)$, $(p_e, p_e^s) = (0.6309, 0.5597)$, $(p_c, p_c^s) = (0.7778, 0.4444)$ *and* $(p_b, p_b^s) = (0.4286, 0.2571)$ *respectively for the cases of* (a), (b), (c) *and* (d).

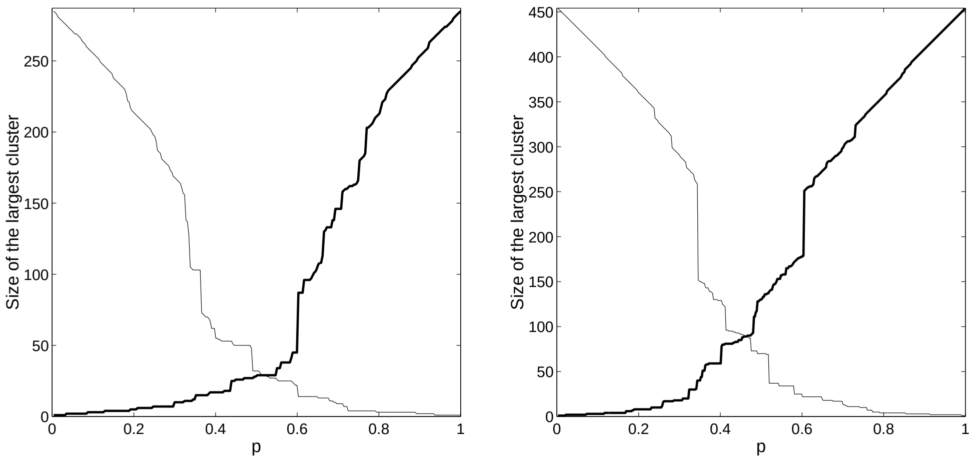
From (2 × 10) simulations we obtain $\bar{p}_v = 0.7374 \pm 0.0500$ and $\bar{p}_e = 0.6328 \pm 0.0595$, whereas from (2×11) simulations $\bar{p}_c = 0.5960 \pm 0.1395$ and $\bar{p}_b$ 0.3922 ± 0.1395.

Then consider the road networks of Brussels in Belgium, as shown in Figure 7.9, together with the fire control area, and the largest cluster sizes in Figure 7.10. Fire control strategy is only one possible application to which the percolation of areas. Many other applications which are similar in nature, for instance emergency evacuation zones, earthquake evacuation zones, zones prepared as measure against a terrorist gas attack, *etc.* There are also other applications, for example strategic areas in market planning and the study of mineral deposits.

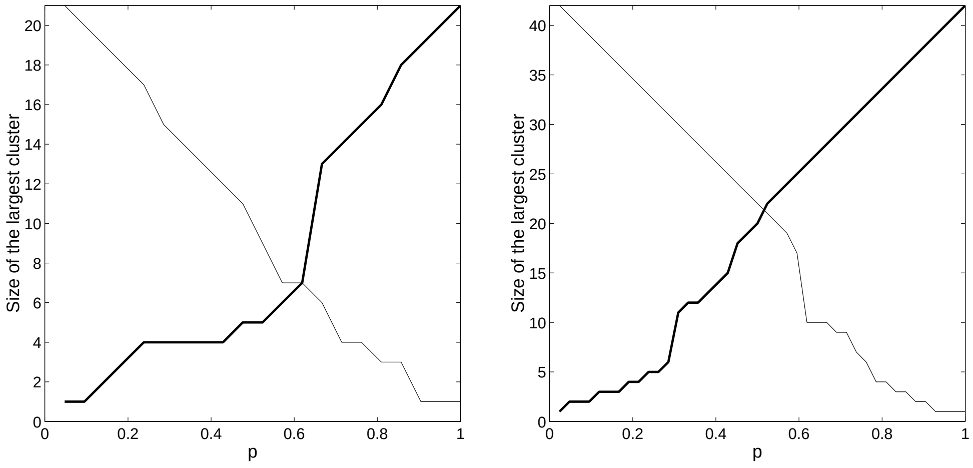


(a) (b)

Figure 7.9 *Brussels has* $n_v = 287$, $n_e = 454$, $n_c = 21$, $n_b = 42$, $x_v = 3.1568$, $x_e = 3.1568$, $x_c = 4.0000$ *and* $x_b = 7.0476$.



(a) (b)

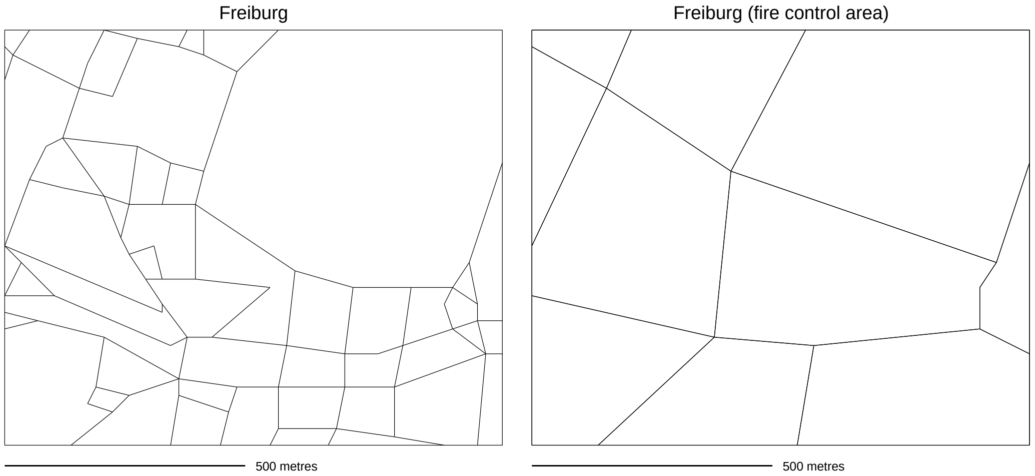


(c) (d)

Figure 7.10 *Brussels. The largest clusters when percolate by means of* (a) *vertices,* (b) *edges,* (c) *cells and* (d). *Here* (p_v, p_v^s) = *(0.6167, 0.6411),* (p_e, p_e^s) = *(0.6057, 0.6586),* (p_c, p_c^s) = *(0.6667, 0.5714) and* (p_b, p_b^s) = *(0.3095,0.4286) respectively for the cases of* (a), (b), (c) *and* (d).

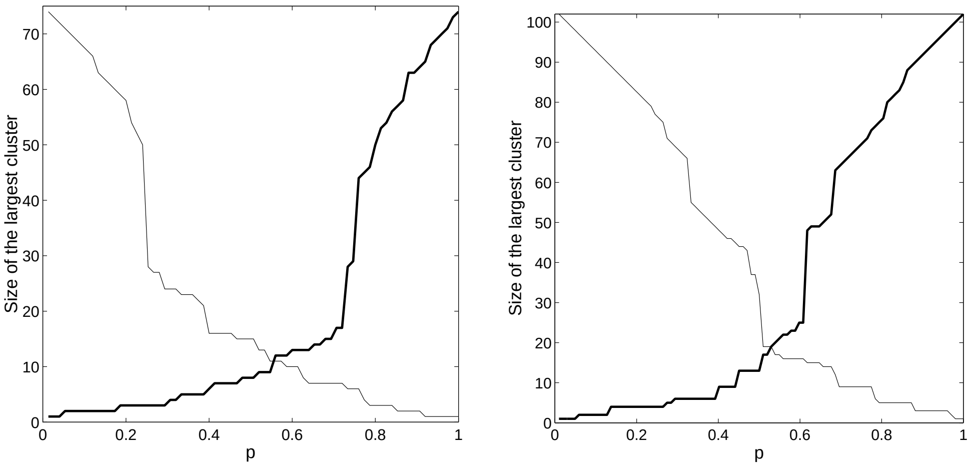
From (2 × 10) simulations on Brussels we obtain $\bar{p}_v = 0.6580 \pm 0.0771$, $\bar{p}_e = 0.6205 \pm 0.0540$, $\bar{p}_c = 0.6286 \pm 0.1043$ and $\bar{p}_b = 0.3786 \pm 0.0924$.

And then consider a small town Freiburg in Germany, where the many roads that are reserved for pedestrians only seem at a first glance to have altered much of the structure. But simulations have shown that the percolation probabilities remain comparable with networks of other towns. The area- and bond coordination numbers obtained for Freiburg are rather low compared with other towns. This could mean that the emergency properties of the town is different from those of others. Its lower connectivity could mean that it is more robust than others against an attack or in the face of catastrophe. But it could also make it more difficult to evacuate from an area. More precise relationship between the valence and the interpretation in terms of physical networks can only be possible by more extensive investigations in the future.

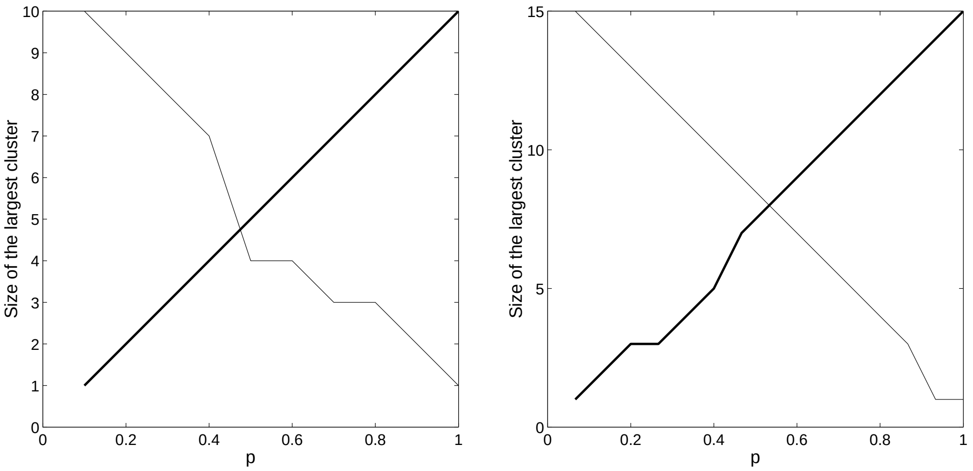


(a) (b)

Figure 7.11 *Freiburg has* $n_v = 75$, $n_e = 102$, $n_c = 10$, $n_b = 15$, $x_v = 2.7200$, $x_e = 4.4510$, $x_c = 3.0000$ *and* $x_b = 4.5333$.



(a) (b)



(c) (d)

Figure 7.10 *Freiburg. The largest clusters when percolate by means of* (a) *vertices,* (b) *edges,* (c) *cells and* (d). *Here* (p_v, p_v^s) = *(0.7333, 0.7733),* (p_e, p_e^s) = *(0.6863, 0.6275),* (p_c, p_c^s) = *(0.4000, 0.5000) and* (p_b, p_b^s) = *(0.4000, 0.3333) respectively for the cases of* (a), (b), (c) *and* (d).

Again, from (2 × 10) simulations we obtain for our Freiburg $\bar{p}_v = 0.7400 \pm 0.1214$, $\bar{p}_e = 0.6490 \pm 0.0787$, $\bar{p}_c = 0.5450 \pm 0.1276$ and $\bar{p}_b$ 0.3767 ± 0.1190.

§ 7.2 Graphs theory and its applications

Gay and Preece (1975) study graphs on square lattices, represent them as matrices and then apply this to the study of networks of fluid distribution through pipes where there are branch flows and nodal pressures. Such a graph has n nodes, $m = b - n + 1$ meshes, where b is the number of branches, and any one of its trees contains $(n - 1)$ branches. The direction of a mesh is the direction of the link connected to it. Define the branch-mesh incidence matrix $C = [C_T; C_L] = [C_T; U]$ and the node-datum incidence matrix $B = [B_T; B_L] = [B_T; 0]$, where c_{ij} is 0, 1, or -1 when the branch i respectively is not in the mesh j, has the same-, or opposite direction; similarly, b_{ij} is 0, 1, or -1 when the branch i is not included in the node to datum path j, is going away from, or towards the datum node. Then $V = E + e$, $J = I + i$, $V = ZJ$, $J = YV$, where E is the vector of branch pressure sources, *i.e.* pumps, e the vector of branch pressure rise, I the vector of branch flows due to the external inputs and outputs, i the vector of branch flows due to other causes. Then the mesh method has the solution $i' = (C_T^T Z_T C_T + Z_L)^{-1}(E_L' - C_T^T Z_T B_T I')$ and $E_L' = C_T^T E_T + E_L$, where $\gamma = [B, C]$, $V' = \gamma^T V$, $J = \gamma J'$, $J' = [I'; i']$ and $V' = [V_T'; V_L']$. The node method, on the other hand, gives $e' = (A^T Y A)^{-1}(I' - A^T Y E)$.

Operation research uses graphs to help in its search for optimum solutions. For example the Evolutionary Operation (EVOP) and its variant Rotating Square Evolutionary Operation (REVOP) use covering graphs on a body-centred lattice in two- or three dimensions (Lowe, 1964) – corresponding to the cases of considering two- or three factors respectively – in finding its solution. Simplex EVOP is another variation, where 2- and 3-factor regular simplexes are used, which has the advantage that it uses the least amount of experimental points. An n-factor regular simplex has $(n + 1)$ points, in particular a 2-factor simplex is the equilateral triangle and a 3-factor simplex is the regular tetrahedron. All of these methods are possible at number of factors higher than three, but the procedure becomes more complicated and can not be shown graphically.

A graph is connected if it cannot be divided into two subgraphs without common vertices and edges. A linear combination of oriented edges, $\sum a_i e_i$ where $e_i = \pm 1$, is a simple cycle if a_i is 1, -1 or 0. It is a cycle if a_i are integers satisfying some linear relations, and a chain if they are arbitrary integers.

If vectors from the origin to vertices A_i follow the relation $A = \sum m_i A_i$, then m_i are the barycentric coordinates of A. Here $m_i \geq 0$, $\sum m_i = 1$ and A is the centre of gravity of the point masses A_i. A convex linear hull of A_i is then the space covered by changing m_i to give all possible A's. It is also known as a rectilinear-, Euclidean- or n-dimensional simplex. Simplices can simply be represented by their vertices. An n-dimensional simplex s_i^n has $n + 1$ $(n - 1)$-dimensional faces the i^{th} one of which is where $m_i = 0$. A k-dimensional face of s^n is the set of those points of s^n for which $n - k$ barycentric coordinates are zero, while the rest change such that the corresponding m_i are non-negative and their sum is equal to unity. The number of k-d faces in s^n is ${}_{k+1}\mathrm{C}_{n+1}$.

The boundary of s^n is the union of all its $(n - 1)$-d faces. The centre of a simplex is a point where $m_i = m$.

8

§ 8. Conclusion

To summarise, we have begun our study from the Voronoi network, and then went on to study continua and then tried to combine them together. I have suggested the idea that a percolation in continua can be represented by a percolation in a lattices the type of which depends on the attrition mechanism of the particles in that continuum, in other words the way they pack together. Even though percolation already models a great many natural phenomena, I think it will prove to underly the fundamental structure of a great many more, even of the Big Bang. Therefore I truly believe that a better percolation theory may be what is behind the working of the universe.

I have written programmes, and have listed here a full half of them which is already quite a lot in any programmer's standard. This is because I try to write everything myself, and not to copy even when it comes to taking codes from programmes that I have written myself in the past. I try to do it this way whenever possible, and as a result the various programmes should have at least a few things out of common with each other.

The results shown in Table 4.7 on page 145 show that the percolation probability depends not only on the valence but also on the way in which the vertices distribute themselves with relative to one another. From Figure 4.12 we can see that the comparative study of tilings is best done by looking at both the tiling itself and its covering lattice. The tilings $5_3[3^5]8_6[3^8]_{\mathrm{I}}$ and $5_3[3^5]7_3[3^7]_{\mathrm{I}}$ (number 24 and 27) look similar to each other and all the values of their p_c are similar to the corresponding values of p_c's of the other; likewise the tilings $5_4[3^5]7_4[3^7]_{\mathrm{I}}$ and $5_3[3^5]8_6[3^8]_{\mathrm{III}}$ (17 and 21). The coverings of the tilings $3_3[3^3]9_3[3^9]_{\mathrm{II}}$ and $3_3[3^3]9_3[3^9]_{\mathrm{I}}$ (2 and 7) look different from each other even though they themselves are similar in appearance, and this is reflected in the values of the cell- and bond-probabilities. Both the tilings $4_3[3^4]10_6[3^{10}]_{\mathrm{I}}$ and $4_3[3^4]10_6[3^{10}]_{\mathrm{II}}$ (6 and 9) as well as their coverings look different from each other, and it is not surprising to see that all their p_c values are quite different.

§ 8.1 Suggestions for future work

Jackson (1994) and Jafferali (1995) study filtration as stochastic process within porous media which they model as a Voronoi Tessellation. Here all the blockages are assumed to be of physical mechanism in nature. Schumacher (1996) asserts how in those cases where particles much smaller than the pores' size come together to form clusters and thereby block the pores, there is involved an effect of Van der Waals electrostatic interaction between particles. Tiyapan (2003, KNT8(iii)) formulates these into a problem of a continuum percolation within a network percolation.

All particles large and small are subjected to the quantum-mechanical effect and exhibit at the same time both particle and wave characteristics. The smaller a particle is, however, the more prominent its wave nature stands out. For particles of the size range in nanometres the wave length involved, depending on their velocity and density, may be in the region of micrometres. If they pass through pores of diameters in the order of 10^{-6} metres their path will deflect from a straight line by the interference effect similar to that experienced by electron passing through slids in a screen. The blockage, therefore, will not be wholly random but depend to a certain degree on their quantum-mechanical properties. Moreover, once inside a pore these particles will undergo the effect of particles in confinement where, depending on their size relative to the size of the pores, their wavefunction may form some standing wave pattern. Then their probable location within the pore, and thus their future trajectory to other pores, will depend on the quantum-mechanical consideration. The de Broglie wave length is $\lambda = h/p$ where h is the Planck's constant and p the particle's momentum. The standing wave equation is $n(\lambda/2) = D$ where n is a positive integer greater than zero and D the diameter of the pore, which gives the wavelength of the standing waves $\lambda = 2D/n$ and the corresponding total energy of the particle $E = h^2n^2/(8mD^2)$. The probability of finding the particle at the area of space δV within the pore is $P = |\psi_{\delta V}|^2\delta V$ (*cf* Walters, 1982) if the pore is totally enclosed. Here ψ is the standing probability wave of the particle. One needs to know this ψ for Voronoi and and irregularly-shaped pores This is what I intend to investigate further in the future. And it is also what I suggest people working in this area bear in mind.

9

§ 9. Bibliography

George O. Abell. The distribution of rich clusters of galaxies. *The Astrophysical Journal. Supplement Series.* **3**, Supplement Number 31, 211–288. May, 1958.

D. A. Aboav and T. G. Langdon. The shape of grains in a polycrystal. *Metallography.* **2**, 171–178. 1969.

D. A. Aboav. The arrangement of grains in a polycrystal. *Metallography.* **3**, 383–390. 1970.

– The stability of grains in a polycrystal. *Metallography.* **4**, 425–441. 1971.

– The arrangement of cells in a net. *Metallography.* **13**, 43–58. 1980.

– The arrangement of cells in a net. II. *Metallography.* **16**, 265–273. 1983.

Joan Adler and Amnon Aharony. Diffusion percolation. I. Infinite time limit and bootstrap percolation. *Journal of Physics A. Mathematical and General.* **21**, 1387–1404. 1988.

P. K Ahnelt., E. Fernandez, O. Martinez, J. A. Bolea, A. Kubber-Heiss. Irregular S-cone mosaics in felid retinas. Spatial interaction with axonless horizontal cells, revealed by cross correlation. *Journal of the Optical Society of America. A, Optics, Image Science, and Vision.* **17**, 3, Mar. 2000, p. 580–8.

John Algeo. *Problems in the origins and development of the English language.* 3rd ed. Harcourt Brace Jovanovich. 1966 (1982).

J. Ammermuller, W. Mockel, P. Rujan. A geometrical description of horizontal cell networks in the turtle retina. *Brain Research.* **616**, 1–2, 9th Jul. 1993, p. 351–6.

Alizabeth Andrews. *Healthy practice for musicians.* Rhinegold. London. 1997.

M. F. Ashby. Cork. in *Encyclopedia of Material Science and Engineering.* 854–858. Pergamon. 1990.

Francis Bacon. *Novum Organum.* (The new organon. or true directions concerning the interpretation of nature.) 1620. Translation. James Spedding, et al. Taggard and Thompson. 1863.

C. Bradford Barber, David P. Dopkin and Hannu Huhdanpaa. The Quickhull algorithm for convex hulls. *ACM Transactions on Mathematical Software.* vol. 22, no. 4, 469–483. December, 1996.

J. M. Benoit, B. Corraze, S. Lefrant, W. J. Blau, P. Bernier and O. Chauvet. Transport properties of PMMA-carbon nanotubes composites. *Synthetic Metals.* **121**, 1–3, 1215–1216. March 2001.

Claude Berge. *The theory of graphs and its applications.* 1958. Translation. Alison Doig. Methuen. Wiley. 1962(1964).

G. Birkhoff and S. M. Lane. *A survey of modern algebra.* A K Peters. 1997.

Reginald Horace Blyth. *Oriental Humour.* Hokuseido. Tokyo. 1959.

A. Bravais. *A. Bravais' Abhandlungen über symmetrische polyeder.* (1849). Ostwald's Klassiker der Exakten Wissenschaften. no. 17, 46–49. Wilhelm Engelmann. Leipzig. 1890.

S. R. Broadbent and J. M. Hammersley. Percolation processes I, Crystals and mazes. *Proceedings of the Cambridge Philosophical Society.* 54, 629–641. 1957.

K. Broderix, H. Lowe, P. Muller and A. Zippelius. Critical dynamics of gelation. *Physical Review E.* **6301**, 1, 1510–1526. January 2001.

G. E. Brown. *Many-body problems.* North-Holland. 1972.

Ronald S. Burt. *Structural holes. The social structures of competition.* Harvard University Press. 1992.

Robert Wolfgang Cahn. *Encyclopedia of Material Science and Engineering.* Pergamon. 1990.

A. G. W. Cameron. Abundance of the elements in the solar system. *Space Science Reviews.* 15, 121–147. 1973.

J. Chalupa, P. L. Leath and G. R. Reich. Bootstrap percolation on a Bethe lattice. *Journal of Physics C, Solid State Physics.* **12**, L31–L35. 1979.

Chamber's Encyclopedia. New revised ed. International Learning Systems. London. 1970.

X. Cheng, A. M. Sastry and B. E. Layton. Transport in stochastic fibrous networks. *Journal of Engineering Materials and Technology, Transactions of the ASME.* **123**, 1, 12–19. January 2001.

Sen Chung Chan. *Studies of separation of dilute dispensions.* Ph.D. thesis. UMIST, Manchester, U.K. 1990.

K. Clarkson and P. Shor. Applications of random sampling in computational geometry, II. *Discrete Computational Geometry.* **4**, 387–421. 1989.

W. D. Clayton. Studies in the *Gramineae.* XXI. *Coelorhachis* and *Rhytachne.* A study in numerical taxonomy. *Kew Bulletin.* **24**, 309–314. 1970.

W. D. Clayton. Studies in the *Gramineae.* XXVI. Numerical taxonomy of the *Arundinelleae. Kew Bulletin.* **26**, 111–123. 1972.

Collins USA *Collins Road Atlas USA. Canada and Mexico.* Rand McNally. 1999.

J. H. Conway and N. J. A. Sloane. *Sphere packings, lattices and groups.* Springer-Verlag, N.Y.. 1988.

Alan Cooper, Russel Stannard and Barrie W. Jones. *The formation and evolution of the stars.* The Open University. 1985.

J. M. Coulson, J. F. Richardson, J. R. Backhurst and J. H. Harker. *Chemical Engineering. V. 2.* 4th ed. Particle technology and separation processes. Pergamon. 1991.

K. G. Cox, N. B. Price and B. Harte. *An introduction to the practical study of crystals, minerals, and rocks.* revised 1st ed. McGrawHill. 1974.

Ken Croswell. *The alchemy of heaven.* Oxford. 1995.

H. M. Cundy and A. P. Rollett. *Mathematical models.* 3rd edition, 1989. Tarquin.

C. A. Curcio, K. R. Sloan. Packing geometry of human cone photoreceptors, variation with eccentricity and evidence for local anisotropy. *Visual Neuroscience.* Vol. 9, No. 2, Aug. 1992, p. 169–80.

James Dwight Dana and Edward Salisbury Dana. *Dana's new mineralogy.* 8th ed. Wiley. 1997.

Paul Davies. Liquid space. *New Scientist.* 30–34. 3rd November 2001 .

P. Dean. A new Monte Carlo method for percolation problems on a lattice. *Proceedings of the Cambridge Philosophical Society.* **59**, 397–410. 1963.

P. G. de Gennes, P. Lafore and J. P. Millot. Amas accidentels dans les solutions solides désordonnées. *Journal of Physics and Chemistry of Solids.* **11**, 105–110. 1959.

Michael de Podesta. *Understanding the properties of matter.* 2nd ed. Taylor & Francis. 1996 (2002).

Renati Des-cartes *Principia philosophiæ.* 1644. Ludovicum Elzevirium. *cf* René Descartes. Œuvres philosophiques. Vol.III. 1973. Garnier Frères. Ferdinard Alquié ed.

G. de Vancouleurs. The extragalactic distance scale VIII. A comparison of distance scales. *The Astrophysical Journal.* **415**, 10–32. 20th September 1993 .

Charles Dickens. *A child's history of England.* 1851(1868).

G. Lejeune Dirichlet. Über die Reduction der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen. *Crelle.* Journal für die reine und angewandte Mathematik. Bd. 40, 209–227. 1850. *cf* Tiyapan (2001)

Ian L. Dryden and Kanti V. Mardia. *Statistical shape analysis.* John Wiley & Sons. 2002.

R. Eisenthal and M. J. Danson. *Enzyme Assays.* 2nd ed. Oxford. 1992 (2002).

J. W. Essam. Percolation theory. *Reports on Progress in Physics.* vol. 43, 835–912, 1980.

Arthur Eugene Fitzgerald, Charles Kingsley Jr. and Alexander Kusko. *Electric Machinery.* McGraw-Hill Kogakusha. 3rd Ed. 1971.

James P. Evans and Frederick M. Chester. Fluid-rock interaction in faults of the San Andreas system. In ferences from San Gabriel fault rock geochemistry and micro structures. *Journal of Geophysical Research.* **100**, B7, 13007–13020. 10 July 1995.

Rachel Farmer. *Beginner's Russian. An easy introduction.* Teach Yourself Books. 1996.

H. L. Frisch, E. Sonnenblick, V. A. Vyssotsky, and J. M. Hammersley. Critical percolation probabilities (site problem). *Physical Review.* vol. 124, no. 4, 1021–1022. 15th November 1961 .

L. Flandin, Y. Brechet and J. Y. Cavaille. Electrically conductive polymer nanocomposites as deformation sensors. *Composites Science and Technology.* **61**, 6, 895–901. 2001.

Mason Florence, Marisa Gierlich and Andrew Dean Nystrom. *Rocky Mountains. Colorado, Wyoming, Montana and Idaho.* Lonely Planet. 3rd ed. 2001.

E. M. Forster. *Howards End.* 1910.

Dennis Freeborn. *From Old English to standard English. A course book in language variation across time.* Macmillan. 1992.

J. Garside and J. W. Mullin. The crystallization of aluminium potassium sulphate, a study in the assessment of crystallizer design data. Part III. Growth and dissolution rates. *Transactions of the Institution of Chemical Engineers.* vol. 46, no. 1, T11–T18. 1968.

B. Gay and P. E. Preece. Matrix methods for the solution of fluid network problems, Part I–mesh methods. *Transactions of the Institution of Chemical Engineers.* vol. 53, no. 1, 12–15. January 1975.

J. W. Giles, C. Hanson and J. G. Marsland. Drop size distributions in agitated liquid-liquid systems with simultaneous interface mass transfer and chemical reaction. in *Proceedings of the International Solvent Extraction Conference, ISEC 71.* vol. 1, 94–111. 1971.

A. S. Glassner. Ed. *Graphics gems.* Academic Press. 1990.

G. Gottardi and E. Galli. *Natural Zeolites.* Berlin. 1985.

Nigel Gotteri and Joanna Michalak-Gray. *Polish.* Teach Yourself Books. 1997.

P. J. Green and R. Sibson. Computing Dirichlet tessellations in the plane. *The Computer Journal.* vol. 21, February to November, 168–173, 1978.

G. Grimmett. *Percolation.* Second edition. Springer. 1999.

Patrick Grosfils, Jean Pierre Boon, E. G. D. Cohen and L. A. Bunimovich. Propagation and organization in lattice random media. *Journal of Statistical Physics.* vol. 97, nos. 3/4, 575–608, 1999.

Branko Grünbaum. *Convex polytopes.* Inter Science, Wiley. Pure and applied mathematics series, vol. XVI. 1967,

Branko Grünbaum and G. C. Shepherd. *Tilings and patterns.* W. H. Freeman. N.Y. 1987.

J. M. Hammersley and W. Morton. Poor man's Monte Carlo. *Journal of the Royal Statistical Society (B).* 16, 23–38. 1954.

J. M. Hammersley. Percolation processes. II. the connective constant. *Proceedings of the Cambridge Philosophical Society.* **53**, 642–645. 1957.

J. M. Hammersley. Comparison of atom and bond percolation processes. *Journal of Mathematical Physics.* vol. 2, no. 5, 728–733. September-October, 1961.

Allen Hammond. *Which World? scenarios for the* 21st *century.* Earthscan. 1998.

C. Hammond. *The basics of crystallography and diffraction.* Oxford. 2001.

Thomas L. Hankins. Triplets and triads. Sir William Rowan Hamilton on the metaphysics of mathematics. *Isis.* **68**, 242, 175–193. 1977.

John Happel. Viscous flow relative to arrays of cylinders. *American Institute of Chemical Engineers Journal.* vol. 5, no. 2, 174–177. June, 1959.

G. H. Hardy and E. M. Wright. *An introduction to the theory of numbers.* 1938. Oxford. or 5th ed. 1979.

H. W. Harrison and P. M. Trotman. *Foundations, basements and external works. Performance, diagnosis, maintenance, repair and the avoidance of defects.* BRE. 2002.

P. M. Heertjes. Filtration. *The Transactions of the Institution of Chemical Engineers.* vol. 42, no. 7, T266–T274, 1964.

H. S. Hele-Shaw. The flow of water. *Nature.* Vol. 58, p. 34–36. May 12, 1898.

– The motion of a perfect liquid. *Nature.* Vol. 60, p. 446–451. September 7, 1899.

M. Hindermann-Bischoff and F. Ehrburger-Dolle. Electrical conductivity of carbon black-polyethylene composites- Experimental evidence of the change of cluster connectivity in the PTC effect. *Carbon.* **39**, 3, 375–382. 2001.

Franz E. Hohn. *Elementary matrix algebra.* 3rd ed. Macmillan. 1958 (1973).

D. Houi and R. Lenormand. Experimental and theoretical study of particle accumulation at the surface of a filter. in *The Proceedings of* 4th *World Filtration Contress.* no. 1, 1.1–1.6. 1986.

Fred Hoyle. *Ten faces of the universe.* Heinemann. London. 1977.

Fred Hoyle and Chandra Wickramasinghe. *Life cloud. The origin of life in the universe.* J. M. Dent & Sons. 1978.

Fred Hoyle and Nalin Chandra Wickramasinghe. *Diseases from space.* J. M. Dent & Sons. 1979.

Q. Huang, Z. W. Li, J. Li and C. K. Ong. The magnetic, electrical transport and magnetoresistance properties of epitaxial $\mathrm{La_{0.7}Sr_{0.3}Mn_{1-x}Fe_xO_3}$ (x=0–0.20) thin films prepared by pulsed laser deposition. *Journal of Physics-Condensed Matter.* **13**, 18, 4033–4047. May 2001.

A. G. Hunt. Applications of percolation theory to porous media with distributed local conductances. *Advances in Water Resources.* **24**, 3–4, 279–307. 2001.

Charlie Ireland. Private communication. Correspondence from Invercargill, New Zealand. 12*th* August 2002

C. Isenberg. *The science of soap films and soap bubbles.* Tieto. 1978.

A. F. Ismail, R. Norida and A. R. Sunarti. Latest development on the membrane formation for gas separation. *Songklanakarin Journal of Science and Technology.* **24** (suppl.), 1025–1043. 2002.

K. J. Ives. Deep bed filtration. in *Solid–Liquid Separation.* 2nd. Butterworths Monographs in Chemistry and Chemical Engineering. Ladislav Svarovsky. ed. 284–301. 1977.

N. M. Jackson. *A mathematical model to simulate the structure and performance of porous media.* PhD Thesis. UMIST, Manchester, 1994.

N. M. Jackson, R. Jafferali, D. J. Bell and G. A. Davies. A study of the structure of micro and ultra filtration membranes. The Voronoi tessellation as a stochastic model to simulate the structure. *Journal of Membrane Science.* **162**, 23–43. 1999.

Riaz Jafferali. *A stochastic model to simulate the structure and performance of microfiltration media and the growth of animal cell cultures.* PhD Thesis. UMIST, Manchester, 1995.

Harold James. *The German slump. Politics and economics 1924–1936.* Clarendon. Oxford. 1986(1987).

Vera Javarek and Miroslava Sudjić *Serbo-Croat.* 2nd ed. Teach Yourself Books. 1963(1972).

G. R. Jerauld, J. C. Hatfield, L. E. Scriven and H. T. Davis. Percolation and conduction on Voronoi and triangular networks. A case study in topological disorder. *Journal of Physics C. Solid State Physics.* **17**, 1519–1529. 1984.

M. M. Jevtic, Z. Stanimirovic and I. Stanimirovic. Evaluation of thick-film resistor structural parameters based on noise index measurements. *Microelectronics Reliability.* **41**, 1, 59–66. January 2001.

A. J. Jones. *Crystalization process sytems.* Butterworth Heinemann. 2002.

JETRO. Nylon 66 Spun Fiber Fabric. *New Technology. Japan.* vol. 29, no. 6, p. 32. Japan External Trade Organisation. September, 2001.

Immanuel Kant. Beantwortung der Frage. Was ist Aufklärung. *Berlinische Monatsschrift.* **4**. December. 1784. 481–494.

F. August Kekulé Sur la constitution des substances aromatiques. *Bulletin de la societé chimique de Paris.* **3**, 98–111. 1865.

Alan R. Kerstein. Equivalence of the void percolation problem for overlapping spheres and a network problem. *Journal of Physics A. Mathematical and General.* **16**, 3071–3075. 1983.

Daniel Keyes. *The minds of Billy Milligan.* Bantam. 1999.

C. Klein and C. S. Hurlbut, Jr. *Manual of mineralogy.* 21st Edition. Wiley. 1993.

D. E. Knuth. *The TEX.* Addison Wesley. 1970.

Donald E. Knuth. *The art of computer programming. V. I Fundamental algorithms.* 3rd ed. Addison-Wesley. 1997

Donald E. Knuth. *The art of computer programming. V. II.* 3rd ed. Addison-Wesley. 1998.

J. W. de Kock and M. R. Judd. The flow pattern within a void in a porous mass. *The Transactions of the Institution of Chemical Engineers.* vol. 43, no. 3, T78–T84. 1965.

J. M. Kosterlitz and D. J. Thouless. Ordering, metastability and phase transitions in two-dimensional systems. *Journal of Physics C. Solid State Physics.* vol. 6, 1181–1203.

John Krige. *Science, revolution and discontinuity.* The Harvester Press. 1980.

Roderic Lakes. Foam structures with a negative Poisson's ratio. *Science.* **235**, 1038–1040. 1987.

Leslie Lamport. *LaTEX document preparation system.* 2nd ed. 1985 (1994).

Russel D. Larsen. Kekulé's benzolfest speech. A fertile resource for the sociology of science. in *The Kekulé riddle. A challenge for chemists and psychologists.* 178–193. Cache River. 1993.

A. Levy, S. Reich and P. Meakin. The shape of clusters on rectangular 2D lattices in a simple "phase separation" computer experiment. *Physics Letters.* vol. 87A, no. 5, 248–252. 11th January 1982 .

Larry S. Liebovitch and Daniela Scheurle. Two lessons from fractals to chaos. *Complexity.* vol. 5, no. 4, 34–43. March-April, 2000.

K. C. Lim, M. A. Hashim and B. Sen Gupta. The effect of volume shape factor on the crystal size distribution of fragments due to attrition. *Crystal Research Technology.* vol. 34, no. 4, 491–502. 1999.

J. Litwiniszyn. Colmatage considered as a certain stochastic process. *Bulletin de L'Academie Polonaise des Sciences.* vol. XI, no. 3, 117–121. 1963.

J. Litwiniszyn. On some mathematical models of the suspension flow in porous medium. *Chemical Engineering Science.* vol. 22, 1315–1324. 1967.

C. W. Lowe. Some techniques of evolutionary operation. *Transactions of the Institution of Chemical Engineers.* vol. 42, no. 9, T334–T344. 1964.

Ian S. Lustick and Dan Miodownik. Deliberative democracy and public discourse. The agent-based argument repertoire model. *Complexity.* vol. 5, no. 4, 13–30. March-April, 2000.

T. Magay and L. Országh. *A concise Hungarian–English dictionary.* Oxford University Press. Akademiai Kiadó 1981 (1990).

K. R. Mecke and A. Seyfried. Strong dependence of percolation thresholds on polydispersity. *Europhysics Letters.* **58** (1), 28–34, 1st April 2002 .

Hans Meinhardt. *The algorithmic beauty of sea shells.* Springer-Verlag. 1995 (1998).

R. K. Merton. *On the shoulders of giants.* A Shandean Postscript, Harvest/Harcourt. 1965.

Stephen A. Miller and Amos Nur. Permeability as a toggle switch in fluid-controlled crustal processes. *Earth and Planetary Science Letters.* **183**, 133–146. 2000.

R. E. Miles. A random division of space. *Special Supplement to Advances in Applied Probability.* 243–266. 1972.

Gian Maria Milesi-Ferretti and Assaf Razin. Current account reversals and currency crises. Empirical regularities. in *Currency Crisis.* Paul Krugman. ed. University of Chicago Press. 2000.

I. Mironi-Harpaz and M. Narki. Electrical behavior and structure of polypropylene/ultrahigh molecular weight polyethylene/carbon black immiscible blends. *Journal of Applied Polymer Science.* **81**, 1, 104–115. July 2001.

U. Mizutani. *Introduction to the electron theory of metals.* Cambridge. 2001.

K. K. Mohanty, J. M. Ottino and H. T. Davis. Reaction and transport in disordered composite media. Introduction of percolation concepts. *Chemical Engineering Science.* vol. 37, no. 6, 905–924. 1982.

J. Møller. Random tessellations in $\Re^d$. *Advances in Applied Probability.* vol. 21, p. 37–73, 1989.

M. Mulder. *Basic principles of membrane technology.* Kluwer. 1992.

T. Mulder and R. Gimbel. On the development of high performance filtration materials for deep bed filters. in *The proceedings for the* 5th *World Filtration Congress.* vol. 3, 45–57. Nice, 1990.

M. Narkis, G. Lidor, A. Vaxman and L. Zuri. Innovative ESD thermoplastic composites structured through melt flow processing. *IEEE Transactions on Electronics Packaging Manufacturing.* **23**, 4, 239–245. October 2000.

Andrew Nathaniel Nelson. *Japanese–English Character Dictionary.* Charles E. Tuttle. 1962.

D. Neilson, J. S. Thakur and E. Tosatti. Characterising the metal-insulator transition in two dimensions. *Australian Journal of Physics.* **53**, 4, 531–535. 2000.

Johann Nittmann, Gérard Daccord and H. Eugene Stanley. Fractal growth of viscous fingers, quantitative characterization of a fluid instability phenomenon. *Nature* Vol. 314, 14 March 1985, p. 141–144

A. Okabe, B. Boots and K. Sugihara. *Spatial tessellations concepts and applications of Voronoi diagrams.* Wiley. 1992.

R. K. Pathria. *Statistical Mechanics.* 2nd ed. Butterworth-Heinemann. 1996 (1997).

C. Pennetta, L. Reggiani, G. Trefan, F. Fantini, A. Scorzoni and I. de Munari. A percolative approach to electromigration in metallic lines. *Journal of Physics D, Applied Physics.* **34**, 9, 1421–1429. May, 2001.

C. Pennetta, G. Trefan and L. Reggiani. Scaling law of resistance fluctuations in stationary random. *Physical Review Letters.* **85**, 24, 5238–5241. December, 2000.

Roger Penrose. *The emperor's new mind. Concerning computers, minds, and the law of physics.* 1989.

R. Penrose. Tilings and quasi-crystals. a non-local growth problem? in *Introduction to the mathematics of quasicrystals.* Marko V. Jarić Ed. 53–79. Aperiodicity and order. Vol. 2. Academic Press. 1989.

Petersburg *Investment projects of St. Petersburg* Information and Consulting Centre. International Finance and Enterprise Week. Manchester International Conference Centre. 31st October–2*nd* November 2001 .

V. Y. Petrovsky and Z. S. Rak. Densification, microstructure and properties of electroconductive Si3N4-TaN composites. Part II *Electrical and mechanical properties.* Journal of the European Ceramic Society.

21, 2, 237–244. February 2001.: F. C. Phillips. *An introduction to crystallography.* Longmans. 3rd ed. 1949 (1963).

H. W. Piekaar and L. A. Clarenburg. Aerosol filters – the tortuosity factor in fibrous filters. *Chemical Engineering Science* vol. 22, 1817–1827. 1967.

Plato. [Aristocles] *Phaedo.* [Phaidon] 360 BC.

Michael de Podesta. *Understanding the properties of matter.* Taylor & Francis. 2nd ed. 2002.

J. B. Poole and D. Doyle. Research in solid-liquid separation. *The Chemical Engineer.* no. 190, CE169–CE172. July–August, 1965.

K. E. Porter. Liquid flow in packed columns. Part I. The rivulet model. *The Transactions of the Institution of Chemical Engineers.* vol. 46, no. 3, T69–T73. 1968.

J. G. Powles and N. Quirke. Fractal geometry and Brownian motion. A new parameter to describe molecular motion. *Physical Review Letters.* Vol. 52, No. 18, 1571–1574. 30 April 1984.

Burkhard A. Prause. *Magnetic resonance imaging of structure and coarsening in three-dimensional foams.* Ph.D. Dissertation, the University of Notre Dame. 2000.

Franco P. Preparata and Michael Ian Shamos. *Computational Geometry. An introduction.* Springer-Verlag. 1985.

Przemyslaw Prusinkiewicz and Aristid Lindenmayer. *The algorithmic beauty of plants.* Springer-Verlag. 1990.

L. R. Pujara and Naresh Shanbhag. Some stability theorems for polygons of polynomials. *IEEE Transactions on Automatic Control.* Vol. 37. No. 11. Nov. 1992. 1845–1849

L. Rai. Pujara. On the pseudoboundary of unstable polytopes of polynomials. *IEEE Transactions on Automatic Control.* **41**, 8. 1188–1190. August 1996.

Annick van Put, Akos Vertes, Darek Wegrzynek, Boris Treiger and René van Grieken. Quantitative characterization of individual particle surfaces by fractal analysis of scanning electron microscope images. *Fresenius Journal Analytical Chemistry.* **350**, 440–447, 1994.

J. H. Raistrick. The nature of adsortive filtration. in *Proceedings of the* 4th *World Filtration Congress.* R. Vanbrabant, J. Hermia and R. A. Weiler, Eds. Part I. p. 1.65–1.72. 1986.

N. Rivier. Recent results on the ideal structure of glasses. *Journal de Physique.* Colloque C9, Supplément au no. 12, **43**, C9-91–C9-95. December, 1982.

Steven Rodelet and Jeffrey Sacs. The onset of the East Asian Financial Crisis. in *Currency Crisis.* Paul Krugman. ed. University of Chicago Press. 2000.

Min Zhi Rong, Ming Qiu Zhang, Yong Xiang Zheng, Han Min Zeng and K. Friedrich. Improvement of tensile properties of nano-SiO_2/PP composites in relation to percolation mechanism. *Polymer.* **42**, 4, 3301–3304. March 2001.

H. E. Rose and J. E. English. The influence of blinding material on the results of test sieving. *Transactions of the Institution of Chemical Engineers.* vol. 51, 14–21. 1973.

P. N. Rowe and R. Collins. The flow pattern within a void in a porous mass, *The Transactions of The Institution of Chemical Engineers.* vol. 43, no. 7, T217–T220. 1965.

P. N. Rowe and C. Yacono. The distribution of bubble size in gas fluidised beds. *Transactions of the Institution of Chemical Engineers.* vol. 53, no. 1, 59–60. January, 1975.

G. S. Rushbrooke and D. J. Morgan. On the magnetically dilute Heisenberg and Ising ferromagnetics. *Molecular Physics.* vol. 4, 1–15, 1961.

Joan Russell. *Swahili.* Teach Yourself Books. 1996.

Oliver Sacks. *Awakenings.* Piccador. Pan Books. 1973(1982).

– *seeing voices. A journey into the world of the deaf.* Piccador. Pan Books. 1989(1990).

Muhammad Sahimi. *Applications of percolation theory.* Taylor & Francis. 1994.

George Salmon. *A treatise on the analytic geometry of three dimensions.* revised by Reginald a. P. Rogers. 5th ed. Longman, Green and Co. 1912.

Vinod K. S. Shante and Scott Kirkpatrick. An introduction to percolation theory. *Advances in Physics.* **20**, 325–357. 1971.

Arnold Schoenberg. *Harmonielehre.* 1978. English translation. Theory of Harmony. Farber. 1983.

Ralf Schumacher. *A stochastic model to simulate the structure and performance of cellular polymeric membranes in dead end filtration.* Diplomarbeit. [Ph.D. thesis] Institute für chemische Verfahrenstechnik der Technischen Universität Clausthal and UMIST. 1996.

B. Sharp, D. Morton and A. E. Clark. Effectiveness of metal surface treatments in controlling microleakage of the acrylic resin-metal framework interface. *Journal of Prosthetic Dentistry.* **84**, 6, 617–622. 2000.

Madan G. Singh. Ed. *Systems & Control Encyclopedia.* Pergamon. 1987.

John Maynard Smith and Eörs Szathmáry. *The major transitions in evolution.* W. H. Freeman. 1995.

Herbert Warington Smyth. *Five years in Siam, from 1891–1896.* J. Murray. London. 1898. in two volumes.

Rupert Snell. *Beginner's Hindi script.* Teach Yourself Books. 2000.

Murray R. Spiegel. *Probability and Statistics.* Schaum's outline. McGraw-Hill. 1975.

David J. Srolovitz. Computer simulation of microstructural evolution. in *Encyclopedia of Material Science and Engineering.* 845–854. Pergamon. 1990.

Dietrich Stauffer and Amnon Aharony. *Introduction to percolation theory.* 1985. 2nd ed. Taylor & Francis. 1992. and 2nd revised ed. 1994 (1998)

O. Stenull, H. K. Janssen, K. Oerding. Effects of surfaces on resistor percolation. *Physical Review E.* **6305**, 5, 6128–6141. May 2001.

R. L. Stevenson. *Dr. Jenkyll and Mr. Hyde.* 1886.

Robert Louis Stevenson. *The amateur emigrant.* 1895.

William C. Stokoe Jr., Dorothy C. Casterline and Carl G. Croneberg. *A Dictionary of American Sign Language on Linguistic Principles.* Gallaudet College Press. 1965

L. Svarovsky. Characterization of particles suspended in liquids. in *Solid-liquid separation.* 2nd ed. L. Svarovsky, Ed. Butterworths. p. 8–32. 1977.

Masaharu Tanemura, Tohru Ogawa and Naofumi Ogita. A new algorithm for three-dimensional Voronoi tessellation. *Journal of Computational Physics.* **51**, 191–207. 1983.

Aspasia Theodosiou. personal communication. 2002.

J. R. R. Tolkien. *Lord of the Rings.* 1955.

C. R. G. Treasure. Fine particle-size classification. *Transactions of the Institution of Chemical Engineers.* vol. 43, no. 6, T199–T205. 1965.

U. of Man. *Mathematics, the University of Manchester, 2000 and 2001 entry.* booklet. the University of Manchester. 2000.

Rüdiger Vaas. Ed. Urknall für Einsteiger. *Bild der Wissenschaft.* May 2002.

L. Verlet. Computer 'experiments' on classical fluids. I. Thermodynamical properties of Lennard-Jones molecules. *Physical Reviews.* **159**, 1, 98–103. 1967.

G. F. Voronoï New application of continuous parameters to the theory of quadratic form. First memoir. On some properties of the perfect positive quadratic forms. *Journal für die reine und angewandte Mathematik.* **133**, 97–178, 1908. *cf* Tiyapan (2001).

G. F. Voronoï New application of continuous parameters to the theory of quadratic form. Second memoir. Research on the primitive parallelohedron. *Journal für die reine und angewandte Mathematik.* **134**, 1908. *cf* Tiyapan (2001).

G. F. Voronoï New application of continuous parameters to the theory of quadratic form. Second memoir. Studies on the primitive parallelohedra. *Journal für die reine und angewandte Mathematik.* **136**, 67–181, 1909. *cf*. Tiyapan (2001).

Tuan Duc Vuong and John Moore. *Colloquial Vietnamese. A complete language course.* Routledge. 1994.

N. Walsh. *Making TeX work.* O'Reilly. 1994.

John Walters. *Quantum mechanics, theory and applications.* Science second level course, S271 Discovering Physics. The Open University. 1982.

David Ward. Language cull could leave people speechless. *The Guardian.* Saturday 25th May 2002 . National news, 13.

D. Weaire. Some remarks on the arrangement of grains in a polycrystal. *Metallography.* **7**, 157–160.

S. H. Wen and D. D. L. Chung. Cement as a thermoelectric material. *Journal of Materials Research.* **15**, 12, 2844–2848. 2000.

Magnus Joseph Wenninger. *Polyhedron models.* Cambridge University Press. 1971.

R. van de Weygaert and V. Icke. Fragmenting the universe II. Voronoi vertices as Abell clusters. *Astronomy and Astrophysics.* **213** 1–9. 1989.

R. van de Weygaert. Fragmenting the universe III. The construction and statistics of 3-d Voronoi tessellation. *Astronomy and Astrophysics.* **283**, 361–406. 1994.

Elaine Wilcock. *A study of mesh mist eliminators.* PhD thesis. UMIST. 1994.

David Wilkinson and Jorge F. Willemsen. Invasion percolation. a new form of percolation theory. *Journal of Physics A. Mathematical and General.* **16**, 3365–3376. 1983.

E. T. Wilkinson, A. R. N. Fairclough and G. A. Davies. The filtration of dilute suspensions using non-woven cloths and membranes. in *Proceedings of the* 4th *World Filtration Congress.* R. Vanbrabant, J. Hermia and R. A. Weiler, Eds. Part I. p. 1.7–1.18. 1986.

Trevor Williams and Rolf Bjerknes. Stochastic model for abnormal clone spread through epithelial basal layer. *Nature.* vol. 236, 19–21. 3rd March 1972 .

John H. Wotiz. *The Kekulé riddle. A challenge for chemists and psychologists.* Cache River. 1993.

P. J. Wray, O. Richmond and H. L. Morrison. Use of the Dirichlet tessellation for characterizing and modeling nonregular dispersions of second-phase particles. *Metallography.* **16**, 39–58. 1983.

X. J. Zhan, J. B. Troy. Modeling cat retinal beta-cell arrays. *Visual Neuroscience.* Vol. 17, No. 1, Jan.–Feb. 2000, p. 23–39.

H. J. Zhang, J. W. Wang, J. Li, S. K. Cao and A. Shan. Toughened polypropylene with balanced rigidity, III, compositions and mechanical properties. *Journal of Applied Polymer Science.* **79**, 8, 1345–1350. February 2001.

§ 9.1 Publication and writings, Kittisak Nui Tiyapan

1991 Antimonytrioxide extraction from stibnum ore by hydrometallurgical method. Senior project. [Final year project]. Mining Engineering, Chulalongkorn University, Thailand. 1991. KNT1(i).

1994 Cyberspace. *Articles Online.* ATSIST. 12th December 1994 . also at www.nectec.or.th/bureaux/atsist KNT2(i).

1995 The End of Stars. *Articles Online.* ATSIST. 20th February 1995 . also at www.nectec.or.th/bureaux/atsist KNT3(i).

1995 Self-tuning Extremum Control. *Design Exercise Report.* under the supervision of Dr. M. B. Zarrop. M.Sc. Course in Control and Information Technology 1994/1995. Control System Centre, UMIST. KNT3(ii).

1995 Computation of Fluid Flow. *M.Sc. dissertation.* under the supervision of Professor G. A. Davies and Professor D. J. Bell. September. UMIST. KNT3(iii).

1995 The story of Andromedra. *Sakkayaphab.* November. **3**, 2, 24–25. ATPIJ, Japan. Thai translation by *Sroemśakd́xi Ụ̀atrongcitṫa.* KNT3(iv).

1996 Let's start at the very beginning. *Sakkayaphab.* January. **3**, 4, 23–25. ATPIJ, Japan. Thai translation by *Sroemśakd́xi Ụ̀atrongcitṫa.* KNT4(i).

1996 To be unkempt. *Sakkayaphab.* April. **3**, 7. ATPIJ, Japan. Thai translation by *Suvanjay Bongṣasukicvaḍhȧna.* KNT4(ii).

1996 I see a white car before me. *Sakkayaphab.* June. **3**, 9, 12–13. ATPIJ, Japan. Thai translation by *Taśȧṇiya Mẹdhạbisiṭh.* KNT4(iii).

1996 Critical probability and other properties of 2-d tessellation *Mathematical Theory of Networks and Systems–96.* 24th –28th June (27th June , Session on Analysis of Queuing Networks). The Ritz-Carlton, St. Louis, Missouri. KNT4(iv).

1996 Some properties of stochastic optimal control. submitted in July 1996 to the 7th International Symposium on Dynamic Games and Applications (16–18 December 1996). Japan. KNT4(v).

1996 On an algorithm for object-location problem. [Object location using Extremum Control] *1996 Advanced Theory and Application of Control Systems.* 6th –8th October . Izu Recreation Centre, Atagawa Heights, Atagawa, Japan. Fu-I1–Fu-I5. KNT4(vi).

1996 On pragmatists and idealists. submitted to the Sakkayaphab journal. 21st October 1996 . KNT4(vii).

1996 [Nhạun khạung Mạu̇rris] (The Morris Worm). *Sakkayaphab.* December. **4**, 3, 20–22. ATPIJ, Japan. KNT4(viii).

1997 Fractals in traffic control. *The Fourth Annual Conference of Thai Researchers in Japan.* abstract. 23rd February . Nippon Seinenkan Hotel, Tokyo. TSAJ and ATPIJ. p. 18 KNT5(i).

1997 Distributed parameter systems. *The Fourth Annual Conference of Thai Researchers in Japan.* abstract. 23rd February . Nippon Seinenkan Hotel, Tokyo. TSAJ and ATPIJ. p. 19 KNT5(ii).

1997 Vision robots. *The Fourth Annual Conference of Thai Researchers in Japan.* abstract. 23rd February . Nippon Seinenkan Hotel, Tokyo. TSAJ and ATPIJ. p. 20 KNT5(iii).

1997 Singular perturbation. *The Fourth Annual Conference of Thai Researchers in Japan.* abstract. 23rd February . Nippon Seinenkan Hotel, Tokyo. TSAJ and ATPIJ. p. 21 KNT5(iv).

1997 Simulation techniques using RLS algorithm for object-locaition problem. *Proceedings of the World Congress on Systems Simulation.* 1st–3rd September . Pan Pacific Hotel, Singapore. 518–522. KNT5(v).

1997 Report for German Literature course. TIT. presented to Professor Ishikawa. 9th September 1997 . KNT5(vi).

1997 Modelling the economics. *1997 Advanced Theory and Application of Control Systems.* 10th –12th October . National Olympic Memorial Youth Centre, Yoyogi, Shibuya-ku, Tokyo. Fu-A1–Fu-A2. KNT5(vii).

1997 Modelling economics as a flow of money within networks. submitted to the Journal of Statistical Physics. 4th November 1997 . KNT5(viii).

1997 Modelling of traffic congestion. submitted to the Journal of Statistical Physics. 4th November 1997 . KNT5(ix).

1998 Simulation techniques using RLS algorithm for object-location problem. Paper presented to the Furuta Laboratory on an in-house seminar. TIT, Tokyo. 4-5pm. 3rd January 1998 KNT6(i).

1998 Critical probability in traffic modelling and control. Paper presented to the Furuta Laboratory on an in-house seminar. TIT, Tokyo. 2nd April 1998 . KNT6(ii).

1998 Technical report number 1. presented to Professor Katsuhisa Furuta, TIT, Japan. 16th July . KNT6(iii).

1998 Technical report number 6. On controlling the synchronous machines. presented to Professor Katsuhisa Furuta, TIT. Japan. 23rd August . KNT6(iv).

1998 Variable structure control for a singularly perturbed system. *1998 Advanced Theory and Application of Control Systems.* 26th–28th September . Hotel Ohashi, Lake Kawaguchi, Yamanashi, Japan. Fu-C1–Fu-C8. KNT6(v).

1998 Technical report number 2 (in System and Control). presented to Professor Katsuhisa Furuta, TIT. Japan. 1st November 1998 . KNT6(vi).

2000 *Interesting English. [Bhạṣạ Angkṛiṣ an nạ̀ soncaị]* Kittix Books. Chulalongkorn University Printing House. Bangkok. KNT7(i).

2000 *Free translation of English. [Plạe kled́ Angkṛiṣ]* Kittix. CU. Bangkok. KNT7(ii).

2001 *The study of Voronoi tessellation.* unpublished. 22nd May 2001 KNT8(i).

2001 *Voronoi Translated, Introduction to Voronoi tessellation and essays by G. L. Dirichlet and G. F. Voronoi.* Kittix Library Networks. CU. Bangkok. KNT8(ii).

2003 *Percolation within percolation and Voronoi Tessellation.* KLN. CU. BKK. KNT8(iii).

2003 *Thai grammar, poetry and dictionary.* KLN. CU. BKK. KNT8(iv).

2003 *A Lanna in town.* KLN. CU. BKK. KNT8(v).

2003 *A Kiwi Lanna.* KLN. CU. BKK. KNT8(vi).

2003 *A British Lanna.* KLN. CU. BKK. KNT8(vii).

2003 *Edokko no Lanna.* KLN. CU. BKK. KNT8(viii).

2003 *The Siamese Lanna.* KLN. CU. BKK. KNT8(ix).

A

§ A. Programs

§ A.1 Object location

```
% Object2dSqua.m
clear all; figure(1); clf; a=1; b=2; c=3;
a1=1; a2=1; c1=10; c2=10; b1=-2*a1*5; b2=-2*a2*7; P=eye(6); lambda=.98;
DithVar=.1; NoiseVar=.1; Limit=10; N=1000; x=2; y=3;
[Object,m] =ObjSqu(x,y);
subplot(2,2,1),Draw(Object,m,Limit,'Object','c');
xlabel('X'),ylabel('Y'); axis equal; axis([-10 10 -10 10]);
subplot(2,2,2),Layout(Object,Limit,m);
Real=RealLin(x,y,N); zeta=[a1,a2,b1,b2,c1,c2]'; t=0; s2=0; s3=0;
for Steps=1:N
  v =dither(DithVar); p =-b1/a1/2 +v; q =-b2/a2/2 +v;
  [Image,n] =ImgMtx(round(p),round(q));
  if Steps==10
    Move1(:,1) =Image(:,1); Move1(:,2) =Image(:,2);
  elseif Steps==100
    Move2(:,1) =Image(:,1); Move2(:,2) =Image(:,2);
  elseif Steps==200
    Move3(:,1) =Image(:,1); Move3(:,2) =Image(:,2);
  elseif Steps==300
    Move4(:,1) =Image(:,1); Move4(:,2) =Image(:,2);
  elseif Steps==400
    Move5(:,1) =Image(:,1); Move5(:,2) =Image(:,2);
  elseif Steps==500
    Move6(:,1) =Image(:,1); Move6(:,2) =Image(:,2);
  elseif Steps==600
    Move7(:,1) =Image(:,1); Move7(:,2) =Image(:,2);
  elseif Steps==700
    Move8(:,1) =Image(:,1); Move8(:,2) =Image(:,2);
  elseif Steps==800
    Move9(:,1) =Image(:,1); Move9(:,2) =Image(:,2);
  end;
  g =Compare(Object,Image,m,n); e =noise(NoiseVar); y =g+e;
  if y>1
    x =[p*p,q*q,p,q,1,1]'; t =t+1; Est(Steps,1) =t;
    Est(Steps,2) =-b1/a1/2; s2=Est(Steps,2);
    Est(Steps,3) =-b2/a2/2; s3=Est(Steps,3);
    P =1/lambda*P*(eye(6)-(x*x'*P)/(lambda+x'*P*x));
    zeta =zeta+P*x*(y-x'*zeta); a1 =zeta(1,:); a2 =zeta(2,:);
    b1 =zeta(3,:); b2 =zeta(4,:);
  else
    t =t+1; Est(Steps,1) =t; Est(Steps,2) =s2; Est(Steps,3) =s3;
    p =RandSrch(Limit); q =RandSrch(Limit); b1 =-2*a1*p; b2 =-2*a2*q;
  end;
end;
Title1 ='X'; Title2 ='Y'; Title3 ='Phase Plane'; Title4 ='Object and Image';
XLab ='Steps'; YLab ='Mag';
subplot(2,2,3),compare2([Est(:,1),Est(:,2)],[Real(:,1),Real(:,2)],Title1,XLab,YLab)
subplot(2,2,4),compare2([Est(:,1),Est(:,3)],[Real(:,1),Real(:,3)],Title2,XLab,YLab)
figure(2); clf;
subplot(3,3,1),Draw(Object,m,Limit,'Object','r'); axis equal; axis([-10 10 -10 10]);
subplot(3,3,2),Draw(Move1,n,Limit,'N = 10','y'); axis equal; axis([-10 10 -10 10]);
subplot(3,3,3),Draw(Move2,n,Limit,'N = 100','y'); axis equal; axis([-10 10 -10 10]);
subplot(3,3,4),Draw(Move3,n,Limit,'N = 200','y'); axis equal; axis([-10 10 -10 10]);
subplot(3,3,5),Draw(Move4,n,Limit,'N = 300','y'); axis equal; axis([-10 10 -10 10]);
subplot(3,3,6),Draw(Move5,n,Limit,'N = 400','y'); axis equal; axis([-10 10 -10 10]);
subplot(3,3,7),Draw(Move6,n,Limit,'N = 500','y'); axis equal; axis([-10 10 -10 10]);
subplot(3,3,8),Draw(Move7,n,Limit,'N = 600','y'); axis equal; axis([-10 10 -10 10]);
subplot(3,3,9),Draw(Move8,n,Limit,'N = 700','y'); axis equal; axis([-10 10 -10 10]);
 % Compare.m
function [y] =Compare(Object,Image,m,n)
y =0;
for i=1:m
  for j=1:n
    if (Object(i,1)==Image(j,1))&(Object(i,2)==Image(j,2))
      y =y +1;
    end;
  end;
end;
 % compare2.m
function [] =compare2(data1,data2,graph_title,x_label,y_label)
plot(data1(:,1),data1(:,2),'-',data2(:,1),data2(:,2),'-');
grid on; title(graph_title); xlabel(x_label); ylabel(y_label);
 % dither.m
```

```
function [dither]=dither(dith_variance)
dith_limit=sqrt(3*dith_variance); dither=dith_limit*2*(.5-rand(1));
 % Draw.m
function []=Draw(Pict,Dim,Limit,Title,Colour)
for i=1:Dim
fill([(Pict(i,1)-.5),(Pict(i,1)+.5),(Pict(i,1)+.5),(Pict(i,1)-.5)],...
  [(Pict(i,2)-.5),(Pict(i,2)-.5),(Pict(i,2)+.5),(Pict(i,2)+.5)],Colour),hold on;
end;
axis([-Limit Limit -Limit Limit]); grid; title(Title);
 % ImgMtx.m
function [yy,n]=ImgMtx(p,q)
i=1;
for x=(p-1):(p+1)
  for y=(q-1):(q+1)
    yy(i,1)=x; yy(i,2)=y;i=i+1;
  end;
end;
n =size(yy,1);
 % Layout.m
function []=Layout(Object,Limit,m)
for i=-Limit:Limit
  x =i+Limit+1;
  for j=-Limit:Limit
    y =j+Limit+1; [Image,n] =ObjMtx(i,j); xx(x,1) =i; yy(y,1) =j;
    z(x,y) =Compare(Object,Image,m,n);
  end;
end;
mesh(xx,yy,z); title('Object Profile'); xlabel('x'); ylabel('y');
zlabel('Pixels overlapped'); grid;
 % noise.m
function [e]=noise(noise_sig)
e=sqrt(noise_sig)*randn(1);
 % ObjSqu.m
function [yy,n]=ObjSqu(p,q)
i=1;
for x=(p-1):(p+1)
  for y=(q-1):(q+1)
    yy(i,1) =x; yy(i,2) =y; i =i+1;
  end;
end;
n =size(yy,1);
 % RealLin.m
function [Real] =RealLin(x,y,N)
for i=1:N
  Real(i,1) =i; Real(i,2) =x; Real(i,3) =y;
end;
% RandSrch.m
function [y]=RandSrch(limit)
y =limit*2*(.5-rand(1));
```

§ A.2 Network percolation, two dimensions

```
% perco
clear all; St=sum(100*clock); rand('state',St); CNa=200; Dim=2; X=rand(CNa,Dim);
[Va,Ca]=voronoin(X); T=delaunayn(X); TN=size(T,1); VNa=size(Va,1); LB=0.05;
UB=0.95; IXa=zeros(VNa,1);  V=[]; Count=0; VCNa=[];
for i=1:CNa,
  VCNa=[VCNa;size(Ca{i},2)];
end
for i=1:VNa,
  if((Va(i,1)>LB & Va(i,1)<UB) & (Va(i,2)>LB & Va(i,2)<UB))
    V=[V;Va(i,:)]; Count=Count+1; IXa(i,1)=Count;
  end
end
VN=size(V,1); VCN=[]; Count=0; Xa=X; X=[];
Tmp=sparse(1,CNa);
for i=1:CNa,
  Include=1;
  for j=1:VCNa(i,1),
    if(IXa(Ca{i}(1,j),1)==0)
      Include=0;
    end
  end
  if(Include==1)
    Count=Count+1; C{Count,1}=[]; VCN=[VCN;VCNa(i,1)];
    for j=1:VCNa(i,1),
      C{Count,1}(1,j)=IXa(Ca{i}(1,j),1);
    end
    X=[X;Xa(i,:)]; Tmp(1,i)=Count;
  end
```

```
end
CN=size(C,1); T2=[]; T3=[];
for i=1:TN,
  TmpA=[];
  for j=1:3,
    if(Tmp(T(i,j)))
      TmpA=[TmpA,Tmp(T(i,j))];
    end
  end
  TmpB=size(TmpA,2);
  if(TmpB==2)
    T2=[T2;TmpA];
  elseif(TmpB==3)
    T3=[T3;TmpA];
  end
end
% for cells
B=[]; BXX=sparse(CN,CN); NeCMat=sparse(CN,CN); Count=0;
for i=1:size(T2,1),
  Count=Count+1; B=[B;[T2(i,1),T2(i,2)]]; BXX(T2(i,1),T2(i,2))=Count;
  BXX(T2(i,2),T2(i,1))=Count; NeCMat(T2(i,1),T2(i,2))=1; NeCMat(T2(i,2),T2(i,1))=1;
end
for i=1:size(T3,1),
  for j=1:Dim,
    for k=(j+1):(Dim+1),
      if(BXX(T3(i,j),T3(i,k))==0)
        Count=Count+1; B=[B;[T3(i,j),T3(i,k)]]; BXX(T3(i,j),T3(i,k))=Count;
        BXX(T3(i,k),T3(i,j))=Count; NeCMat(T3(i,j),T3(i,k))=1;
        NeCMat(T3(i,k),T3(i,j))=1;
      end
    end
  end
end
BN=Count; A=X; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N); LBc=0.2; UBc=1-LBc;
for i=1:N,
  if(A(i,1)<=LBc)
    LMat(1,i)=1;
  elseif(A(i,1)>=UBc)
    UMat(1,i)=1;
  end
end
NeMat=NeCMat; Blocked=randperm(CN);
% for bonds
NeBMat=sparse(BN,BN);
for i=1:CN,
  [a,b,c]=find(BXX(i,:)); nc=size(c,2);
  for j=1:(nc-1),
    for k=(j+1):nc,
      NeBMat(c(1,j),c(1,k))=1; NeBMat(c(1,k),c(1,j))=1;
    end
  end
end
A=B; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:N,
  if((X(A(i,1),1)<=LBc) | (X(A(i,2),1)<=LBc))
    LMat(1,i)=1;
  elseif((X(A(i,1),1)>=UBc) | (X(A(i,2),1)>=UBc))
    UMat(1,i)=1;
  end
end
NeMat=NeBMat; Blocked=randperm(BN);
%for vertices
Tmp=sparse(1,VN);
for i=1:CN,
  for j=1:VCN(i,1),
    Tmp(1,C{i}(1,j))=1;
  end
end
Vv=[];
Count=0;
for i=1:VN,
  if(Tmp(1,i))
    Count=Count+1; Vv=[Vv;V(i,:)]; Tmp(1,i)=Count;
  end
end
VvN=size(Vv,1);
for i=1:CN,
  for j=1:VCN(i,1),
    Cv{i}(1,j)=Tmp(1,C{i}(1,j));
  end
end
E=[]; EVV=sparse(VN,VN); EVVv=sparse(VvN,VvN); NeVMat=sparse(VvN,VvN);
```

```
Countv=0; Count=0;
for i=1:CN,
  Tmp=[Cv{i}(1,1:VCN(i,1)),Cv{i}(1,1)];
  for j=1:VCN(i,1),
    V1=Tmp(1,j); V2=Tmp(1,(j+1));
    if(NeVMat(V1,V2)==0)
      Countv=Countv+1; NeVMat(V1,V2)=1; NeVMat(V2,V1)=1;
    end
  end
  Tmp=[C{i}(1,1:VCN(i,1)),C{i}(1,1)];
  for j=1:VCN(i,1),
    V1=Tmp(1,j); V2=Tmp(1,(j+1));
    if(EVV(V1,V2)==0)
      Count=Count+1; E=[E;[V1,V2]]; EVV(V1,V2)=Count; EVV(V2,V1)=Count;
    end
  end
end
EN=Count; A=Vv; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
LBv=2*LB; UBv=(UB-LB);
for i=1:N,
  if(A(i,1)<=LBv)
    LMat(1,i)=1;
  elseif(A(i,1)>=UBv)
    UMat(1,i)=1;
  end
end
NeMat=NeVMat; Blocked=randperm(VvN);
%for edges
NeEMat=sparse(EN,EN); MEV=sparse(EN,VN); [a,b,c]=find(EVV); nc=size(c,1);
for i=1:nc,
  MEV(c(i),a(i))=1; MEV(c(i),b(i))=1;
end
for i=1:VN,
  a=find(MEV(:,i));
  if(~isempty(a))
    TmpN=size(a,1); Tmp=[a;a(1,1)]';
    for j=1:TmpN,
      for k=(j+1):TmpN,
        NeEMat(Tmp(1,j),Tmp(1,k))=1; NeEMat(Tmp(1,k),Tmp(1,j))=1;
      end
    end
  end
end
A=E; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:N,
  if((V(A(i,1),1)<=LBv) | (V(A(i,2),1)<=LBv))
    LMat(1,i)=1;
  elseif((V(A(i,1),1)>=UBv) | (V(A(i,2),1)>=UBv))
    UMat(1,i)=1;
  end
end
NeMat=NeEMat; Blocked=randperm(EN);
% percol
clear ClusA ClusB TSeries; NClusA=0; Perco=0;
for i=1:N,
  Joined=0;
  for j=1:NClusA,
    if(ClusA{j,3}(1,Blocked(1,i))~=0)
      ClusA{j,1}=ClusA{j,1}+1; ClusA{j,2}(1,Blocked(1,i))=1;
      ClusA{j,3}=ClusA{j,3} | NeMat(Blocked(1,i),:); Joined=1;
    end
    if(Joined==1)
      for k=1:4,
        ClusB{1,k}=ClusA{j,k};
      end
      NClusB=1;
      if(j==1)
        Tmp=ClusA; clear ClusA;
        for k=1:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{(k+1),l};
          end
        end
      elseif(j==NClusA)
        Tmp=ClusA; clear ClusA;
        for k=1:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{k,l};
          end
        end
      else
        Tmp=ClusA; clear ClusA;
        for k=1:(j-1),
          for l=1:4,
            ClusA{k,l}=Tmp{k,l};
          end
        end
        for k=j:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{(k+1),l};
          end
        end
      end
      for k=1:(NClusA-1),
        if(sum(ClusA{k,2} & ClusB{1,3}) ~= 0)
          ClusB{1,1}=ClusB{1,1}+ClusA{k,1}; ClusB{1,2}=ClusB{1,2} | ClusA{k,2};
          ClusB{1,3}=ClusB{1,3} | ClusA{k,3}; ClusB{1,4}=ClusB{1,4} | ClusA{k,4};
        else
          NClusB=NClusB+1;
          for l=1:4,
            ClusB{NClusB,l}=ClusA{k,l};
          end
        end
      end
      if((sum(full(LMat & ClusB{1,2}))~=0) & (sum(full(UMat & ClusB{1,2}))~=0))
        ClusB{1,4}=1; Perco=1;
      end
      NClusA=NClusB; ClusA=ClusB; clear ClusB; break;
    end
  end
  if(Joined==0)
    NClusA=NClusA+1; ClusA{NClusA,1}=1;
    ClusA{NClusA,2}=sparse(1,Blocked(1,i),1,1,N);
    ClusA{NClusA,3}=NeMat(Blocked(1,i),:); ClusA{NClusA,4}=0;
  end
  TSeries{i,1}=ClusA; TSeries{i,2}=Perco;
end
% Reverse
Tmp=Blocked; Blocked=[];
for i=1:N,
  Blocked=[Blocked,Tmp(1,(N-i+1))];
end
% simulations
Nc=0; TSnap=[];
for i=1:N,
  if(TSeries{i,2})
    Nc=i; break;
  end
end
Pc=Nc/N; Cord=mean(sum(NeMat,2));
```

§ A.3 Network percolation, three dimensions

```
% perco3d.m
clear all; St=sum(100*clock); rand('state',St); CNa=300; Dim=3;
X=rand(CNa,Dim); [Va,Ca]=voronoin(X); T=delaunayn(X); TN=size(T,1);
VNa=size(Va,1); LB=0.05; UB=0.95; IXa=zeros(VNa,1); VCNa=[];
for i=1:CNa,
  VCNa=[VCNa;size(Ca{i},2)];
end
MVCa=[];
for i=1:CNa,
  Tmp=ones(1,VCNa(i,1)); MVCa=[MVCa;sparse(Tmp,Ca{i},Tmp,1,VNa)];
end
Vin=zeros(1,VNa); Count=0;
for i=1:VNa,
  if((max(Va(i,:))<1) & (min(Va(i,:))>0))
    Count=Count+1; Vin(1,i)=1; IXa(i,1)=Count;
  end
end
Tmp=~Vin; Cin=ones(1,CNa);
for i=1:CNa,
  if(sum(Tmp & MVCa(i,:)))
    Cin(1,i)=0;
  end
end
C=[]; count=0; VCN=[];
for i=1:CNa,
  if(Cin(i))
    count=count+1; TmpN=size(Ca{i},2);
    for j=1:TmpN,
      C{count,1}(1,j)=IXa(Ca{i}(1,j));
    end
    VCN(count,1)=TmpN;
  end
end
CN=size(C,1); MidBCx=sparse(CNa,CNa); MidBCy=sparse(CNa,CNa);
MidBCz=sparse(CNa,CNa); BLng=sparse(CNa,CNa);
for i=1:TN,
  Tmp=[T(i,:),T(i,1)];
  for j=1:(Dim+1),
    for k=(j+1):(Dim+1),
      if((Cin(1,Tmp(1,j)) | Cin(1,Tmp(1,k))) & ~BLng(j,k))
        MidBCx(Tmp(1,j),Tmp(1,k))=(X(k,1)+X(j,1))/2;
        MidBCx(Tmp(1,k),Tmp(1,j))=(X(k,1)+X(j,1))/2;
        MidBCy(Tmp(1,j),Tmp(1,k))=(X(k,2)+X(j,2))/2;
        MidBCy(Tmp(1,k),Tmp(1,j))=(X(k,2)+X(j,2))/2;
        MidBCz(Tmp(1,j),Tmp(1,k))=(X(k,3)+X(j,3))/2;
        MidBCz(Tmp(1,k),Tmp(1,j))=(X(k,3)+X(j,3))/2;
        dx=X(k,1)-X(j,1); dy=X(k,2)-X(j,2); dz=X(k,3)-X(j,3);
        TmpA=sqrt(dx*dx + dy*dy + dz*dz);
        BLng(Tmp(1,j),Tmp(1,k))=TmpA; BLng(Tmp(1,k),Tmp(1,j))=TmpA;
      end
    end
  end
end
Fa=[]; Count=0; FaC=[];
for i=1:CNa,
  if(Cin(1,i))
    FaC{i,1}=0; FaC{i,2}=[];
  end
end
for i=1:(CNa-1),
  for j=(i+1):CNa,
    TmpA=0; TmpB=0;
    if(Cin(1,i))
      TmpA=1;
    end
    if(Cin(1,j))
      TmpB=1;
    end
    if(TmpA | TmpB)
      Tmp=MVCa(i,:) & MVCa(j,:);
      if(sum(Tmp))
        [a,b]=find(Tmp); Count=Count+1; Fa{Count,1}=size(b,2); Fa{Count,2}=b;
        Fa{Count,3}=[MidBCx(i,j),MidBCy(i,j),MidBCz(i,j)];
        if(TmpA)
          FaC{i,1}=FaC{i,1} + 1; FaC{i,2}=[FaC{i,2},Count];
          FaC{i,3}{1,1}=i; FaC{i,3}{1,2}=j;
        end
        if(TmpB)
          FaC{j,1}=FaC{j,1} + 1; FaC{j,2}=[FaC{j,2},Count];
          FaC{j,3}{1,1}=i; FaC{j,3}{1,2}=j;
        end
      end
    end
  end
end
FaN=size(Fa,1); V=[];
for i=1:VNa,
  if(Vin(1,i))
    V=[V;[Va(i,:),i]];
  end
end
VN=size(V,1); Tmp=sparse(VNa,1);
for i=1:VN,
  Tmp(V(i,4),1)=i;
end
F=Fa;
for i=1:FaN,
  for j=1:F{i,1},
    F{i,2}(1,j)=Tmp(Fa{i,2}(1,j),1);
  end
end
FN=FaN; FC=[];
count=0;
for i=1:CNa,
  if(Cin(i))
    count=count+1; FC{count,1}=FaC{i,1};
    FC{count,2}=FaC{i,2}; FC{count,3}=FaC{i,3};
  end
end
NghV=sparse(VN,VN); Tmp=F; TmpN=FN;
for i=1:TmpN,
  TmpA=Tmp{i,2}; x=[]; y=[]; z=[]; TmpB=Tmp{i,1};
  if(TmpB==3)
    for j=1:2,
      for k=(j+1):3,
        NghV(TmpA(1,j),TmpA(1,k))=1; NghV(TmpA(1,k),TmpA(1,j))=1;
      end
    end
  else
    for j=1:TmpB,
      x=[x;V(TmpA(1,j),1)]; y=[y;V(TmpA(1,j),2)]; z=[z;V(TmpA(1,j),3)];
    end
    a=y(1)*(z(2)-z(3))+y(2)*(z(3)-z(1))+y(3)*(z(1)-z(2));
    b=z(1)*(x(2)-x(3))+z(2)*(x(3)-x(1))+z(3)*(x(1)-x(2));
    c=x(1)*(y(2)-y(3))+x(2)*(y(3)-y(1))+x(3)*(y(1)-y(2));
    K=1/sqrt(a*a + b*b + c*c); Th{1}=K*a; Th{2}=K*b; Th{3}=K*c; Max=0;
    for j=1:3,
      if(Th{j}<Max)
        Max=Th{j}; jMax=j;
      end
    end
    if(jMax==1)
      p=y; q=z;
    elseif(jMax==2)
      p=x; q=z;
    else
      p=x; q=y;
    end
    t=delaunay(p,q);
    for j=1:size(t,1),
      for k=1:3,
        t(j,k)=TmpA(1,t(j,k));
      end
    end
    Nt=size(t,1); TmpC=sparse(VN,VN);
    for j=1:Nt,
      TmpT=[t(j,:),t(j,1)];
      for k=1:3,
        TmpD=sort([TmpT(1,k),TmpT(1,(k+1))]);
        TmpC(TmpD(1,1),TmpD(1,2))=TmpC(TmpD(1,1),TmpD(1,2))+1;
      end
    end
    [k,l,m]=find(TmpC);
    for j=1:size(k,1),
      if(m(j)==1)
        NghV(k(j),l(j))=1; NghV(l(j),k(j))=1;
      end
    end
  end
end
Fed=[];
for i=1:FN,
  for j=1:2,
    Fed{i,j}=F{i,j};
  end
end
for i=1:FN,
  Count=0; TmpN=Fed{i,1};
  if(TmpN>3)
    Tmp=Fed{i,2}; TmpA=Tmp(1,1); Tmp=Tmp(1,2:TmpN); TmpN=TmpN-1; Count=Count+1;
    while(TmpN)
      a=TmpA(1,Count); TmpB=[]; Found=0;
      for j=1:TmpN,
        TmpC=Tmp(1,j);
        if(NghV(a,TmpC) & ~Found)
          TmpA=[TmpA,TmpC]; Found=1;
        else
          TmpB=[TmpB,TmpC];
        end
      end
      Tmp=TmpB; TmpN=TmpN-1; Count=Count+1;
    end
    Fed{i,2}=TmpA;
  end
end
LB2=2*LB; UB2=(UB-LB);
% cells II
Tmp=ones(1,CNa);
for i=1:CNa,
  if((max(X(i,:))>UB) | (min(X(i,:))<LB))
    Tmp(1,i)=0;
  end
end
```

```
a=find(Tmp); TmpB=sparse(1,CNa); x=[];
for i=1:size(a,2),
  TmpB(1,a(1,i))=i; x(i,:)=X(a(i),:);
end
xn=size(x,1); TmpA=zeros(size(T)); TmpN=size(T,1);
for i=1:TmpN,
  for j=1:4,
    if((Tmp(1,T(i,j))))
      TmpA(i,j)=TmpB(1,T(i,j));
    end
  end
end
nghc=sparse(xn,xn);
for i=1:TmpN,
  [a,b,c]=find(TmpA(i,:)); TmpB=size(c,2);
  if(TmpB>1)
    for j=1:(TmpB-1),
      for k=(j+1):TmpB,
        nghc(c(1,j),c(1,k))=1; nghc(c(1,k),c(1,j))=1;
      end
    end
  end
end
A=x; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:N,
  if(A(i,1)<=LB2)
    LMat(1,i)=1;
  elseif(A(i,1)>=UB2)
    UMat(1,i)=1;
  end
end
NeMat=nghc; Blocked=randperm(xn);
% bonds II
[a,b,c]=find(triu(nghc)); b=[a,b]; bn=size(b,1); Tmp=sparse(bn,xn);
for i=1:bn,
  Tmp(i,b(i,1))=1; Tmp(i,b(i,2))=1;
end
nghb=sparse(bn,bn);
for i=1:xn,
  a=find(Tmp(:,i));
  if(~isempty(a))
    TmpN=size(a,1);
    for j=1:(TmpN-1),
      for k=(j+1):TmpN,
        nghb(a(j,1),a(k,1))=1; nghb(a(k,1),a(j,1))=1;
      end
    end
  end
end
A=b; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:N,
  if((x(A(i,1),1)<=LB2) | (x(A(i,2),1)<=LB2))
    LMat(1,i)=1;
  elseif((x(A(i,1),1)>=UB2) | (x(A(i,2),1)>=UB2))
    UMat(1,i)=1;
  end
end
NeMat=nghb; Blocked=randperm(N);
% vertices II
[a,b,c]=find((NghV)); Tmp=sparse(1,VN);
for i=1:size(a,1),
  Tmp(1,b(i,1))=1;
end
d=find(Tmp); TmpN=size(d,2);
for i=1:TmpN,
  Tmp(1,d(1,i))=i;
end
for i=1:size(a,1),
  a(i,1)=Tmp(1,a(i,1)); b(i,1)=Tmp(1,b(i,1));
end
nghv=sparse(a,b,c,TmpN,TmpN); TmpA=[];
for i=1:TmpN,
  TmpA(Tmp(1,d(i)),:)=V(i,1:3);
end
A=TmpA; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:N,
  if(A(i,1)<=LB2)
    LMat(1,i)=1;
  elseif(A(i,1)>=UB2)
    UMat(1,i)=1;
  end
end
```

```
NeMat=nghv; Blocked=randperm(N);
% edges II
[a,b,c]=find(triu(NghV)); E=[a,b]; EN=size(E,1); Tmp=sparse(EN,VN);
for i=1:EN,
  Tmp(i,E(i,1))=1; Tmp(i,E(i,2))=1;
end
NghE=sparse(EN,EN);
for i=1:VN,
  a=find(Tmp(:,i));
  if(~isempty(a))
    TmpN=size(a,1);
    for j=1:(TmpN-1),
      for k=(j+1):TmpN,
        NghE(a(j,1),a(k,1))=1; NghE(a(k,1),a(j,1))=1;
      end
    end
  end
end
A=E; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:N,
  if((V(A(i,1),1)<=LB2) | (V(A(i,2),1)<=LB2))
    LMat(1,i)=1;
  elseif((V(A(i,1),1)>=UB2) | (V(A(i,2),1)>=UB2))
    UMat(1,i)=1;
  end
end
NeMat=NghE; Blocked=randperm(N);
% percolation
clear ClusA ClusB TSeries; NClusA=0; Perco=0;
for i=1:N,
  Joined=0;
  for j=1:NClusA,
    if(ClusA{j,3}(1,Blocked(1,i))~=0)
      ClusA{j,1}=ClusA{j,1}+1; ClusA{j,2}(1,Blocked(1,i))=1;
      ClusA{j,3}=ClusA{j,3} | NeMat(Blocked(1,i),:); Joined=1;
    end
    if(Joined==1)
      for k=1:4,
        ClusB{1,k}=ClusA{j,k};
      end
      NClusB=1;
      if(j==1)
        Tmp=ClusA; clear ClusA;
        for k=1:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{(k+1),l};
          end
        end
      elseif(j==NClusA)
        Tmp=ClusA; clear ClusA;
        for k=1:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{k,l};
          end
        end
      else
        Tmp=ClusA; clear ClusA;
        for k=1:(j-1),
          for l=1:4,
            ClusA{k,l}=Tmp{k,l};
          end
        end
        for k=j:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{(k+1),l};
          end
        end
      end
      for k=1:(NClusA-1),
        if(sum(ClusA{k,2} & ClusB{1,3}) ~= 0)
          ClusB{1,1}=ClusB{1,1}+ClusA{k,1}; ClusB{1,2}=ClusB{1,2} | ClusA{k,2};
          ClusB{1,3}=ClusB{1,3} | ClusA{k,3}; ClusB{1,4}=ClusB{1,4} | ClusA{k,4};
        else
          NClusB=NClusB+1;
          for l=1:4,
            ClusB{NClusB,l}=ClusA{k,l};
          end
        end
      end
      if((sum(full(LMat & ClusB{1,2}))~=0) & (sum(full(UMat & ClusB{1,2}))~=0))
        ClusB{1,4}=1; Perco=1;
      end
      NClusA=NClusB; ClusA=ClusB; clear ClusB; break;
    end
  end
  if(Joined==0)
    NClusA=NClusA+1; ClusA{NClusA,1}=1;
    ClusA{NClusA,2}=sparse(1,Blocked(1,i),1,1,N);
    ClusA{NClusA,3}=NeMat(Blocked(1,i),:); ClusA{NClusA,4}=0;
  end
  TSeries{i,1}=ClusA; TSeries{i,2}=Perco;
end
% Reverse
Tmp=Blocked; Blocked=[];
for i=1:N,
  Blocked=[Blocked,Tmp(1,(N-i+1))];
end
% simulations
Nc=0;
for i=1:N,
  if(TSeries{i,2})
    Nc=i; break;
  end
end
Pc=Nc/N; Cord=mean(sum(NeMat,2));
```

§ A.4 Network percolation, 2–d section

```
% section
MVC=[];
for i=1:CN,
  Tmp=ones(1,VCN(i,1)); MVC=[MVC;sparse(Tmp,C{i},Tmp,1,VN)];
end
CE=[];
for i=1:EN,
  Tmp=MVC(:,E(i,1)) & MVC(:,E(i,2));
  if(sum(Tmp))
    TmpA=find(Tmp)'; TmpN=size(TmpA,2); CE{i,1}=TmpN; CE{i,2}=TmpA;
  end
end
ie=[]; je=[]; ke=[];
for i=1:EN,
  ie(i,1)=V(E(i,2),1)-V(E(i,1),1); je(i,1)=V(E(i,2),2)-V(E(i,1),2);
  ke(i,1)=V(E(i,2),3)-V(E(i,1),3);
end
a=1; b=.01; cc=0; d=-.5; v=[]; vC=[]; count=0;
for i=1:EN,
  Tmp=(a*ie(i)+b*je(i)+cc*ke(i));
  if(Tmp)
    v1=E(i,1); x1=V(v1,1); y1=V(v1,2); z1=V(v1,3); TmpA=(a*x1+b*y1+cc*z1+d);
    v2=E(i,2); x2=V(v2,1); y2=V(v2,2); z2=V(v2,3); t=-TmpA/Tmp;
    if((t>=0) & (t<=1))
      x=x1+(x2-x1)*t; y=y1+(y2-y1)*t; z=z1+(z2-z1)*t; count=count+1;
      v(count,:)=[x,y,z]; vC{count,1}=CE{i,1}; vC{count,2}=CE{i,2};
    end
  else
    if(~TmpA)  % both nom and denom = 0
      count=count+1; v(count,:)=[x1,y1,z1]; vC{count,1}=CE{i,1};
      vC{count,2}=CE{i,2}; count=count+1; v(count,:)=[x2,y2,z2];
      vC{count,1}=CE{i,1}; vC{count,2}=CE{i,2};
    end
  end
end
vn=count; cC=sparse(CN,1); count=0;
for i=1:vn,
  for j=1:vC{i,1},
    if(~cC(vC{i,2}(j)))
      count=count+1; cC(vC{i,2}(j),1)=count;
    end
  end
end
cn=count; vc=vC;
for i=1:vn,
  for j=1:vc{i,1},
    vc{i,2}(1,j)=cC(vC{i,2}(j));
  end
end
c=[];
for i=1:cn,
  c{i,1}=0; c{i,2}=[];
end
for i=1:vn,
  for j=1:vc{i,1},
    c{vc{i,2}(j),1}=c{vc{i,2}(j),1}+1; c{vc{i,2}(j),2}=[c{vc{i,2}(j),2},i];
  end
end
for i=1:cn,
  Tmp=[];
  for j=1:c{i,1},
    Tmp=[Tmp;v(c{i,2}(j),:),c{i,2}(j)];
  end
  TmpA=min(Tmp,[],1); TmpB=max(Tmp,[],1); [TmpC,TmpD]=min(TmpB-TmpA);
  if(TmpD==1)
    TmpA=Tmp(:,2); TmpB=Tmp(:,3);
  elseif(TmpD==2)
    TmpA=Tmp(:,1); TmpB=Tmp(:,3);
  else
    TmpA=Tmp(:,1); TmpB=Tmp(:,2);
  end
  TmpC=delaunay(TmpA,TmpB); TmpN=size(TmpC,1);
  for j=1:TmpN,
    for k=1:3,
      TmpC(j,k)=Tmp(TmpC(j,k),4);
    end
  end
  TmpA=sparse(vn,vn);
  for j=1:TmpN,
    for k=1:2,
      for m=(k+1):3,
        TmpA(TmpC(j,k),TmpC(j,m))=TmpA(TmpC(j,k),TmpC(j,m))+1;
        TmpA(TmpC(j,m),TmpC(j,k))=TmpA(TmpC(j,m),TmpC(j,k))+1;
      end
    end
  end
  [x,y,z]=find(TmpA); TmpB=[]; TmpC=[];
  for j=1:size(x,1),
    if(z(j)==1)
      TmpB=[TmpB;x(j),y(j)]; TmpC(y(j),1)=1;
    end
  end
  TmpA=[];
  for j=1:size(TmpC,1),
    TmpA{j,1}=[];
  end
  for j=1:size(TmpB,1),
    TmpA{TmpB(j,1),1}=[TmpA{TmpB(j,1),1},TmpB(j,2)];
  end
  Tmp=Tmp(1,4); TmpB=Tmp;
  TmpC=sparse(Tmp,1,1,vn,1); count=c{i,1}-1;
  while(count>0),
    if(~(TmpC(TmpA{Tmp}(1),1)))
      Tmp=TmpA{Tmp}(1); TmpB=[TmpB,Tmp]; TmpC(Tmp,1)=1;
    else
      Tmp=TmpA{Tmp}(2); TmpB=[TmpB,Tmp]; TmpC(Tmp,1)=1;
    end
    count=count-1;
  end
  c{i,3}=TmpB;
end
for i=1:cn,
  Tmp=[0,0,0];
  for j=1:c{i,1},
    Tmp=Tmp+v(c{i,2}(j),:);
  end
  Tmp=Tmp/c{i,1}; c{i,4}=Tmp;
end
Tmp=sqrt(a*a+b*b+cc*cc); u=[a/Tmp,b/Tmp,cc/Tmp]; uzp=u; ux=[1,0,0];
Tmp=cross(u,ux); TmpA=sqrt(Tmp(1)*Tmp(1)+Tmp(2)*Tmp(2)+Tmp(3)*Tmp(3));
uyp=Tmp/TmpA; uxp=cross(uyp,uzp); R=[uxp,0;uyp,0;uzp,0;0,0,0,1];
vp=(R*[v';ones(1,vn)])'; vp=vp(:,1:2); ad=min(vp,[],1);
vp=vp-[ad(1)*ones(vn,1),ad(2)*ones(vn,1)];; cs=[];
for i=1:cn,
  Tmp=R*[c{i,4}';1]; Tmp=Tmp(1:2)'-ad; c{i,5}=Tmp; cs=[cs;Tmp];
end
LB=min(vp(:,1)); UB=max(vp(:,1)); Tmp=UB-LB; LBv=LB+0.1*Tmp;
UBv=UB-LBv; Tmp=min(cs(:,1)); LB=min(cs(:,1)); UB=max(cs(:,1));
Tmp=UB-LB; LBc=LB+0.1*Tmp; UBc=UB-LBc;
% cell
cvm=sparse(cn,vn);
for i=1:cn,
  for j=1:c{i,1},
    cvm(i,c{i,2}(j))=1;
  end
end
```

```
nghc=sparse(cn,cn);
for i=1:(cn-1),
  for j=(i+1):cn,
    Tmp=find(cvm(i,:) & cvm(j,:));
    if(~isempty(Tmp))
      TmpN=size(Tmp,2);
      if(TmpN>1)
        for k=1:TmpN,
          nghc(i,j)=1; nghc(j,i)=1;
        end
      end
    end
  end
end
N=cn; LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:cn,
  if(cs(i,1)<=LBc)
    LMat(1,i)=1;
  end
  if(cs(i,1)>=UBc)
    UMat(1,i)=1;
  end
end
NeMat=nghc; Blocked=randperm(N);
% bond
[p,q,r]=find(triu(nghc)); b=[p,q]; bn=size(b,1); bcm=sparse(bn,bn);
for i=1:bn,
  bcm(i,b(i,1))=1; bcm(i,b(i,2))=1;
end
nghb=sparse(bn,bn);
for i=1:cn,
  Tmp=find(bcm(:,i));
  if(~isempty(Tmp))
    TmpN=size(Tmp,1);
    for j=1:(TmpN-1),
      for k=(j+1):TmpN,
        nghb(Tmp(j),Tmp(k))=1; nghb(Tmp(k),Tmp(j))=1;
      end
    end
  end
end
N=bn; LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:bn,
  if((cs(b(i,1),1)<=LBc) | (cs(b(i,2),1)<=LBc))
    LMat(1,i)=1;
  end
  if((cs(b(i,1),1)>=UBc) | (cs(b(i,2),1)>=UBc))
    UMat(1,i)=1;
  end
end
NeMat=nghb; Blocked=randperm(N);
% vertice
nghv=sparse(vn,vn);
for i=1:cn,
  Tmp=[c{i,3},c{i,3}(1)];
  for j=1:c{i,1},
    nghv(Tmp(j),Tmp(j+1))=1; nghv(Tmp(j+1),Tmp(j))=1;
  end
end
LMat=sparse(1,vn); UMat=sparse(1,vn);
for i=1:vn,
  if(vp(i,1)<=LBv)
    LMat(1,i)=1;
  end
  if(vp(i,1)>=UBv)
    UMat(1,i)=1;
  end
end
N=vn; NeMat=nghv; Blocked=randperm(N);
% edge
[p,q,r]=find(triu(nghv)); e=[p,q]; en=size(e,1); evm=sparse(en,en);
for i=1:en,
  evm(i,e(i,1))=1; evm(i,e(i,2))=1;
end
nghe=sparse(en,en);
for i=1:vn,
  Tmp=find(evm(:,i));
  if(~isempty(Tmp))
    TmpN=size(Tmp,1);
    for j=1:(TmpN-1),
      for k=(j+1):TmpN,
        nghe(Tmp(j),Tmp(k))=1; nghe(Tmp(k),Tmp(j))=1;
      end
    end
  end
end
N=en; LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:en,
  if((vp(e(i,1),1)<=LBc) | (vp(e(i,2),1)<=LBc))
    LMat(1,i)=1;
  end
  if((vp(e(i,1),1)>=UBc) | (vp(e(i,2),1)>=UBc))
    UMat(1,i)=1;
  end
end
NeMat=nghe; Blocked=randperm(N);
```

§ A.5 Continuum percolation of n-gons

```
% regpoly.m
clear all; St=sum(100*clock); rand('state',St); Size=10; N=40; n=11; Angle=2*pi/n;
X=Size*rand(N,2); TwoPi=2*pi; Rad=TwoPi*rand(N,1);
R=sqrt(1/(n*sin(Angle/2)*cos(Angle/2)));
for i=1:N,
  for j=0:(n-1),
    Tmp=Rad(i,1)+j*Angle; V{i}{1,(j+1)}=[(X(i,1)+R*cos(Tmp)),(X(i,2)+R*sin(Tmp))];
  end
end
Tmp=V{1}{1,1}+(V{1}{1,2}-V{1}{1,1})/2; dx=Tmp(1,1)-X(1,1); dy=Tmp(1,2)-X(1,2);
r=sqrt(dx*dx + dy*dy); clf; hold on;
for i=1:N,
  x=[]; y=[];
  for j=1:n,
    x=[x;V{i}{1,j}(1,1)]; y=[y;V{i}{1,j}(1,2)];
  end
  x=[x;V{i}{1,1}(1,1)]; y=[y;V{i}{1,1}(1,2)]; plot(x,y);
end
plot([0,Size,Size,0,0],[0,0,Size,Size,0]); axis equal; axis off;
for i=1:N,
  Tmp=[];
  for j=1:n,
    Tmp=[Tmp;[j,V{i}{1,j}(1,2)]]; Tmp=sortrows(Tmp,2);
  end
end
T=delaunay(X(:,1),X(:,2)); NT=size(T,1); D=sparse(N,N); Ov=sparse(N,N);
Ov1=sparse(N,1); Pair=[]; Limbo=[]; Oclock=sparse(NT,NT);
for i=1:NT,
  Tmp=[T(i,:),T(i,1)];
  for j=1:3,
    c1=Tmp(1,j); c2=Tmp(1,(j+1)); dx=X(c2,1)-X(c1,1); dy=X(c2,2)-X(c1,2);
    TmpA=sqrt(dx*dx + dy*dy); D(c1,c2)=TmpA; D(c2,c1)=TmpA;
    Pair=[Pair;[c1,c2;c2,c1]];
    if(TmpA<=(2*r))
      Ov(c1,c2)=1; Ov(c2,c1)=1; Ov1(c1,1)=1; Ov1(c2,1)=1;
    elseif(TmpA<=(2*R))
      Limbo=[Limbo;[c1,c2;c2,c1]];
    end
  end
  TmpB=atan(abs(dy/dx));
  if(dx>=0)
    if(dy>=0)  % Quadrant 1
      Oclock(c1,c2)=TmpB; Oclock(c2,c1)=pi+TmpB;
    else  % Quadrant 4
      Oclock(c1,c2)=TwoPi-TmpB; Oclock(c2,c1)=pi-TmpB;
    end
  else
    if(dy>=0)  % Quadrant 2
      Oclock(c1,c2)=pi-TmpB; Oclock(c2,c1)=TwoPi-TmpB;
    else  % Quadrant 3
      Oclock(c1,c2)=pi+TmpB; Oclock(c2,c1)=TmpB;
    end
  end
end
Tmp=Angle/2; Star=[];
for i=1:N,
  TmpA=[]; TmpB=Rad(i,1);
  for j=1:n,
    TmpA=[TmpA,mod((TmpB + (j-1)*Angle + Tmp),TwoPi)];
  end
  Star=[Star;TmpA];
end
Wobble=sparse(N,N); jWobble=sparse(N,N); TmpN=size(Limbo,1);
```

```
for i=1:TmpN,
  Min=10; TmpA=Limbo(i,1); TmpB=Limbo(i,2); jMin=j;
  for j=1:n,
    Tmp=Star(TmpA,j)-Oclock(TmpA,TmpB);
    if(abs(Tmp)<abs(Min))
      Min=Tmp; jMin=j;
    end
  end
  Wobble(TmpA,TmpB)=Min; jWobble(TmpA,TmpB)=jMin;
end
Tmp=Angle/2;
for i=1:2:TmpN,
  TmpA=Limbo(i,1); TmpB=Limbo(i,2);
  if(abs(Wobble(TmpB,TmpA)) >= abs(Wobble(TmpA,TmpB)))
    TmpA=Limbo((i+1),1); TmpB=Limbo((i+1),2);
  end
  J=jWobble(TmpA,TmpB); v{1}=V{TmpA}{1,J};
  if(J==n)
    v{2}=V{TmpA}{1,1};
  else
    v{2}=V{TmpA}{1,(J+1)};
  end
  J=jWobble(TmpB,TmpA); v{3}=V{TmpB}{1,J};
  if(J==n)
    v{4}=V{TmpB}{1,1};
  else
    v{4}=V{TmpB}{1,(J+1)};
  end
  Max=0;
  if(Wobble(TmpA,TmpB)>=0)
    vMin=v{1};
  else
    vMin=v{2};
  end
  d1=R*cos(Tmp-abs(Wobble(TmpA,TmpB))); TmpD=(X(TmpA,1)-X(TmpB,1));
  a=(X(TmpA,2)-X(TmpB,2))/TmpD; b=(X(TmpA,1)*X(TmpB,2)-X(TmpB,1)*X(TmpA,2))/TmpD;
  a1=a; b1=vMin(1,2)-a1*vMin(1,1); x3=v{3}(1,1); y3=v{3}(1,2); x4=v{4}(1,1);
  y4=v{4}(1,2); TmpD=x3-x4; p=(y3-y4)/TmpD; q=(x3*y4-x4*y3)/TmpD; TmpD=a1-p;
  x=(q-b1)/TmpD; y=(a1*q-b1*p)/TmpD; dx=x-X(TmpB,1); dy=y-X(TmpB,2);
  r2=sqrt(dx*dx + dy*dy); d2=r2*cos(Tmp-abs(Wobble(TmpB,TmpA))); d=D(TmpA,TmpB);
  if((d1+d2)>=d)
    Ov(TmpA,TmpB)=1; Ov(TmpB,TmpA)=1; Ov1(TmpA,1)=1; Ov1(TmpB,1)=1;
  end
end
Clus=Ov;
for i=1:N,
  Clus(i,i)=1;
end
NClus=size(Clus,1); ClusA=Clus(1,:); NClusA=1;
for i=2:NClus,
  Joined=0;
  for j=1:NClusA,
    TmpC=Clus(i,:) | ClusA(j,:);
    if(sum(Clus(i,:) & ClusA(j,:)))
      ClusA(j,:)=TmpC; ClusB=ClusA; ClusA=TmpC; NClusB=NClusA;
      NClusA=1; Joined=1; break;
    end
  end
  if(~Joined)
    ClusA=[ClusA;Clus(i,:)]; NClusA=NClusA+1;
  else
    for j=1:NClusB,
      if(sum(ClusA(1,:) & ClusB(j,:)))
        ClusA(1,:)=ClusA(1,:) | ClusB(j,:);
      else
        ClusA=[ClusA;ClusB(j,:)]; NClusA=NClusA+1;
      end
    end
  end
end
Left=sparse(1,N); Right=sparse(1,N); Margin=0.1*Size;
for i=1:N,
  if(X(i,1)<=Margin)
    Left(1,i)=1;
  elseif(X(i,1)>=(Size-Margin))
    Right(1,i)=1;
  end
end
Plated=0;
for i=1:NClusA,
  if(sum(Left & ClusA(i,:)) & sum(Right & ClusA(i,:)))
    Plated=1; break;
  end
end
```

§ A.6 Tilings

```
% tiling.m;  NB. must run one of the data below first.
qn=size(q,1); in2n=size(in2,1); in3n=size(in3,1); p=[];
p{1,1}=[o;o]'; p{1,2}=sz; r=[]; s=[];
for i=1:sz,
  r(1,i)=dx(m(i)); s(1,i)=dy(n(i));
end
p{1,3}=r'; p{1,4}=s'; iin=size(ii,1); Tmp=ones(iin,1);
TmpA=sparse(Tmp,ii(:,1),Tmp,1,sz); Tmp=[]; TmpB=[]; TmpC=[]; count=0;
for i=1:sz,
  if(~TmpA(i))
    count=count+1; Tmp=[Tmp;i,count]; TmpB=[TmpB;r(i)]; TmpC=[TmpC;s(i)];
  end
end
p{2,1}=Tmp; p{2,2}=count; p{2,3}=TmpB; p{2,4}=TmpC; iiin=size(iii,1);
Tmp=ones(iiin,1); TmpA=sparse(Tmp,iii(:,1),Tmp,1,sz);
Tmp=[]; TmpB=[]; TmpC=[]; count=0;
for i=1:sz,
  if(~TmpA(i))
    count=count+1; Tmp=[Tmp;i,count]; TmpB=[TmpB;r(i)]; TmpC=[TmpC;s(i)];
  end
end
p{3,1}=Tmp; p{3,2}=count; p{3,3}=TmpB; p{3,4}=TmpC; iv=[ii,2*ones(iin,1)];
for i=1:iiin,
  Tmp=0;
  for j=1:iin,
    if(iii(i,1)==ii(j,1))
      Tmp=1;
    end
  end
  if(~Tmp)
    iv=[iv;iii(i,:),3];
  end
end
ivn=size(iv,1); Tmp=ones(size(ivn,1),1);
TmpA=sparse(Tmp,iv(:,1),Tmp,1,sz); Tmp=[]; TmpB=[]; TmpC=[]; count=0;
for i=1:sz,
  if(~TmpA(i))
    count=count+1; Tmp=[Tmp;i,count]; TmpB=[TmpB;r(i)]; TmpC=[TmpC;s(i)];
  end
end
p{4,1}=Tmp; p{4,2}=count; p{4,3}=TmpB; p{4,4}=TmpC; map=[]; Tmp=ones(sz,1);
map{1,1}=sparse(p{1,1}(:,1),Tmp,p{1,1}(:,2),sz,1); count=p{1,2}; e=q;
v=[p{1,3},p{1,4}]; Tmp=ones(p{2,2},1);
for i=2:nx,
  map{i,1}=sparse(p{2,1}(:,1),Tmp,count*Tmp+p{2,1}(:,2),sz,1);
  for k=1:iin,
    map{i,1}(ii(k,1),1)=map{(i-1),1}(ii(k,2),1);
  end
  count=count+p{2,2}; % +iin;
  for k=1:qn,
    e=[e;map{i,1}(q(k,1)),map{i,1}(q(k,2))];
  end
  for k=1:in2n,
    e=[e;map{i,1}(in2(k,1)),map{(i-1),1}(in2(k,2))];
  end
  v=[v;(i-1)*dim1*Tmp+p{2,3},p{2,4}];
end
Tmp=ones(p{3,2},1);
for j=2:ny,
  map{1,j}=sparse(p{3,1}(:,1),Tmp,count*Tmp+p{3,1}(:,2),sz,1);
  for k=1:iiin,
    map{1,j}(iii(k,1),1)=map{1,(j-1)}(iii(k,2),1);
  end
  count=count+p{3,2}; % +iiin;
  for k=1:qn,
    e=[e;map{1,j}(q(k,1)),map{1,j}(q(k,2))];
  end
  for k=1:in3n,
    e=[e;map{1,j}(in3(k,1)),map{1,(j-1)}(in3(k,2))];
  end
  v=[v;p{3,3},(j-1)*dim2*Tmp+p{3,4}];
end
Tmp=ones(p{4,2},1);
for i=2:nx,
  for j=2:ny,
    map{i,j}=sparse(p{4,1}(:,1),Tmp,count*Tmp+p{4,1}(:,2),sz,1);
    for k=1:ivn,
      if(iv(k,3)==2)
        map{i,j}(iv(k,1),1)=map{(i-1),j}(iv(k,2),1);
      else
        map{i,j}(iv(k,1),1)=map{i,(j-1)}(iv(k,2),1);
      end
    end
    count=count+p{4,2}; % +ivn;
    for k=1:qn,
      e=[e;map{i,j}(q(k,1)),map{i,j}(q(k,2))];
    end
    for k=1:in2n,
      e=[e;map{i,j}(in2(k,1)),map{(i-1),j}(in2(k,2))];
    end
    for k=1:in3n,
      e=[e;map{i,j}(in3(k,1)),map{i,(j-1)}(in3(k,2))];
    end
    v=[v;(i-1)*dim1*Tmp+p{4,3},(j-1)*dim2*Tmp+p{4,4}];
  end
end
en=size(e,1); vn=size(v,1); cn=size(c,1);
for i=1:cn,
  c{i,2}=size(c{i},2);
end
C=[]; count=0;
for i=1:nx,
  for j=1:ny,
    for k=1:cn,
      count=count+1;
      C{count,1}=[];
      for m=1:c{k,2},
        C{count,1}=[C{count},map{i,j}(c{k,1}(m))];
      end
    end
  end
end
ciin=size(cii,1);
for i=1:ciin,
  cii{i,3}=size(cii{i,1},2); cii{i,4}=size(cii{i,2},2);
end
ciiin=size(ciii,1);
for i=1:ciiin,
  ciii{i,3}=size(ciii{i,1},2); ciii{i,4}=size(ciii{i,2},2);
end
for i=2:nx,
  for j=1:ciin,
    count=count+1; C{count,1}=[];
    for k=1:cii{j,3},
      C{count,1}=[C{count},map{i,1}(cii{j,1}(k))];
    end
    for k=1:cii{j,4},
      C{count,1}=[C{count},map{(i-1),1}(cii{j,2}(k))];
    end
  end
end
for i=2:ny,
  for j=1:ciiin,
    count=count+1;
    C{count,1}=[];
    for k=1:ciii{j,3},
      C{count,1}=[C{count},map{1,i}(ciii{j,1}(k))];
    end
    for k=1:ciii{j,4},
      C{count,1}=[C{count},map{1,(i-1)}(ciii{j,2}(k))];
    end
  end
end
civn=size(civ,1);
for i=1:civn,
  civ{i,5}=size(civ{i,1},2); civ{i,6}=size(civ{i,2},2);
  civ{i,7}=size(civ{i,3},2); civ{i,8}=size(civ{i,4},2);
end
for i=2:nx,
  for j=2:ny,
    for m=1:ciin,
      count=count+1; C{count,1}=[];
      for k=1:cii{m,3},
        C{count,1}=[C{count},map{i,j}(cii{m,1}(k))];
      end
      for k=1:cii{m,4},
        C{count,1}=[C{count},map{(i-1),j}(cii{m,2}(k))];
      end
    end
    for m=1:ciiin,
      count=count+1;
      C{count,1}=[];
      for k=1:ciii{m,3},
        C{count,1}=[C{count},map{i,j}(ciii{m,1}(k))];
      end
      for k=1:ciii{m,4},
        C{count,1}=[C{count},map{i,(j-1)}(ciii{m,2}(k))];
      end
    end
    for m=1:civn,
      count=count+1;
      C{count,1}=[];
      for k=1:civ{m,5},
        C{count,1}=[C{count},map{i,j}(civ{m,1}(k))];
      end
      for k=1:civ{m,6},
        C{count,1}=[C{count},map{(i-1),j}(civ{m,2}(k))];
      end
      for k=1:civ{m,7},
        C{count,1}=[C{count},map{(i-1),(j-1)}(civ{m,3}(k))];
      end
      for k=1:civ{m,8},
        C{count,1}=[C{count},map{i,(j-1)}(civ{m,4}(k))];
      end
    end
  end
end
Cn=size(C,1);
for i=1:Cn,
  C{i,2}=size(C{i,1},2);
end
x=[];
for i=1:Cn,
  Tmp=[0,0];
  for j=1:C{i,2},
    Tmp=Tmp+v(C{i,1}(j),:);
  end
  x=[x;Tmp/C{i,2}];
end
xn=size(x,1);
% for Cells
B=[]; Bxx=sparse(Cn,Cn); NeCMat=sparse(Cn,Cn); CVMat=sparse(Cn,vn);
for i=1:Cn,
  for j=1:C{i,2},
    CVMat(i,C{i,1}(j))=1;
  end
end
count=0;
for i=1:vn,
  Tmp=find(CVMat(:,i)); TmpN=size(Tmp,1);
  for j=1:(TmpN-1),
    for k=(j+1):TmpN,
      if(~NeCMat(Tmp(j),Tmp(k)))
        count=count+1; NeCMat(Tmp(j),Tmp(k))=1; NeCMat(Tmp(k),Tmp(j))=1;
        B=[B;Tmp(j),Tmp(k)]; Bxx(Tmp(j),Tmp(k))=count; Bxx(Tmp(k),Tmp(j))=count;
      end
    end
  end
end
Bn=count; A=x; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
LB=min(x(:,1)); UB=max(x(:,1)); rng=UB-LB;
% LBc=.05*rng+LB;
LBc=.1*rng+LB; UBc=UB-LBc;
for i=1:N,
  if(A(i,1)<=LBc)
    LMat(1,i)=1;
  elseif(A(i,1)>=UBc)
    UMat(1,i)=1;
  end
end
NeMat=NeCMat; Blocked=randperm(Cn);
% for Bonds
NeBMat=sparse(Bn,Bn);
for i=1:Cn,
  [p,q,r]=find(Bxx(i,:)); nc=size(r,2);
  for j=1:(nc-1),
    for k=(j+1):nc,
      NeBMat(r(1,j),r(1,k))=1; NeBMat(r(1,k),r(1,j))=1;
    end
  end
end
A=B; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:N,
  if((x(A(i,1),1)<=LBc) | (x(A(i,2),1)<=LBc))
    LMat(1,i)=1;
  elseif((x(A(i,1),1)>=UBc) | (x(A(i,2),1)>=UBc))
    UMat(1,i)=1;
  end
end
NeMat=NeBMat; Blocked=randperm(Bn);
% for cells
b=[]; bxx=sparse(Cn,Cn); NecMat=sparse(Cn,Cn); count=0;
for i=1:(Cn-1),
  for j=(i+1):Cn,
    if(sum(CVMat(i,:) & CVMat(j,:))==2)
      count=count+1; NecMat(i,j)=1; NecMat(j,i)=1; b=[b;i,j];
      bxx(i,j)=count; bxx(j,i)=count;
    end
  end
end
bn=count; A=x; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:N,
  if(A(i,1)<=LBc)
    LMat(1,i)=1;
  elseif(A(i,1)>=UBc)
    UMat(1,i)=1;
  end
end
NeMat=NecMat; Blocked=randperm(Cn);
% for bonds
NebMat=sparse(bn,bn);
for i=1:Cn,
  [a,bb,c]=find(bxx(i,:)); nc=size(c,2);
  for j=1:(nc-1),
    for k=(j+1):nc,
      NebMat(c(1,j),c(1,k))=1; NebMat(c(1,k),c(1,j))=1;
    end
  end
end
A=b; N=size(A,1); LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:N,
  if((x(A(i,1),1)<=LBc) | (x(A(i,2),1)<=LBc))
    LMat(1,i)=1;
  elseif((x(A(i,1),1)>=UBc) | (x(A(i,2),1)>=UBc))
    UMat(1,i)=1;
  end
end
NeMat=NebMat; Blocked=randperm(bn);
% for vertices
NeVMat=sparse(vn,vn);
for i=1:en,
  NeVMat(e(i,1),e(i,2))=1; NeVMat(e(i,2),e(i,1))=1;
end
A=v; N=vn; LMat=sparse(1,N); UMat=sparse(1,N); LB=min(v(:,1));
UB=max(v(:,1)); rng=UB-LB; LBv=.05*rng+LB; UBv=UB-LBv;
for i=1:vn,
  if(v(i,1)<LBv)
    LMat(1,i)=1;
  end
  if(v(i,1)>UBv)
    UMat(1,i)=1;
  end
end
NeMat=NeVMat; Blocked=randperm(vn);
% for edges
EVMat=sparse(en,vn);
for i=1:en,
  EVMat(i,e(i,1))=1; EVMat(i,e(i,2))=1;
end
NeEMat=sparse(en,en);
for i=1:vn,
  Tmp=find(EVMat(:,i)); TmpN=size(Tmp,1);
  for j=1:(TmpN-1),
    for k=(j+1):TmpN,
      NeEMat(Tmp(j),Tmp(k))=1; NeEMat(Tmp(k),Tmp(j))=1;
    end
  end
end
A=e; N=en; LMat=sparse(1,N); UMat=sparse(1,N);
for i=1:N,
  if((v(A(i,1),1)<=LBv) | (v(A(i,2),1)<=LBv))
```

```
    LMat(1,i)=1;
  elseif((v(A(i,1),1)>=UBv) | (v(A(i,2),1)>=UBv))
    UMat(1,i)=1;
  end
end
NeMat=NeEMat; Blocked=randperm(N);
% At the end of the day.
figure(1); clf; hold on;
for i=1:en,
  plot([v(e(i,1),1),v(e(i,2),1)],[v(e(i,1),2),v(e(i,2),2)]);
end
figure(2); clf; hold on;
for i=1:bn,
  plot([x(b(i,1),1),x(b(i,2),1)],[x(b(i,1),2),x(b(i,2),2)]);
end
figure(3); clf; hold on;
for i=1:Bn,
  plot([x(B(i,1),1),x(B(i,2),1)],[x(B(i,1),2),x(B(i,2),2)]);
end
% Here come data to be run first, though listed last.
% (1)  3_3[3^3]7_1[3^7]
clear all; sz=20; nx=7; ny=4; dx=0.5*[0;1;2;4;6;7;8;10;12]; Tmp=sqrt(3)/2;
dy=Tmp*[0;1;2;4;5;6;7;8;10;11;12]; dim1=max(dx); dim2=max(dy);
q=[1,2; 1,4; 2,3; 3,5; 4,7; 5,6; 5,8; 7,8; 7,9; 8,10; 9,12; 10,11; 10,14; 12,13;...
 13,14;,13,15; 14,16; 15,19; 16,17; 16,18; 18,20];
m=[3,4,6,1,7,9,3,5,2,7,8,2,3,5,1,7,9,6,3,4];
n=[1,1,2,3,3,3,4,4,5,6,6,7,8,8,9,9,9,10,11,11];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20];
ii=[4,6;15,17]; iii=[1,19;2,20]; in2=[9,11;12,11]; in3=[3,18];
c={[1,2,3,5,8,7,4];[7,8,10,14,13,12,9];[13,14,16,18,20,19,15]};
cii={[4,7,9],[11,10,8,5];[9,12],[11];[12,13,15],[16,14,10,11]};
ciii={[2,3],[18]}; civ={[1,4],[3,5],[18,16,17],[]};
% (2)  3_3[3^3]9_3[3^9]_II
clear all; sz=19; nx=4; ny=5; Tmp=1/sqrt(3); dx=Tmp*[0,1,2,3,4,5,6,7,8,9,10,11,12];
Tmp=1/3; dy=Tmp*[0,2,4,6,7,8,9,11,13,15,16,17,18]; dim1=max(dx); dim2=max(dy);
q=[1,3;2,4;3,5;3,6;4,7;4,9;5,6;5,9;6,8;7,10;8,11;9,12;10,13;12,15;...
 12,16;13,17;14,19;15,16;15,18;16,19;17,18];
m=[10,4,10,4,9,11,2,12,7,1,13,7,1,13,6,8,3,5,10];
n=[1,2,3,4,5,5,6,6,7,8,8,9,10,10,11,11,12,12,13];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19];
ii=[10,11;13,14]; iii=[1,19]; in2=[7,8]; in3=[2,17;2,18];
c={[3,6,5];[4,9,12,15,18,17,13,10,7];[12,16,15];[11,14,19,16,12,9,5,6,8]};
cii={[7,10],[8];}; ciii={[2],[17,18];[1,3,5,9,4,2],[18,15,16]};
civ={[2,4,7],[8,6,3,1],[14],[]};
% (3)  4_4[3^4]8_4[3^8]
clear all; sz=8; nx=10; ny=10; Tmp=1/sqrt(2); dx=[0,Tmp,1+Tmp,1+2*Tmp];
dy=[0,Tmp,1+Tmp,1+2*Tmp]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,3;2,4;3,5;5,7;6,8]; m=[2,3,1,4,1,4,2,3]; n=[1,1,2,2,3,3,4,4];
o=[1,2,3,4,5,6,7,8]; ii=[3,4;5,6]; iii=[1,7;2,8]; in2=[]; in3=[];
c={[1,2,4,6,8,7,5,3]}; cii={}; ciii={}; civ={[1,3],[2],[6],[]};
% (4)  3_3[3^3]8_2[3^8]
clear all; sz=25; nx=3; ny=6; dx=[0,1,2,3,4,5,6,7,8,9,10,11,12];
dy=.5*[0,2,4,5,6,8,10,11,12]; dim1=max(dx); dim2=max(dy);
q=[1,4;2,6;3,5;4,7;4,8;5,9;5,10;6,12;6,13;7,8;7,11;8,12;9,10;9,13;10,14;11,17;...
 12,15;13,16;15,19;15,20;16,21;16,22;17,23;18,25;19,20;19,23;20,24;21,22;21,24;22,25];
m=[3,7,11,3,11,7,2,4,10,12,1,5,9,13,5,9,1,13,4,6,8,10,3,7,11];
n=[1,1,1,2,2,3,4,4,4,4,5,5,5,5,6,6,7,7,8,8,8,8,9,9,9];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25];
ii=[11,14;17,18]; iii=[1,23;2,24;3,25]; in2=[]; in3=[];
c={[4,8,7];[5,10,9];[7,8,12,15,19,23,17,11];[6,13,16,21,24,20,15,12];...
 [9,10,14,18,25,22,16,13];[15,20,19];[16,22,21]}; cii={};
ciii={[2,6,12,8,4,1],[19,20];[3,5,9,13,6,2],[21,22]};
civ={[1,4,7,11],[10,5,3],[18],[]};
% (5)  3_3[3^3]9_3[3^9]_III
clear all; sz=12; nx=6; ny=4; Tmp=sqrt(3)/2; dx=Tmp*[0,1,2,3,4];
dy=.5*[0,3,5,6,9,11,12]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,3;2,3;3,4;4,5;4,6;5,7;6,8;7,9;9,11;10,12];
m=[2,4,3,3,2,4,1,5,1,5,2,4]; n=[1,1,2,3,4,4,5,5,6,6,7,7];
o=[1,2,3,4,5,6,7,8,9,10,11,12]; ii=[7,8;9,10]; iii=[1,11;2,12]; in2=[5,6]; in3=[];
c={[1,2,3];[4,6,8,10,12,11,9,7,5]}; cii={[5,7],[6]}; ciii={};
civ={[1,3,4,5],[6,4,3,2],[10],[]};
% (6)  4_3[3^4]10_6[3^10]_I
clear all; sz=20; nx=8; ny=4; dx=[0,2,3,5,6]; dy=3*[0,1,2,3,4];
dim1=max(dx); dim2=max(dy);
q=[1,2;1,6;2,3;2,7;3,4;4,5;6,7;6,9;7,10;9,10;10,11;11,12;11,14;12,13;12,15;14,15;14,18;15,19];
m=[1,2,3,4,5,1,2,5,1,2,3,4,5,3,4,1,2,3,4,5];
n=[1,1,1,1,1,2,2,2,3,3,3,3,3,4,4,5,5,5,5,5];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20];
ii=[1,5;6,8;9,13;16,20]; iii=[1,16;2,17;3,18;4,19;5,20]; in2=[]; in3=[];
c={[1,2,7,6];[2,3,4,5,8,13,12,11,10,7];[6,7,10,9];[11,12,15,14];[14,15,19,18]};
cii={[9,10,11,14,18,17,16],[19,15,12]}; ciii={}; civ={};
% (7)  3_3[3^3]9_3[3^9]_I
```

```
clear all; sz=37; nx=5; ny=3; x=2*sqrt(3)/(2+sqrt(3)); i=1/2; j=sqrt(3)/2;
dx=x*[0, i, i+j, 2*i+j, 3*i+j, 3*i+2*j, 4*i+2*j, 5*i+2*j, 5*i+3*j,...
 6*i+3*j, 7*i+3*j, 7*i+4*j, 8*i+4*j]; y=1/2/sqrt(3);
dy=x*[0, 2*y, 2*y+1, 4*y+1, 5*y+1, 5*y+1.5, 6*y+1.5, 8*y+1.5, 8*y+2.5, 10*y+2.5,...
 11*y+2.5, 11*y+3, 12*y+3, 14*y+3, 14*y+4, 16*y+4, 17*y+4, 17*y+4.5, 18*y+4.5,...
 20*y+4.5, 20*y+5.5, 22*y+5.5, 23*y+5.5, 23*y+6, 24*y+6]; dim1=max(dx); dim2=max(dy);
q=[1,4;2,3;3,5;3,6;4,8;4,9;5,6;5,7;6,9;7,10;8,11;9,14;10,12;12,15;13,16;14,17;14,19;...
 15,19;16,18;17,18;17,20;18,20;19,22;20,21;21,23;21,24;22,25;22,27;23,24; 23,26;...
 24,28;25,26;25,29;26,29;27,31;29,30;30,33;30,34;31,35;32,37;33,34;33,36;34,37;35,36];
m=[10,4,4,10,3,5,2,12,7,1,13,1,13,7,2,12,9,11,4,10,10,4,9,...
 11,6,8,1,13,7,7,1,13,6,8,3,5,10];
n=[1,2,3,4,5,5,6,6,7,8,8,9,9,10,11,11,12,12,13,14,15,16,17,...
 17,18,18,19,19,20,21,22,22,23,23,24,24,25];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,...
 24,25,26,27,28,29,30,31,32,33,34,35,36,37]; ii=[10,11;12,13;27,28;31,32];
iii=[1,37]; in2=[7,8;15,16]; in3=[2,35;2,36];
c={[3,6,5];[5,6,9,14,19,15,12,10,7];[4,8,11,13,16,18,17,14,9];...
 [14,17,20,21,23,26,25,22,19];[17,18,20];[21,24,23];[22,25,29,30,33,36,35,31,27];...
 [23,24,28,32,37,34,30,29,26];[25,26,29];[30,34,33]};
cii={[7,10],[8];[12,15],[16];[15,19,22,27],[24,21,20,18,16]};
ciii={[2],[35,36];[1,4,9,6,3,2],[36,33,34]}; civ={[2,3,5,7],[8,4,1],[32],[35]};
% (8)  4_2[3^4]10_4[3^10]
clear all; sz=42; nx=3; ny=6; i=sqrt(3)-1; j=1/2; k=sqrt(3);
dx=[0, i, k-j, k, k+j, k+i, 2*k-i, 2*k-j, 2*k, 2*k+j, 3*k-i, 3*k, 3*k+i,...
 4*k-j, 4*k, 4*k+j, 4*k+i, 5*k-i, 5*k-j, 5*k, 5*k+j, 6*k-i, 6*k];
i=(2/3)*(3-sqrt(3)); j=1/sqrt(3); k=1/2/sqrt(3);
dy=[0, i, 2-k, 2, 2+j, 3, 3+i, 5-k, 5, 5+j, 6]; dim1=max(dx); dim2=max(dy);
q=[1,11;2,5;3,6;5,7;5,8;6,9;6,10;7,12;7,15;8,12;8,17;9,13;9,18;10,13;...
 10,20;11,21;12,16;13,19;14,23;15,16;15,21;16,17;17,22;18,19;18,22;19,20;...
 20,23;21,24;22,31;23,25;24,26;24,27;25,28;25,29;26,30;26,33;27,30;27,35;...
 28,32;28,36;29,32;29,38;30,34;31,40;31,41;32,37;33,34;33,39;...
 34,35;35,40;36,37;36,41;37,38;38,42];
m=[1,9,15,23,9,15,8,10,14,16,1,9,15,23,6,9,11,13,15,18,4,12,20,4,...
 20,3,5,19,21,4,12,20,2,4,7,17,20,22,1,9,15,23];
n=[1,1,1,1,2,2,3,3,3,3,4,4,4,4,5,5,5,5,5,5,6,6,6,7,...
  7,8,8,8,8,9,9,9,10,10,10,10,10,10,11,11,11,11];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,...
 25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42];
ii=[1,4;11,14;39,42]; iii=[1,39;2,40;3,41;4,42]; in2=[]; in3=[];
c={[5,8,12,7];[6,10,13,9];[7,12,16,15];[8,17,16,12];[9,13,19,18];[10,20,19,13];...
 [15,16,17,22,31,40,35,27,24,21];[18,19,20,23,25,28,36,41,31,22];[24,27,30,26];...
 [25,29,32,28];[26,30,34,33];[27,35,34,30];[28,32,37,36];[29,38,37,32]};
cii={[11,21,24,26,33,39],[38,29,25,23]};
ciii={[2,5,7,15,21,11,1],[33,34,35];[3,6,9,18,22,17,8,5,2],[31];...
 [4,14,23,20,10,6,3],[36,37,38]}; civ={};
% (9)  4_3[3^4]10_6[3^10]_II
clear all; sz=56; nx=5; ny=3; i=2/(2-1/sqrt(3)); n=sqrt(3);
j=.5*(1.5*i-sqrt(3)); k=i/2; p=n-n*(1-i/n)-j; q=i;
dx=[0,j,k,p,q,n,n+j,n+k,n+p,n+q,2*n,2*n+j,2*n+k,2*n+p,2*n+q,3*n,3*n+j,...
 3*n+k,3*n+p,3*n+q,4*n]; m=3; j=i*(1/2+1/n)-1; tmp=1-i/sqrt(3);
p=i; k=p-tmp; q=p+tmp; r=q+i/2; s=2*i;
dy=[0,j,k,p,q,r,s,m,m+j,m+k,m+p,m+q,m+r,m+s,2*m,2*m+j,2*m+k,2*m+p,2*m+q,2*m+r,...
 2*m+s,3*m,3*m+j,3*m+k,3*m+p,3*m+q,3*m+r,3*m+s,4*m]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,7;2,8;3,6;5,6;6,10;7,8;7,12;8,13;10,11;10,14;11,18;11,19;12,13;13,16;...
 14,17;14,19;15,18;16,17;16,21;17,22;18,20;19,20;20,23;21,22;21,25;22,26;23,27;...
 24,28;25,26;25,28;26,29;27,30;27,31;28,32;29,30;29,33;30,34;32,35;33,34;33,38;...
 34,39;35,36;35,37;36,41;36,44;37,44;38,39;38,41;39,42;40,43;41,45;42,43;42,46;...
 43,47;44,45;45,48;46,47;46,51;47,52;48,49;48,50;49,55;50,54;51,52;51,55;52,56];
m=[1,5,15,21,9,13,1,5,21,13,17,1,5,11,21,6,10,20,14,18,6,10,18,2,6,10,16,...
  5,11,15,19,3,11,15,3,7,1,11,15,21,10,16,20,4,8,16,20,8,12,6,16,20,1,5,15,21];
n=[1,1,1,1,2,3,4,4,4,5,6,7,7,7,7,8,8,8,9,10,11,11,12,13,14,14,14,...
 15,15,15,16,17,18,18,19,20,21,21,21,21,22,22,22,23,24,25,25,26,27,28,28,28,29,29,29,29];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,...
 28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56];
ii=[1,4;7,9;12,15;37,40;53,56]; iii=[1,53;2,54;3,55;4,56];
in2=[24,23;24,31;32,31]; in3=[5,49;5,50];
c={[1,2,8,7];[7,8,13,12];[10,11,19,14];[11,18,20,19];[14,19,20,23,27,30,29,26,22,17];...
 [16,17,22,21];[21,22,26,25];[25,26,29,33,38,41,36,35,32,28];[29,30,34,33];...
 [33,34,39,38];[35,36,44,37];[36,41,45,44];[38,39,42,46,51,55,49,48,45,41];...
 [42,43,47,46];[46,47,52,51]};
cii={[12,13,16,21,25,28,24],[23,20,18];[24],[31,27,23];[24,28,32],[31];...
 [32,35,37],[43,42,39,34,30,27,31];[37,44,45,48,50,54,53],[52,47,43]};
ciii={[5,6,10,14,17,16,13,8,2],[50];[5],[50,48,49];[5,6,3],[49];...
 [4,9,15,18,11,10,6,3],[51,52]};civ={};
% (10)  4_2[3^4]8_2[3^8]
clear all; sz=32; nx=4; ny=6; i=1/2; j=sqrt(3);
dx=[0,i,2*i,j,2*j,3*j-2*i,3*j-i,3*j,3*j+i,3*j+2*i,4*j,5*j,6*j-2*i,6*j-i,6*j];
i=(2*(sqrt(3)-1))/sqrt(3); j=(sqrt(3)/2)+i; k=(1/2/sqrt(3))+j; m=i+sqrt(3);
n=3; dy=[0,i,j,k,m,n,n+i,n+j,n+k,n+m,2*n]; dim1=max(dx); dim2=max(dy);
q=[1,5;2,10;3,11;5,7;6,8;7,9;7,14;8,12;8,15;9,13;10,17;10,18;11,18;11,19;13,14;...
 14,17;15,16;15,19;17,23;18,20;19,25;20,21;20,22;21,24;21,26;22,24;22,28;23,29;...
 23,30;24,27;25,31;25,32;26,27;26,30;27,28;28,31];
```

```
m=[1,5,11,15,1,15,2,14,1,5,11,15,1,3,13,15,4,8,12,8,7,9,4,8,12,6,8,10,1,5,11,15];
n=[1,1,1,1,2,2,3,3,4,4,4,4,5,5,5,5,6,6,6,7,8,8,9,9,9,10,10,10,11,11,11,11];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,...
 24,25,26,27,28,29,30,31,32];
ii=[1,4;5,6;9,12;13,16;29,32]; iii=[1,29;2,30;3,31;4,32]; in2=[]; in3=[];
c={[7,14,13,9];[8,12,16,15];[10,18,20,21,26,30,23,17];[11,19,25,31,28,22,20,18];...
 [20,22,24,21];[21,24,27,26];[22,28,27,24]}; cii={[5,7,9],[8];[13,14,17,23,29],[25,19,15]};
ciii={[2,10,17,14,7,5,1],[23];[3,11,18,10,2],[26,27,28]};
civ={[1],[6,8,15,19,11,3],[25],[]};
% (11)  4_3[3^4]8_3[3^8]_I
clear all; sz=22; nx=9; ny=3; m=sqrt(3); i=2/(2-1/m); n=m/2; tmp=(m-i)/2;
tmp1=i/4; j=n-tmp1; k=n-tmp; p=n+tmp; q=n+tmp1;
dx=[0,j,k,p,q,m,m+j,m+k,m+p,m+q,2*m]; m=2; n=1; tmp=.5*(1-i+sqrt(3)*i/2); tmp1=2*tmp;
j=m-tmp1; k=j+i/2; p=m+tmp; q=m+n-tmp; r=p+i/2; s=m+n+tmp1; u=2*m+n; t=u+n;
dy=[0,j,k,p,q,r,s,u,t,t+j,t+k,t+p,t+q,t+r,t+s,t+u,2*t]; dim1=max(dx); dim2=max(dy);
q=[1,3;3,5;4,6;5,7;5,8;6,7;7,9;8,9;9,10;10,11;10,12;11,13;13,16;14,15;15,17;...
 15,18;16,17;17,19;18,19;19,20;20,21;20,22];
m=[1,11,1,11,5,9,8,2,6,6,1,11,1,11,7,3,4,10,6,6,1,11];
n=[1,1,2,2,3,4,5,6,7,8,9,9,10,10,11,12,13,14,15,16,17,17];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22];
ii=[1,2;3,4;11,12;13,14;21,22]; iii=[1,21;2,22]; in2=[8,6;16,18]; in3=[];
c={[5,7,9,8];[10,12,14,15,17,16,13,11];[15,18,19,17]};
cii={[3,5,8],[6];[8,9,10,11],[10,9,7,6];[13,16],[18,15];[16,17,19,20,21],[20,19,18]};
ciii={[2,4,6,7,5,3,1],[20]}; civ={};
% (12)  4_3[3^4]8_3[3^8]_II
clear all; sz=41; nx=5; ny=3; dx=[0,1,2,3,4,5,6,7,8,9,10,11,12];
dy=2.5*[0,1,2,3,4,5,6,7,8]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,10;2,3;3,4;4,5;4,8;5,6;5,9;6,7;8,9;8,12;9,13;10,11;11,12;11,20;12,13;13,14;...
 14,15;14,17;15,16;15,18;17,18;17,22;18,23;19,20;19,25;20,21;21,22;21,30;22,23;23,24;...
 24,26;25,27;26,32;27,28;28,29;28,33;29,30;29,34;30,31;31,32;31,40;33,34;33,36;34,37];
m=[1,3,5,6,8,10,13,6,8,1,4,6,8,9,11,13,9,11,2,4,7,9,...
 11,12,2,12,2,3,5,7,10,12,3,5,1,3,5,6,8,10,13];
n=[1,1,1,1,1,1,1,2,2,3,3,3,3,3,3,3,4,4,5,5,5,5,5,5,6,6,7,7,7,7,7,7,8,8,9,9,9,9,9,9,9];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,...
 28,29,30,31,32,33,34,35,36,37,38,39,40,41]; ii=[1,7;10,16;35,41];
iii=[1,35;2,36;3,37;4,38;5,39;6,40;7,41]; in2=[19,24;25,26;27,32]; in3=[];
c={[1,2,3,4,8,12,11,10];[4,5,9,8];[5,6,7,16,15,14,13,9];[8,9,13,12];...
 [11,12,13,14,17,22,21,20];[14,15,18,17];[17,18,23,22];[19,20,21,30,29,28,27,25];...
 [21,22,23,24,26,32,31,30,21];[28,29,34,33];[33,34,37,36]};
cii={[10,11,20,19],[24,23,18,15,16];[19,25],[26,24];[25,27],[32,26]};
ciii={[3,4,5,6],[34,29,30,31]}; civ={[1],[6],[31,32],[27,28,33,36]};
% (13)  4_3[3^4]8_3[3^8]_III
clear all; sz=22; nx=10; ny=3; x=16*sqrt(3)/(4*(sqrt(3)+1)); i=x/4; j=2*sqrt(3);
dx=[0,i,2*i,j-2*i,j-i,j,j+i,j+2*i,2*j-2*i,2*j-i,2*j]; i=(sqrt(3)/4)*x; j=12;
dy=[0,x,x+i,x+2*i,j-2-x-2*i,j-2-x-i,j-2-x,j-2,j,j+x,j+x+i,j+x+2*i,2*j-2-x-2*i,...
 2*j-2-x-i,2*j-2-x,2*j-2,2*j]; dim1=max(dx); dim2=max(dy);
q=[1,3;3,7;4,5;5,6;6,7;7,8;8,9;9,10;10,11;10,12;11,13;13,15;14,17;15,16;16,17;17,18;...
 18,19;19,20;20,21;20,22]; m=[1,11,1,11,10,9,8,7,6,6,1,11,1,11,2,3,4,5,6,6,1,11];
n=[1,1,2,2,3,4,5,6,7,8,9,9,10,10,11,12,13,14,15,16,17,17];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22];
ii=[1,2;3,4;11,12;13,14;21,22]; iii=[1,21;2,22]; in2=[8,5;9,6;15,18;16,19]; in3=[];
c={[10,12,14,17,16,15,13,11]};
cii={[3,7,8],[5];[8,9],[6,5];[9,10,11],[10,9,8,7,6];[13,15],[17,18];[15,16],[18,19];,,,
 [16,17,18,19,20,21],[19,20]}; ciii={[2,4,5,6,7,3,1],[20]}; civ={}
% (14)  4_4[3^4]7_2[3^7]_II
clear all; sz=33; nx=6; ny=5; dx=[0,1,2,3,4,5,6,7,8,9,10,11,12];
dy=4*[0,1,2,3,4]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,8;2,3;3,4;4,5;4,10;5,6;5,11;6,7;8,9;9,10;9,16;10,11;11,12;12,13;12,18;...
 13,14;13,19;15,16;15,21;16,17;17,18;17,24;18,19;19,20;20,26;21,22;22,23;22,28;...
 23,24;23,29;24,25;25,26;25,32];
m=[1,3,5,6,8,10,13,1,4,6,8,9,11,13,2,4,7,9,11,12,2,3,5,7,10,12,1,3,5,6,8,10,13];
n=[1,1,1,1,1,1,1,2,2,2,2,2,2,2,3,3,3,3,3,3,4,4,4,4,4,4,5,5,5,5,5,5,5];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,...
 27,28,29,30,31,32,33]; ii=[1,7;8,14;27,33]; iii=[1,27;2,28;3,29;4,30;5,31;6,32;7,33];
in2=[15,20;21,26]; in3=[];
c={[1,2,3,4,10,9,8];[4,5,11,10];[5,6,7,14,13,12,11];[9,10,11,12,18,17,16];[12,13,19,18];...
 [15,16,17,24,23,22,21];[17,18,19,20,26,25,24];[22,23,29,28];[23,24,25,32,31,30,29]};
cii={[8,9,16,15],[20,19,13];[15,21],[26,20];[21,22,28,27],[32,25,26]}; ciii={}; civ={};
% (15)  3_3[3^3]12_6[3^12]
clear all; sz=17; nx=8; ny=5; i=2+sqrt(3); dx=[0,1,2,i,i+1,i+2,2*i];
i=sqrt(3); j=3+2*i; dy=[0,i,i+2,2*i+2,j,j+i,j+i+2,j+2*i+2,2*j]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,4;2,4;4,5;5,6;5,7;6,7;7,9;8,10;9,10;9,11;10,11;11,12;12,13;12,14;13,14;13,16;14,17];
m=[1,3,7,2,2,1,3,7,4,6,5,5,4,6,1,3,7]; n=[1,1,1,2,3,4,4,4,5,5,6,7,8,8,9,9,9];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17]; ii=[1,3;6,8;15,17]; iii=[1,15;2,16;3,17];
in2=[]; in3=[]; c={[1,2,4];[5,7,6];[9,10,11];[12,14,13]};
cii={[6,7,9,11,12,13,16],[17,14,12,11,10]}; ciii={};
civ={[1,4,5,6],[10,9,7,5,4,2],[13,14],[]};
% (16)  4_4[3^4]7_2[3^7]_I
clear all; sz=35; nx=6; ny=6; dx=[0,1,2,3,4,5,6]; dy=[0,1,2,3,4,5,6];
dim1=max(dx); dim2=max(dy);
q=[1,2;1,7;2,3;3,4;3,12;4,5;5,8;7,10;8,9;8,14;10,11;10,16;11,12;11,17;...
 12,13;13,14;13,18;14,19;16,17;17,22;18,19;18,23;19,20;20,21;20,28;22,23;...
 22,25;23,26;24,25;24,30;25,31;26,27;26,33;27,28;27,34;28,29];
m=[1,2,3,4,5,7,1,5,7,1,2,3,4,5,7,1,2,4,5,6,7,2,4,1,2,4,5,6,7,1,2,3,4,5,7];
n=[1,1,1,1,1,1,2,2,2,3,3,3,3,3,3,4,4,4,4,4,4,5,5,6,6,6,6,6,6,7,7,7,7,7,7];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,...
 27,28,29,30,31,32,33,34,35]; ii=[7,9;16,21;24,29];
iii=[1,30;2,31;4,33;5,34]; in2=[]; in3=[];
c={[1,2,3,12,11,10,7];[3,4,5,8,14,13,12];[13,14,19,18];[11,12,13,18,23,22,17];...
 [10,11,17,16];[18,19,20,28,27,26,23];[24,25,31,30];[26,27,34,33]};
cii={[16,17,22,25,24],[28,20];[10],[21,20,19,14,8,9]};
ciii={[2,3,4],[25,22,23,26];}; civ={[1],[9,8,5],[27,28],[24]};
% (17)  5_4[3^5]7_4[3^7]_I
clear all; sz=75; nx=2; ny=3; i=sqrt(3)/2;
dx=i*[0,1,2,3,4,6,7,8,9,10,12,13,14,15,16,18,19,20,21,22,24];
dy=.5*[0,1,2,4,6,7,8,10,12,13,14,16,18,19,20,22,24];
dim1=max(dx); dim2=max(dy);
q=[1,7;1,10;2,10;2,11;3,11;3,12;4,12;4,13;5,13;5,14;6,8;6,15;7,9;7,25;8,14;9,18;...
 10,19;11,20;12,16;13,17;14,23;16,17;16,21;17,22;18,27;19,25;19,28;20,26;20,29;...
 21,29;21,30;22,30;22,31;23,31;23,32;24,32;25,27;26,28;26,41;27,35;28,36;29,37;...
 30,38;31,33;32,34;33,34;33,39;34,40;35,43;35,44;36,44;36,45;37,41;37,46;38,42;...
 38,47;39,47;39,48;40,48;40,49;41,45;42,46;42,60;43,50;44,51;45,55;46,56;47,57;...
 48,58;50,51;50,53;51,54;53,62;54,62;54,63;55,63;55,64;56,64;56,65;57,60;57,66;...
 58,61;58,67;59,67;60,65;61,66;62,70;63,68;64,69;65,73;66,74;67,75;68,69;68,71;69,72];
m=[3,6,10,13,16,20,2,19,1,5,8,11,15,18,21,11,15,1,5,8,11,15,18,21,4,7,3,6,10,13,...
 16,20,16,20,3,6,10,13,16,20,9,12,1,5,8,11,15,18,21,1,5,21,1,5,8,11,15,18,21,14,...
 17,3,6,10,13,16,20,6,10,3,6,10,13,16,20];
n=[1,1,1,1,1,1,2,2,3,3,3,3,3,3,3,4,4,5,5,5,5,5,5,5,6,6,7,7,7,7,7,7,8,8,9,9,9,9,9,...
 9,10,10,11,11,11,11,11,11,11,12,12,12,13,13,13,13,13,13,13,14,14,15,15,15,15,15,...
 15,16,16,17,17,17,17,17,17];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,...
 30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,...
 56,57,58,59,60,61,62,63,64,65,66,67,68,69,70,71,72,73,74,75];
ii=[9,15;18,24;43,49;50,52;53,59]; iii=[1,70;2,71;3,72;4,73;5,74;6,75];
in2=[]; in3=[8,61];
c={[1,10,19,25,7];[2,11,20,26,28,19,10];[3,12,16,21,29,20,11];[4,13,17,16,12];...
 [5,14,23,31,22,17,13];[6,15,24,32,23,14,8];[7,25,27,18,9];[16,17,22,30,21];...
 [19,28,36,44,35,27,25];[20,29,37,41,26];[21,30,38,42,46,37,29];[22,31,33,39,47,38,30];...
 [23,32,34,33,31];[26,41,45,36,28];[33,34,40,48,39];[35,44,51,50,43];...
 [36,45,55,63,54,51,44];[37,46,56,64,55,45,41];[38,47,57,60,42];[39,48,58,61,66,57,47];...
 [40,49,52,59,67,58,48];[42,60,65,56,46];[50,51,54,62,53];[55,64,69,68,63]};
cii={[18,27,35,43],[40,34,32]};
ciii={[2,10,1],[62,54,63,68];[2,11,3],[69,68];[4,12,3],[69,64,56,65];...
 [5,13,4],[65,60,57,66];[5,14,8],[61,66];[8,6],[67,58,61]};
civ={[1,7,9],[6],[67,59],[62]};
% (18)  3_1[4^3]5_1[4^5]_I
clear all; sz=27; nx=4; ny=8; dx=[0,1,2,3,4,5,6,7,8]; dy=[0,1,2,3,4];
dim1=max(dx); dim2=max(dy);
q=[1,2;1,8;2,3;2,8;3,4;3,12;4,5;4,9;5,6;5,16;6,7;6,10;8,11;9,13;9,14;9,15;11,12;11,18;...
 12,13;12,23;13,14;13,19;14,15;14,19;15,16;15,19;16,17;16,25;18,21;18,22;19,24;20,26];
m=[1,2,3,5,7,8,9,1,5,9,1,3,4,5,6,7,9,1,5,9,1,2,3,5,7,8,9];
n=[1,1,1,1,1,1,1,2,2,2,3,3,3,3,3,3,3,4,4,4,5,5,5,5,5,5,5];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27];
ii=[1,7;8,10;11,17;18,20;21,27]; iii=[1,21;2,22;3,23;4,24;5,25;6,26;7,27]; in2=[]; in3=[];
c={[1,2,8];[2,3,12,11,8];[3,4,9,13,12];[4,5,16,15,9];[5,6,10,17,16];[6,7,10];...
 [9,14,13];[9,15,14];[11,12,23,22,18];[12,13,19,24,23];[13,14,19];[14,15,19];...
 [15,16,25,24,19];[16,17,20,26,25];[18,22,21]}; cii={}; ciii={[6,7],[20]}; civ={};
% (19)  3_1[4^3]5_1[4^5]_II
clear all; sz=13; nx=8; ny=8; dx=[0,1,2,3,4]; dy=[0,1,2,3,4]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,5;2,3;2,4;4,6;4,7;4,8;5,6;5,11;6,7;6,10;7,8;7,10;8,9;8,10;10,12];
m=[1,3,5,3,1,2,3,4,5,3,1,3,5]; n=[1,1,1,2,3,3,3,3,3,4,5,5,5];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13]; ii=[1,3;5,9;11,13]; iii=[1,11;2,12;3,13]; in2=[]; in3=[];
c={[1,2,4,6,5];[2,3,9,8,4];[4,7,6];[4,8,7];[5,6,10,12,11];[6,7,10];[7,8,10];[8,9,13,12,10]};
cii={}; ciii={}; civ={};
% (20)  5_3[3^5]8_6[3^8]_II
clear all; sz=110; nx=3; ny=2; i=sqrt(3)/2;
dx=[0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16];
dy=0.5*[0,1,2,4,6,7,8,10,12,13,14,16,18,19,20,22,24,25,26,28,30,31,32,34,...
 36,37,38,40,42,43,44,46,48]; dim1=max(dx); dim2=max(dy);
q=[1,5;1,8;2,6;2,9;3,9;3,10;4,10;4,11;5,7;5,19;6,8;7,14;8,15;9,12;10,13;12,13;...
 12,16;13,17;14,21;15,19;15,22;16,22;16,23;17,23;17,24;18,20;19,21;20,24;21,27;...
 22,25;23,26;24,30;25,26;25,28;26,29;27,31;27,34;28,34;28,35;29,35;29,36;30,32;...
 30,37;31,33;32,36;32,46;33,40;34,38;35,39;36,43;38,39;38,41;39,42;40,47;41,47;...
 41,48;42,48;42,49;43,45;43,50;44,46;45,49;45,58;46,50;47,51;48,52;49,55;50,56;...
 51,52;51,53;52,54;53,59;53,60;54,60;54,61;55,57;55,62;56,58;56,63;57,61;57,73;...
 58,62;59,64;60,65;61,69;62,70;64,65;64,67;65,68;67,74;68,74;68,75;69,72;69,76;...
 70,73;70,77;71,77;72,75;72,85;73,76;74,78;75,81;76,82;77,79;78,80;79,83;80,86;...
 80,87;81,84;81,88;82,85;82,89;83,89;83,90;84,87;84,100;85,88;86,91;87,95;88,96;...
 89,92;91,94;92,93;92,97;94,101;95,99;95,102;96,100;96,103;97,103;97,104;98,104;...
 99,101;100,102;101,107;102,108;103,105;104,106;105,106;105,109;106,110];
m=[3,7,11,15,2,6,1,5,9,13,17,9,13,1,5,9,13,17,4,16,3,7,11,15,7,11,3,7,...
 11,15,2,14,1,5,9,13,17,5,9,1,5,9,13,17,12,16,3,7,11,15,3,7,3,7,...
 11,15,10,14,1,5,9,13,17,1,5,17,1,5,9,13,17,8,12,3,7,11,15,3,15,3,...
 7,11,15,6,10,1,5,9,13,17,1,13,17,1,5,9,13,17,4,8,3,7,11,15,...
 11,15,3,7,11,15];
n=[1,1,1,1,2,2,3,3,3,3,3,4,4,5,5,5,5,5,6,6,7,7,7,7,8,8,9,...
 9,9,9,10,10,11,11,11,11,11,12,12,13,13,13,13,13,14,14,15,15,15,15,16,16,17,...
 17,17,17,18,18,19,19,19,19,19,20,20,20,21,21,21,21,21,22,22,23,23,23,23,24,24,...
 25,25,25,25,26,26,27,27,27,27,27,28,28,28,29,29,29,29,29,30,30,31,31,31,...
 31,32,32,33,33,33,33];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,...
 28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,...
 53,54,55,56,57,58,59,60,61,62,63,64,65,66,67,68,69,70,71,72,73,74,75,76,77,...
 78,79,80,81,82,83,84,85,86,87,88,89,90,91,92,93,94,95,96,97,98,99,100,101,...
 102,103,104,105,106,107,108,109,110];
ii=[7,11;14,18;33,37;40,44;59,63;64,66;67,71;86,90;91,93;94,98];
iii=[1,107;2,108;3,109;4,110]; in2=[31,20;78,79]; in3=[6,99];
c={[1,8,15,19,5];[2,9,12,16,22,15,8,6];[3,10,13,12,9];[4,11,18,20,24,17,13,10];...
 [5,19,21,14,7];[12,13,17,23,16];[15,22,25,28,34,27,21,19];[16,23,26,25,22];...
 [17,24,30,32,36,29,26,23];[25,26,29,35,28];[27,34,38,41,47,40,33,31];...
 [28,35,39,38,34];[29,36,43,45,49,42,39,35];[30,37,44,46,32];[32,46,50,43,36];...
 [38,39,42,48,41];[41,48,52,51,47];[42,49,55,57,61,54,52,48];[43,50,56,58,45];...
 [45,58,62,55,49];[51,52,54,60,53];[53,60,65,64,59];[54,61,69,72,75,68,65,60];...
 [55,62,70,73,57];[57,73,76,69,61];[64,65,68,74,67];[68,75,81,84,87,80,78,74];...
 [69,76,82,85,72];[70,77,79,83,89,82,76,73];[72,85,88,81,75];...
 [80,87,95,99,101,94,91,86];[81,88,96,100,84];[82,89,92,97,103,96,88,85];...
 [83,90,93,92,89];[84,100,102,95,87];[92,93,98,104,97];[97,104,106,105,103]};
cii={[14,21,27,31],[20];[31,33],[30,24,20];[40,47,51,53,59],[56,50,46];...
 [67,74,78],[79,77];[64],[71,77,70,62,58,56,63];[78,80,86],[83,79]};
ciii={[1,8,6],[99,101];[2,6],[99,95,102];[2,9,3],[105,103,96,100,102];[3,10,4],[106,105]};
civ={[1,5,7],[4],[106,104,98],[101]};
% (21)  5_3[3^5]8_6[3^8]_III
clear all; sz=110; nx=4; ny=2; i=sqrt(3)/2;
dx=i*[0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16];
dy=0.5*[0,1,2,4,6,7,8,10,12,13,14,16,18,19,20,22,24,25,26,28,30,31,32,34,...
 36,37,38,40,42,43,44,46,48]; dim1=max(dx); dim2=max(dy);
q=[1,7;1,8;2,5;2,9;3,6;3,10;4,10;4,11;5,8;6,9;6,20;7,12;8,13;9,17;10,18;12,13;...
 12,15;13,16;15,22;16,22;16,23;17,23;17,24;18,20;18,25;19,21;20,24;21,25;22,28;...
 23,26;24,27;25,31;26,27;26,29;27,30;28,32;28,35;29,33;29,36;30,36;30,37;31,37;31,38;...
 32,34;33,35;33,47;34,39;35,43;36,44;37,40;39,42;40,41;40,45;42,49;43,49;43,50;44,47;...
 44,51;45,48;45,52;46,52;47,50;48,51;48,60;49,53;50,54;51,57;52,58;53,54;53,55;54,56;...
 55,59;55,62;56,62;56,63;57,63;57,64;58,60;58,65;59,61;59,73;60,64;61,68;62,69;63,66;...
 64,67;66,67;66,70;67,71;68,75;69,73;69,76;70,74;70,77;71,77;71,78;72,78;73,75;74,76;...
 74,85;75,79;76,82;77,83;78,80;79,81;80,84;81,87;81,88;82,88;82,89;83,85;83,90;84,86;...
 84,91;85,89;86,90;86,100;87,94;88,92;89,93;90,97;92,93;92,95;93,96;94,101;95,99;...
 95,102;96,102;96,103;97,103;97,104;98,100;99,101;100,104;101,107;102,108;103,105;...
 104,106;105,106;105,109;106,110];
m=[3,7,11,15,6,10,1,5,9,13,17,1,5,17,1,5,9,13,17,12,16,3,7,11,15,7,11,3,7,11,15,2,...
 6,1,5,9,13,17,1,13,17,1,5,9,13,17,8,12,3,7,11,15,3,7,3,7,11,15,2,14,1,5,9,13,17,9,...
 13,1,5,9,13,17,4,8,3,7,11,15,3,15,3,7,11,15,10,14,1,5,9,13,17,5,9,1,5,9,13,17,4,...
 16,3,7,11,15,11,15,3,7,11,15];
n=[1,1,1,1,2,2,3,3,3,3,3,4,4,4,5,5,5,5,5,6,6,7,7,7,7,8,8,9,9,9,9,10,10,11,11,11,...
 11,11,12,12,12,13,13,13,13,13,14,14,15,15,15,15,16,16,17,17,17,17,18,18,19,19,19...
 ,19,19,20,20,21,21,21,21,21,22,22,23,23,23,23,24,24,25,25,25,25,26,26,27,27,27,...
 27,27,28,28,29,29,29,29,29,30,30,31,31,31,31,32,32,33,33,33,33];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,...
 31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56,57,58,...
 59,60,61,62,63,64,65,66,67,68,69,70,71,72,73,74,75,76,77,78,79,80,81,82,83,84,85,86,...
 87,88,89,90,91,92,93,94,95,96,97,98,99,100,101,102,103,104,105,106,107,108,109,110];
ii=[7,11;12,14;15,19;34,38;39,41;42,46;61,65;68,72;87,91;94,98];
iii=[1,107;2,108;3,109;4,110]; in2=[32,21;79,80]; in3=[5,99];
c={[1,8,13,12,7];[2,9,17,23,16,13,8,5];[3,10,18,20,6];[6,20,24,17,9];...
 [12,13,16,22,15];[16,23,26,29,33,35,28,22];[17,24,27,26,23];[18,25,31,37,30,27,24,20];...
 [26,27,30,36,29];[28,35,43,49,42,39,34,32];[29,36,44,47,33];[30,37,40,45,48,51,44,36];...
 [31,38,41,40,37];[33,47,50,43,35];[40,41,46,52,45];[43,50,54,53,49];...
 [44,51,57,63,56,54,50,47];[45,52,58,60,48];[48,60,64,57,51];[53,54,56,62,55];...
 [55,62,69,73,59];[56,63,66,70,74,76,69,62];[57,64,67,66,63];[58,65,72,78,71,67,64,60];...
 [59,73,75,68,61];[66,67,71,77,70];[69,76,82,88,81,79,75,73];[70,77,83,85,74];...
 [71,78,80,84,86,90,83,77];[74,85,89,82,76];[81,88,92,95,99,101,94,87];...
 [82,89,93,92,88];[83,90,97,103,96,93,89,85];[84,91,98,100,86];[86,100,104,97,90];...
 [92,93,96,102,95];[97,104,106,105,103]};
cii={[15,22,28,32],[21];[12],[19,21,25,18,10,4,11];[32,34],[31,25,21];...
 [42,49,53,55,59,61],[58,52];[68,75,79],[80,78];[79,81,87],[84,80]};
ciii={[1,8,5],[99,101];[5,2],[102,95,99];[2,9,6,3],[105,103,96,102];[3,10,4],[106,105]};
civ={[1,7],[4],[106,104,100,98],[101]};
% (22)  5_2[3^5]12_12[3^12]
clear all; sz=34; nx=5; ny=3; i=sqrt(3)/2; dx=i*[0,1,2,3,4,5,6,7,8];
dy=[0,.5,1,3,3.5,4,5,6,6.5,7,9,9.5,10,11,12]; dim1=max(dx); dim2=max(dy);
q=[1,3;1,6;2,4;2,6;3,5;3,11;4,7;4,12;5,8;6,9;8,13;9,11;9,12;10,14;11,13;12,14;...
 13,15;14,16;15,17;16,18;17,19;17,21;18,20;18,23;19,22;19,27;20,22;20,28;21,24;...
 22,25;24,27;25,29;25,30;26,28;27,29;28,30;29,31;30,32;31,32;31,33;32,34];
m=[3,7,2,8,1,5,9,1,5,9,4,6,3,7,3,7,3,7,4,6,1,5,9,1,5,9,2,8,3,7,3,7,3,7];
n=[1,1,2,2,3,3,3,4,4,4,5,5,6,6,7,7,8,8,9,9,10,10,10,11,11,11,12,12,13,13,14,14,15,15];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,...
 28,29,30,31,32,33,34]; ii=[5,7;8,10;21,23;24,26]; iii=[1,33;2,34]; in2=[15,16]; in3=[];
c={[1,6,9,11,3];[2,4,12,9,6];[3,11,13,8,5];[4,7,10,14,12];...
```

```
 [9,12,14,16,18,20,22,19,17,15,13,11];[17,19,27,24,21];[18,23,26,28,20];...
 [19,22,25,29,27];[20,28,30,25,22];[25,30,32,31,29]};
cii={[8,13,15],[16,14];[15,17,21],[18,16]}; ciii={[1,6,2],[32,31]};
civ={[1,3,5],[4,2],[32,30,28,26],[27,29,31]};
% (23)  5_4[3^5]7_4[3^7]_II
clear all; sz=21; nx=3; ny=5; i=sqrt(3)/2; dx=i*[0,1,2,3,4,6,7,8,9,10,12];
dy=[0,.5,1,3,3.5,4,6]; dim1=max(dx); dim2=max(dy);
q=[1,4;1,6;2,5;2,7;3,8;3,9;4,7;4,14;5,8;6,10;7,11;8,12;10,14;11,16;11,17;12,15;...
 12,17;13,18;14,16;15,18;16,19;17,20;18,21];
m=[3,6,10,4,7,1,5,8,11,1,5,8,11,2,9,3,6,10,3,6,10];
n=[1,1,1,2,2,3,3,3,3,4,4,4,4,5,5,6,6,6,7,7,7];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21];
ii=[6,9;10,13]; iii=[1,19;2,20;3,21]; in2=[]; in3=[5,15];
c={[1,4,14,10,6];[2,5,8,12,17,11,7];[3,9,13,18,15,12,8];[4,7,11,16,14]};
cii={}; ciii={[1,4,7,2],[17,11,16];[2,5],[15,12,17];[5,8,3],[18,15]};
civ={[1,6],[3],[18,13],[14,16]};
% (24)  5_3[3^5]8_6[3^8]_I
clear all; sz=30; nx=8; ny=6; i=sqrt(3)/2; dx=[0,1,2,3,4,5,6,7,8];
dy=[0,.5,1,3,3.5,4,6,6.5,7,9,9.5,10,12]; dim1=max(dx); dim2=max(dy);
q=[1,3;1,6;2,4;2,7;3,5;3,11;4,6;5,8;6,9;8,13;9,11;9,14;10,12;11,13;12,14;13,15;...
 14,16;15,17;15,20;16,18;16,21;17,19;18,20;18,26;19,22;20,23;21,24;22,27;23,25;...
 23,28;24,26;25,27;26,28;27,29;28,30];
m=[3,7,2,6,1,5,9,1,5,9,4,8,3,7,3,7,2,6,1,5,9,1,5,9,4,8,3,7,3,7];
n=[1,1,2,2,3,3,3,4,4,4,5,5,6,6,7,7,8,8,9,9,9,10,10,10,11,11,12,12,13,13];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30];
ii=[5,7;8,10;19,21;22,24]; iii=[1,29;2,30]; in2=[17,12]; in3=[4,25];
c={[1,6,9,11,3];[2,7,10,12,14,9,6,4];[3,11,13,8,5];[9,14,16,18,20,15,13,11];...
 [15,20,23,25,27,22,19,17];[16,21,24,26,18];[18,26,28,23,20]};
cii={[8,13,15,17],[12];[17,19],[16,14,12]};
ciii={[1,6,4],[25,27];[4,2],[28,23,25]}; civ={[1,3,5],[2],[28,26,24],[27]};
% (25)  4_2[3^4]12_6[3^12]
clear all; sz=21; nx=10; ny=6; i=sqrt(3)/2;
dx=[0,1-i,.5,1,1.5,1+i,2,1.5+i,2+i,2.5+i,2+2*i]; i=1/sqrt(3); j=sqrt(3)/6;
d=sqrt(3)+1.5; dy=[0,i,i+j,3*i,3*i+1,d,d+i,d+i+j,d+3*i,d+3*i+1,2*d];
dim1=max(dx); dim2=max(dy);
q=[1,2;1,6;2,3;2,5;3,7;5,6;5,7;6,8;7,8;8,9;9,10;9,11;11,12;11,14;12,13;13,14;...
 13,15;14,16;15,16;16,17;17,20;17,21];
m=[1,4,7,11,4,3,5,4,4,2,6,9,9,8,10,9,9,1,4,7,11];
n=[1,1,1,1,2,3,3,4,5,6,6,6,7,8,8,9,10,11,11,11,11];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21];
ii=[1,4;18,21]; iii=[1,18;2,19;3,20;4,21]; in2=[10,12;10,15]; in3=[];
c={[1,2,5,6];[2,3,7,5];[5,7,8,6];[11,12,13,14];[13,15,16,14]};
cii={[10],[15,13,12]}; ciii={};
civ={[1,6,8,9,10],[12,11,9,8,7,3],[17],[18];[3,2],[4],[17,16,15],[10,9,11,14,16,17,20]};
% (26)  4_2[3^4]18_12[3^18]
clear all; sz=36; nx=6; ny=3; i=sqrt(3); j=6/i; d=2*j+i;
dx=[0,i,j,d-j,d-i,d,d+i,d+j,2*d-j,2*d-i,2*d];
dy=[0,3,4,6,7,9,10,13,15,18,19,21,22,24,25,28,30]; dim1=max(dx); dim2=max(dy);
q=[1,3;2,4;3,5;3,8;4,6;4,9;5,7;7,8;8,11;9,10;9,13;11,12;11,15;12,13;12,14;13,16;...
 14,15;14,16;15,17;16,17;17,18;18,19;18,20;19,21;19,22;20,21;20,24;21,23;22,23;...
 22,26;23,24;24,27;25,26;25,29;26,31;27,28;27,32;29,31;30,32;31,33;32,34;33,35;34,36];
m=[1,11,2,10,1,11,1,3,9,11,4,6,8,6,5,7,6,6,5,7,6,4,6,8,1,3,9,11,1,11,2,10,1,11,1,11];
n=[1,1,2,2,3,3,4,4,4,4,5,5,5,6,7,7,8,9,10,10,11,12,12,12,...
 13,13,13,13,14,14,15,15,16,16,17,17];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,...
 25,26,27,28,29,30,31,32,33,34,35,36];
ii=[1,2;5,6;7,10;25,28;29,30;33,34;35,36]; iii=[1,35;2,36]; in2=[]; in3=[];
c={[3,8,7,5];[4,6,10,9];[11,12,14,15];[12,13,16,14];[14,16,17,15];[18,20,21,19];...
 [19,21,23,22];[20,24,23,21];[25,26,31,29];[27,28,30,32]};
cii={[1,3,5],[4];[7,8,11,15,17,18,19,22,26,25],[27,24,20,18,17,16,13,9];[29,31,33],[32]};
ciii={[1,3,8,11,12,13,9,4,2],[34,32,27,24,23,22,26,31,33]}; civ={};
% (27)  5_3[3^5]7_3[3^7]_I
clear all; sz=60; nx=4; ny=2; x=sqrt(3); dx=x*[0,1,2,3,4,5,6];
dy=[0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18]; dim1=max(dx); dim2=max(dy);
q=[1,5;2,5;2,6;3,6;3,7;4,7;5,8;6,11;7,9;8,10;9,12;10,13;10,14;11,14;11,15;12,15;...
 12,16;13,17;14,18;15,22;17,18;17,20;18,21;20,24;21,24;21,25;22,25;22,26;23,26;...
 24,27;25,28;26,31;27,28;27,29;28,30;29,32;29,33;30,33;30,34;31,34;31,35;32,38;...
 33,36;34,37;36,37;36,39;37,40;38,42;39,42;39,43;40,43;40,44;41,44;42,47;43,45;...
 44,46;45,46;45,48;46,49;47,50;47,51;48,51;48,52;49,52;49,53;50,54;...
 51,58;52,55;54,57;55,56;55,59];
m=[1,3,5,7,2,4,6,2,6,2,4,6,1,3,5,7,1,3,7,1,3,5,7,2,4,6,2,4,2,4,6,1,3,5,7,3,...
 5,1,3,5,7,2,4,6,4,6,2,4,6,1,3,5,7,1,5,7,1,3,5,7];
n=[1,1,1,1,2,2,2,3,3,4,4,4,5,5,5,5,6,6,6,7,7,7,7,8,8,8,9,9,10,10,10,11,11,11,...
 11,12,12,13,13,13,13,14,14,14,15,15,16,16,16,17,17,17,17,18,18,18,19,19,19,19];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,...
 31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48,49,50,51,52,53,54,55,56,57,58,59,60];
ii=[1,4;13,16;17,19;20,23;32,35;38,41;50,53;54,56;57,60];
iii=[1,57;2,58;3,59;4,60]; in2=[8,9]; in3=[];
c={[2,6,11,14,10,8,5];[3,7,9,12,15,11,6];[10,14,18,17,13];[11,15,22,25,21,18,14];...
 [17,18,21,24,20];[21,25,28,27,24];[22,26,31,34,30,28,25];[27,28,30,33,29];...
 [29,33,36,39,42,38,32];[30,34,37,36,33];[31,35,41,44,40,37,34];[36,37,40,43,39];...
 [39,43,45,48,51,47,42];[40,44,46,45,43];[45,46,49,52,48];[49,53,56,55,52]};
```

```
cii={[1,5,8],[9,7];[8,10,13],[12,9];[17],[23,26,22,15,12,16];...
 [20,24,27,29,32],[31,26];[38,42,47,50],[49,46,44]};
ciii={[1,5,2],[51,47,50,54];[2,6,3],[55,52,48,51];[3,7,4],[56,55]}; civ={};
% (28)  5_3[3^5]7_3[3^7]_II
clear all; sz=32; nx=5; ny=3; nx=3; ny=2; x=sqrt(3)/2; dx=[0,2,3,4,6,7,8];
dy=0.5*[0,2,4,5,6,10,11,12,14,16,17,18,22,23,24]; dim1=max(dx); dim2=max(dy);
q=[1,4;2,8;4,5;4,7;5,10;7,11;8,10;8,12;9,12;10,11;11,13;12,14;13,16;13,17;14,15;
 14,17;15,18;15,20;16,21;17,19;19,20;19,22;20,24;21,25;22,25;22,26;23,24;24,26;
 25,27;26,28;27,29;27,30;28,31;28,32;29,31];
m=[1,4,7,1,2,7,1,4,7,3,2,5,2,5,6,1,4,7,4,5,1,4,7,6,2,5,2,5,3,1,4,7];
n=[1,1,1,2,2,2,3,3,3,4,5,5,6,6,7,8,8,8,9,9,10,10,10,11,12,12,13,13,14,15,15,15];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,29,30,31,32];
ii=[1,3;4,6;7,9;16,18;21,23;30,32]; iii=[1,30;2,31;3,32]; in2=[]; in3=[5,29];
c={[4,5,10,11,7];[8,12,14,17,13,11,10];[13,17,19,22,25,21,16];[14,15,20,19,17];...
 [15,18,23,24,20];[19,20,24,26,22];[22,26,28,31,29,27,25]};
cii={[7,11,13,16],[15,14,12];[21,25,27,30],[28,26,24]};
ciii={[1,4,5],[29,27];[2,8,10,5],[29];[2,8,12,9,6,3],[28]}; civ={[1],[],[28,26,24,23],[25,27]};
% (29)  3_2[4^3]5_2[4^5]_I
clear all; sz=15; nx=6; ny=6; dx=[0,1,2,3,4]; dy=[0,1,2,3,4]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,6;2,3;2,5;3,4;3,5;5,7;5,8;6,7;6,10;7,8;7,10;8,9;8,13;10,12;11,14];
m=[1,3,4,5,3,1,2,3,5,1,5,1,3,4,5]; n=[1,1,1,1,2,3,3,3,3,4,4,5,5,5,5];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15]; ii=[1,4;6,9;10,11;12,15]; iii=[1,12;2,13;3,14;4,15];
in2=[]; in3=[];
c={[1,2,5,7,6];[2,3,5];[3,4,9,8,5];[5,8,7];[6,7,10];[7,8,13,12,10];[8,9,11,14,13]};
cii={}; ciii={}; civ={[1],[3],[11],[]};
% (30)  3_2[4^3]5_2[4^5]_II
clear all; sz=41; nx=3; ny=3; dx=[0,1,2,3,4,5,6,7,8]; dy=[0,1,2,3,4,5,6,7,8];
dim1=max(dx); dim2=max(dy);
q=[1,6;2,3;2,8;3,4;3,13;4,5;4,9;5,9;6,7;6,15;7,8;7,11;8,11;8,12;9,10;9,14;11,12;...
 11,16;12,13;12,22;13,14;13,17;14,17;14,18;15,16;15,20;16,20;16,21;17,18;17,23;...
 18,19;18,27;20,21;21,22;21,29;22,23;22,25;23,25;23,26;24,28;25,26;25,30;26,27;26,35;...
 27,28;27,31;28,31;28,32;29,30;29,33;30,33;30,34;31,32;31,36;32,40;33,34;33,37;34,35;...
 35,36;35,38;36,38;36,39];
m=[1,5,6,8,9,1,3,4,8,9,3,4,6,7,1,2,6,7,9,1,2,4,5,9,4,5,7,8,2,3,7,8,2,3,5,6,1,5,6,8,9];
n=[1,1,1,1,1,2,2,2,2,2,3,3,3,3,4,4,4,4,4,5,5,5,5,5,6,6,6,6,7,7,7,7,8,8,8,8,9,9,9,9,9];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25,26,27,28,...
 29,30,31,32,33,34,35,36,37,38,39,40,41];
ii=[1,5;6,10;15,19;20,24;37,41]; iii=[1,37;2,38;3,39;4,40;5,41]; in2=[29,32]; in3=[7,34];
c={[2,3,13,12,8];[3,4,9,14,13];[4,5,9];[5,10,9];[6,7,11,16,15];[7,8,11];[8,12,11];...
 [9,10,19,18,14];[11,12,22,21,16];[12,13,17,23,22];[13,14,17];[14,18,17];[15,16,20];...
 [16,21,20];[17,18,27,26,23];[18,19,24,28,27];[21,22,25,30,29];[22,23,25];[23,26,25];...
 [25,26,35,34,30];[26,27,31,36,35];[27,28,31];[28,32,31];[29,30,33];[30,34,33];...
 [31,32,40,39,36];[35,36,38];[36,39,38]};
cii={[20,21,29],[32,28];[29,33,37],[40,32]}; ciii={[1,6,7],[34,33];[7,8,2],[35,34]}; civ={};
% (31)  3_3[4^3]5_3[4^5]
clear all; sz=9; nx=6; ny=7; dx=[0,2,3,4,6]; dy=[0,2.5,5]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,6;2,3;2,5;3,4;3,5;5,7;5,8]; m=[1,2,4,5,3,1,2,4,5]; n=[1,1,1,1,2,3,3,3,3];
o=[1,2,3,4,5,6,7,8,9]; ii=[1,4;6,9]; iii=[2,7;3,8;4,9]; in2=[]; in3=[];
c={[1,2,5,7,6];[2,3,5];[5,8,7]}; cii={[1,6],[8,5,3]}; ciii={}; civ={};
% (32)  3_1[4^3]6_2[4^6]_I
clear all; sz=21; nx=3; ny=3; dx=[0,2,3,5,7,8,10];
dy=[0,2,3,5,7,8,10]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,7;2,3;2,7;3,4;3,6;4,5;4,8;6,10;6,11;6,12;7,9;9,10;9,15;10,11;10,14;11,12;...
 11,14;12,13;12,14;14,19;15,17;15,18;16,20];
m=[1,3,4,5,7,4,1,7,1,2,4,6,7,4,1,7,1,3,4,5,7]; n=[1,1,1,1,1,2,3,3,4,4,4,4,4,6,5,5,7,7,7,7,7];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21]; ii=[1,5;7,8;9,13;15,16;17,21];
iii=[1,17;2,18;3,19;4,20;5,21]; in2=[]; in3=[];
c={[1,2,7];[2,3,6,10,9,7];[3,4,8,13,12,6];[4,5,8];[6,11,10];[6,12,11];[9,10,14,19,18,15];...
 [10,11,14];[11,12,14];[12,13,16,20,19,14];[15,18,17]}; cii={[15,17],[20]}; ciii={}; civ={};
% (33)  3_1[4^3]6_2[4^6]_II
clear all; sz=12; nx=6; ny=7; dx=[0,1,3,5,6]; dy=[0,2,3,5]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,8;2,3;2,6;3,4;3,6;4,5;4,6;6,7;7,9;7,10;7,11]; m=[1,2,3,4,5,3,3,1,2,3,4,5];
n=[1,1,1,1,1,2,3,4,4,4,4,4]; o=[1,2,3,4,5,6,7,8,9,10,11,12]; ii=[1,5;8,12];
iii=[1,8;2,9;3,10;4,11;5,12]; in2=[]; in3=[];
c={[1,2,6,7,9,8];[2,3,6];[3,4,6];[7,10,9];[7,11,10]}; cii={[1,8],[11,7,6,4]}; ciii={}; civ={};
% (34)  3_2[4^3]6_4[4^6]
clear all; sz=7; nx=10; ny=7; dx=[0,1,2,4]; dy=[0,3,6]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,6;2,3;2,4;4,5;4,6]; m=[1,3,4,2,1,3,4]; n=[1,1,1,2,3,3,3]; o=[1,2,3,4,5,6,7];
ii=[5,7;1,3]; iii=[1,5;2,6;3,7]; in2=[]; in3=[]; c={[1,2,4];[4,5,6]};
cii={[1,4,5],[6,4,2]}; ciii={}; civ={};
% (35)  3_3[4^3]6_6[4^6]
clear all; sz=11; nx=8; ny=5; dx=[0,1,2,3,4]; y=sqrt(3); dy=y*[0,1,2,3,4]; dim1=max(dx);
dim2=max(dy); q=[1,2;2,3;2,4;3,4;4,6;4,7;5,6;5,8;6,7;6,8;8,9;8,10];
m=[1,3,5,4,1,3,5,2,1,3,5]; n=[1,1,1,2,3,3,3,4,5,5,5]; o=[1,2,3,4,5,6,7,8,9,10,11];
ii=[1,3;5,7;9,11]; iii=[1,9;2,10;3,11]; in2=[]; in3=[]; c={[2,3,4];[4,7,6];[5,6,8];[8,10,9]};
cii={[1,2,4,6,5],[4];[5,8,9],[10,8,6]}; ciii={}; civ={};
% (36)  3_1[4^3]8_4[4^8]
clear all; sz=12; nx=7; ny=7; dx=[0,2,3,5]; dy=[0,2,3,5]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,5;2,3;2,5;3,4;3,6;5,7;7,9;7,10;8,11]; m=[1,2,3,4,1,4,1,4,1,2,3,4];
n=[1,1,1,1,2,2,3,3,4,4,4,4]; o=[1,2,3,4,5,6,7,8,9,10,11,12]; ii=[1,4;5,6;7,8;9,12];
iii=[1,9;2,10;3,11;4,12]; in2=[]; in3=[]; c={[1,2,5];[2,3,6,8,11,10,7,5];[3,4,6];[7,10,9]};
cii={[7,9],[11]}; ciii={}; civ={};
```

```
% (37)  3_1[5^3]4_2[5^4]_I
clear all; sz=8; nx=16; ny=4; dx=[0,1,2]; y=sqrt(3); dy=[0,2,y+2,y+4,2*y+4]; dim1=max(dx);
dim2=max(dy); q=[1,2;1,3;3,4;3,5;4,5;5,6;6,7;6,8]; m=[1,3,1,3,2,2,1,3]; n=[1,1,2,2,3,4,5,5];
o=[1,2,3,4,5,6,7,8]; ii=[1,2;3,4;7,8]; iii=[1,7;2,8]; in2=[5,5;6,6]; in3=[];
c={[1,2,4,3];[3,4,5];[6,8,7]}; cii={[3,5,6,7],[6,5]}; ciii={}; civ={};
% (38)  3_1[5^3]4_2[5^4]
clear all; sz=25; nx=3; ny=3; dx=[0,1,2,3,4]; dy=[0,1,2,3,4]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,6;2,3;2,6;2,7;3,4;3,7;3,8;4,5;4,9;6,7;6,11;7,8;7,12;8,9;8,12;8,13;9,10;9,13;...
 9,14;11,12;11,16;12,13;12,17;13,14;13,18;14,15;14,18;14,19;15,19;16,17;16,21;17,18;...
 17,21;17,22;18,19;18,23;19,20;19,24;20,24];
m=[1,2,3,4,5,1,2,3,4,5,1,2,3,4,5,1,2,3,4,5,1,2,3,4,5];
n=[1,1,1,1,1,2,2,2,2,2,3,3,3,3,3,4,4,4,4,4,5,5,5,5,5];
o=[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24,25];
ii=[1,5;6,10;11,15;16,20;21,25]; iii=[1,21;2,22;3,23;4,24;5,25]; in2=[]; in3=[];
c={[1,2,6];[2,7,6];[2,3,7];[3,8,7];[3,4,9,8];[4,5,10,9];[6,7,12,11];[7,8,12];[8,13,12];...
 [8,9,13];[9,14,13];[9,10,15,14];[11,12,17,16];[12,13,18,17];[13,14,18];[14,19,18];...
 [14,15,19];[15,20,19];[16,17,21];[17,22,21];[17,18,23,22];[18,19,24,23];[19,20,24]};
cii={[16,21],[24]}; ciii={}; civ={};
% (39)  3_2[5^3]4_4[5^4]
clear all; sz=12; nx=4; ny=4; t=37*pi/180; y=cos(t/2)/tan(t/2);
dx=[0,1,2,y+1,y+2,y+3,2*y+2]; dy=[0,1,y,y+1,y+2,2*y+1,2*y+2]; dim1=max(dx); dim2=max(dy);
q=[1,2;1,5;2,4;2,5;3,4;4,6;4,7;5,6;5,8;6,7;6,8;6,9;7,9;8,10;8,11;9,11;9,12];
m=[1,3,7,5,2,4,6,2,5,1,3,7]; n=[1,1,1,2,3,4,4,5,6,7,7,7]; o=[1,2,3,4,5,6,7,8,9,10,11,12];
ii=[1,3;10,12]; iii=[1,10;2,11;3,12]; in2=[5,7;8,7]; in3=[4,9];
c={[1,2,5];[2,4,6,5];[4,7,6];[5,6,8];[6,7,9];[6,9,11,8]};
cii={[1,5],[7,4];[5,8],[7];[8,10],[9,7];[8,11],[12]}; ciii={[2,4],[9];[4,3],[9]}; civ={};
```

§ A.7 Covering lattices

```
% cover contour
clear all; St=sum(100*clock); rand('state',St); CaN=100; X=rand(CaN,2);
[Va,Ca]=voronoin(X); VaN=size(Va,1); Vin=sparse(VaN,1);
for i=1:VaN,
  if((Va(i,1)<=1) & (Va(i,1)>=0) & (Va(i,2)<=1) & (Va(i,2)>=0))
    Vin(i,1)=1;
  end
end
C0=[]; Xn=[]; count=0;
for i=1:CaN,
  TmpN=size(Ca{i},2); Tmp=1;
  for j=1:TmpN,
    if(~Vin(Ca{i}(1,j),1))
      Tmp=0; break;
    end
  end
  if(Tmp)
    count=count+1; C0{count,1}=TmpN; C0{count,2}=Ca{i}; Xn=[Xn;i];
  end
end
CN=size(C0,1); C1=[];
for i=1:CN,
  Tmp=[C0{i,2},C0{i,2}(1,1)];
  for j=1:C0{i,1},
    C1{i,j}(1,1)=(Va(Tmp(1,j),1)+Va(Tmp(1,(j+1)),1))/2;
    C1{i,j}(1,2)=(Va(Tmp(1,j),2)+Va(Tmp(1,(j+1)),2))/2;
  end
end
C{1}=C1; figure(1); clf; hold on;
for i=1:CN,
  x=[]; y=[];
  for j=1:C0{i,1},
    x=[x,C{1}{i,j}(1,1)]; y=[y,C{1}{i,j}(1,2)];
  end
  x=[x,x(1,1)]; y=[y,y(1,1)]; plot(x,y);
end
axis equal;
n=8;
for k=2:n,
  Cn=[];
  for i=1:CN,
    TmpX=[]; TmpY=[];
    for j=1:C0{i,1},
      TmpX=[TmpX,C{k-1}{i,j}(1,1)]; TmpY=[TmpY,C{k-1}{i,j}(1,2)];
    end
    TmpX=[TmpX,TmpX(1,1)]; TmpY=[TmpY,TmpY(1,1)];
    for j=1:C0{i,1},
      Cn{i,j}(1,1)=(TmpX(1,j)+TmpX(1,(j+1)))/2;
      Cn{i,j}(1,2)=(TmpY(1,j)+TmpY(1,(j+1)))/2;
    end
  end
  C{k}=Cn;
end
figure(2); clf; hold on;
for i=1:CN,
  x=[]; y=[];
  for j=1:CO{i,1},
    x=[x,C{n}{i,j}(1,1)]; y=[y,C{n}{i,j}(1,2)];
  end
  x=[x,x(1,1)]; y=[y,y(1,1)]; plot(x,y);
end
axis equal; DEVV=sparse(VaN,VaN);
DVO=[];
for i=1:CN,
  TmpN=CO{i,1}; Tmp=[CO{i,2},CO{i,2}(1,1)];
  for j=1:TmpN,
    V1=Tmp(1,j); V2=Tmp(1,(j+1));
    if(~DEVV(V1,V2))
      dx=Va(V2,1)-Va(V1,1); dy=Va(V2,2)-Va(V1,2);
      TmpA=sqrt(dx*dx + dy*dy); DEVV(V1,V2)=TmpA; DEVV(V2,V1)=TmpA;
    end
    DVO{i,1}{1,1}(1,j)=DEVV(V1,V2); dx=Va(V1,1)-X(Xn(i,1),1);
    dy=Va(V1,2)-X(Xn(i,1),2); TmpA=sqrt(dx*dx + dy*dy); DVO{i,1}{2,1}(1,j)=TmpA;
  end
end
AO=[];
for i=1:CN,
  TmpN=CO{i,1}; Tmp=[DVO{i,1}{2,1},DVO{i,1}{2,1}(1,1)]; At=0;
  for j=1:TmpN,
    a=DVO{i,1}{1,1}(1,j); b=Tmp(1,j); c=Tmp(1,(j+1)); s=(a+b+c)/2;
    Ai=sqrt(s*(s-a)*(s-b)*(s-c)); At=At+Ai;
  end
  AO=[AO;At];
end
DV=[]; A=[];
for i=1:n,
  TmpA=[];
  for j=1:CN,
    TmpN=CO{j,1}; TmpB=[];
    for k=1:TmpN,
      TmpB=[TmpB;C{i}{j,k}];
    end
    TmpB=[TmpB;TmpB(1,:)];
    for k=1:TmpN,
      x1=TmpB(k,1); y1=TmpB(k,2); x2=TmpB((k+1),1); y2=TmpB((k+1),2);
      dx=x2-x1; dy=y2-y1; TmpA{j,1}(1,k)=sqrt(dx*dx + dy*dy);
      dx=x1-X(Xn(j,1),1); dy=y1-X(Xn(j,1),2); TmpA{j,2}(1,k)=sqrt(dx*dx+dy*dy);
    end
  end
  DV{i}=TmpA; TmpA=[];
  for j=1:CN,
    TmpN=CO{j,1}; TmpB=[DV{i}{j,2},DV{i}{j,2}(1,1)]; At=0;
    for k=1:TmpN,
      a=DV{i}{j,1}(1,k); b=TmpB(1,k); c=TmpB(1,(k+1)); s=(a+b+c)/2;
      Ai=sqrt(s*(s-a)*(s-b)*(s-c)); At=At+Ai;
    end
    TmpA=[TmpA;At];
  end
  A{i}=TmpA;
end
p=[sum(AO)];
for i=1:n,
  p=[p,sum(A{i})];
end
p=p*100/p(1); figure(3); clf; plot(0:n,p);
```

§ A.8 Covering contour

```
% gxy.m, cover contour (c) 2002, Kit Tiyapan@UMIST
clear all; St=sum(100*clock); rand('state',St); CaN=120;
X=rand(CaN,2); [Va,Ca]=voronoin(X); VaN=size(Va,1); Vin=sparse(VaN,1);
for i=1:VaN,
  if((Va(i,1)<=1) & (Va(i,1)>=0) & (Va(i,2)<=1) & (Va(i,2)>=0))
    Vin(i,1)=1;
  end
end
CO=[]; Xn=[]; count=0;
for i=1:CaN,
  TmpN=size(Ca{i},2); Tmp=1;
  for j=1:TmpN,
    if(~Vin(Ca{i}(1,j),1))
```

```
      Tmp=0; break;
    end
  end
  if(Tmp)
    count=count+1; C0{count,1}=TmpN; C0{count,2}=Ca{i}; Xn=[Xn;i];
  end
end
CN=size(C0,1); C1=[];
for i=1:CN,
  Tmp=[C0{i,2},C0{i,2}(1,1)];
  for j=1:C0{i,1},
    C1{i,j}(1,1)=(Va(Tmp(1,j),1)+Va(Tmp(1,(j+1)),1))/2;
    C1{i,j}(1,2)=(Va(Tmp(1,j),2)+Va(Tmp(1,(j+1)),2))/2;
  end
end
C{1}=C1;

n=8;
for k=2:n,
  Cn=[];
  for i=1:CN,
    TmpX=[]; TmpY=[];
    for j=1:C0{i,1},
      TmpX=[TmpX,C{k-1}{i,j}(1,1)]; TmpY=[TmpY,C{k-1}{i,j}(1,2)];
    end
    TmpX=[TmpX,TmpX(1,1)]; TmpY=[TmpY,TmpY(1,1)];
    for j=1:C0{i,1},
      Cn{i,j}(1,1)=(TmpX(1,j)+TmpX(1,(j+1)))/2;
      Cn{i,j}(1,2)=(TmpY(1,j)+TmpY(1,(j+1)))/2;
    end
  end
  C{k}=Cn;
end
```

§ A.9 Number of vertices

```
% numofvertices.m
clear all; dimmin=2; dimmax=9; batches=5; dvn=[]; cpu=[];
nmax=1000; rand('state',sum(100*clock));
for i=dimmin:dimmax,
  for j=1:batches,
    n=round(nmax/i); x=rand(n,i); t=cputime; [v,c]=voronoin(x);
    cpu(i,j)=(cputime-t)/n; lend=~floor(v); hend=~(ceil(v)-ones(size(v)));
    lhend=lend & hend; in=min(lhend,[],2); dvn(i,j)=sum(in)/n;
  end
end
dvn=[(1:dimmax)',dvn]; dvn=dvn(2:dimmax,:); figure(1); clf;
for i=1:batches,
    semilogy(dvn(:,1),dvn(:,(i+1)),'.','LineWidth',2); hold on;
end
edvn=[dvn(:,1),sum(dvn(:,2:(batches+1)),2)/batches]; tmp=edvn(:,2)./exp(edvn(:,1));
A=sum(tmp)/(dimmax-1); m=[dimmin,dimmax]; semilogy(m,A*exp(m));
cpu=[(1:dimmax)',cpu]; cpu=cpu(2:dimmax,:); figure(2); clf;
for i=1:batches,
    semilogy(cpu(:,1),cpu(:,(i+1)),'.','LineWidth',2); hold on;
end
ecpu=[cpu(:,1),sum(cpu(:,2:(batches+1)),2)/batches]; tmp=ecpu(:,2)./exp(ecpu(:,1));
B=sum(tmp)/(dimmax-1); m=[dimmin,dimmax]; semilogy(m,(B/35)*(exp(1)+2).^m);
```

§ A.10 Vertices per cell and cell ratio

```
% numveachcell.m
clear all; dimmin=2; dimmax=6; batches=5; nmax=3000;
rand('state',sum(100*clock));
for i=dimmin:dimmax,
  for j=1:batches,
    n=round(nmax*2/i); x=rand(n,i); [v,c]=voronoin(x);
    fleet{i,j,1}=v; fleet{i,j,2}=c; fleet{i,j,3}=n;
  end
end
for i=dimmin:dimmax,
  for j=1:batches,
    v=fleet{i,j,1}; c=fleet{i,j,2}; n=fleet{i,j,3}; lend=~floor(v);
    hend=~(ceil(v)-ones(size(v))); lhend=lend & hend; in=min(lhend,[],2);
    numvc=[]; vcin=[];
    for p=1:n,
      numvc=[numvc,size(c{p},2)]; flag=1;
      for q=1:numvc(p),
        if(~in(c{p}(q)))
          flag=0; break;
        end
      end
      if(flag)
        vcin=[vcin,numvc(p)];
      end
    end
    tmpn=size(vcin,2); rcin(i,j)=tmpn/n; vec(i,j)=sum(vcin)/tmpn;
  end
end
dum=rcin; str={'c_{in} / c_{all}'}; dum=dum(2:dimmax,:); tmp=[];
for i=dimmin:dimmax,
  tmp=[tmp;i*ones(batches,1),dum((i-1),:)'];
end
figure(1); semilogy(tmp(:,1),tmp(:,2),'.','LineWidth',2); hold on;
[p,s,mu]=polyfit(tmp(:,1),tmp(:,2),4); x=(dimmin:.02:dimmax)';
y=polyval(p,x,[],mu); semilogy(x,y); dum=vec; dum=dum(2:dimmax,:); tmp=[];
for i=dimmin:dimmax,
  tmp=[tmp;i*ones(batches,1),dum((i-1),dimmin:dimmax)'];
end
figure(2); semilogy(tmp(:,1),tmp(:,2),'.','LineWidth',2); hold on;
edum=[dum(:,1),sum(dum(:,2:(batches+1)),2)/batches];
tmp=edum(:,2)./exp(edum(:,1)); A=sum(tmp)/(dimmax-1); m=[dimmin,dimmax];
semilogy(m,(A/70)*(exp(1)+4).^m); xlabel('Dimension','FontSize',14);
ylabel(str,'FontSize',14);
```

§ B.5 TEXnicalities

```
% thshead.tex, Kit Tiyapan 15 October 2003
\def\Ordinate{\ifnum\day>30 1 \else\ifnum\day>20 \day-20 \else\day\fi\fi}
\def\date{{\number\day\/{\ifcase\Ordinate\or $^{st}$\or $^{nd}$\or $^{rd}$
 \else $^{th}$\fi}} {\ifcase\month\or January\or February\or March\or April\or
 May\or June\or July\or August\or September\or October\or November\else
 December\fi}, {\number\year}}
\def\dat[#1:#2:#3]{\begingroup\tmp=#1 \ifnum\tmp>30 \tmp=1 \else\ifnum\tmp>20
 \advance\tmp-20 \fi\fi
 {#1\/{\ifcase\tmp\or $^{st}$\or $^{nd}$\or $^{rd}$ \else $^{th}$\fi}} #2 #3
 \endgroup}
\input manmac  % \tracingall
% \proofmodefalse
\ifproofmode\else\overfullrule=0pt\fi \hsize=6in  \vsize=9.70820393249937in
\maxdepth=2pt \parindent=2pc \pagewidth=\hsize \pageheight=\vsize
\font\titlefont=cmbx10 at 15pt \font\inchhigh=cminch at 20pt
\input epsf \input rotate \input ukhyphen \def\home{/home/mjkpjkt2}
\def\dry#1{\ifcase#1\or \home/ar/thes1\or \home/ts\or
 \home/tiyapan/Extremum/DsgnExReport\or \home/xfg\or \home/ths\or
 \home/tiyapan/Perco/Papers\or \home/tiyapan/Extremum/Matlab\or
 \home/ar/rc1\or \home/ar/trans\or \home/ar/wn21\or \home/ar/wn23\or
 \home/ar/wn7\or \home/phy\or \home/rds\or \home/obj\or \home/voy\or
 \home/cum\or \home/flt\or \home/pcl\or \home/stp\or \home/cnm\fi}
\def\o[#1:#2]{$#1^{\hbox{\sevenrm #2}}$}
\def\Title{Ph.D. Thesis, UMIST. K N Tiyapan.}
\font\tenbm=cmmib10 \font\ninett=cmtt9 \font\sixit=cmti6 \font\elevenbf=cmbx11
\font\twelvebf=cmbx12 \font\elevenex=cmex11 \font\eleveni=cmmi11
\font\elevenit=cmti11 \font\elevenrm=cmr11 \font\twelverm=cmr12
\font\elevensl=cmsl11 \font\elevensy=cmsy11 \font\eleventt=cmtt11
\def\elevenbig#1{{\hbox{$\left#1\vbox to9.35pt{}\right.\n@space$}}}
\def\elevenpoint{\def\rm{\fam0\elevenrm}
  \textfont0=\elevenrm \scriptfont0=\sevenrm \scriptscriptfont0=\fiverm
  \textfont1=\eleveni \scriptfont1=\seveni \scriptscriptfont1=\fivei
  \textfont2=\elevensy \scriptfont2=\sevensy \scriptscriptfont2=\fivesy
  \textfont3=\elevenex \scriptfont3=\tenex \scriptscriptfont3=\tenex
  \def\it{\fam\itfam\elevenit} \textfont\itfam=\elevenit
  \def\sl{\fam\slfam\elevensl} \textfont\slfam=\elevensl
  \def\bf{\fam\bffam\elevenbf} \textfont\bffam=\elevenbf
  \scriptfont\bffam=\sevenbf \scriptscriptfont\bffam=\fivebf
  \def\tt{\fam\ttfam\eleventt} \textfont\ttfam=\eleventt
  \tt \ttglue=.5em plus.25em minus.15em \normalbaselineskip=12.5pt
  \def\MF{{\manual META}\-{\manual FONT}} \let\sc=\eightrm \let\big=\elevenbig
  \setbox\strutbox=\hbox{\vrule height8.8pt depth3.8pt width\z@}
  \normalbaselines\rm}
\def\[#1]{{\it #1}} \def\(#1){{\bf #1}} \def\r[#1]{{\rm #1}}
  \def\s[#1]{{\sl #1}}
\def\beginsection#1\par{\vskip\z@ plus.05\vsize\penalty-10 % 250
 \vskip\z@ plus-.05\vsize\bigskip\vskip\parskip
 \message{#1}\leftline{\bf#1}\smallskip\noindent}
\def\ifundefined#1{\expandafter\ifx\csname#1\endcsname\relax}
\def\ix[#1](#2){\expandafter\def\csname#1\endcsname{#2}}
\def\x[#1]{\ifundefined{#1}\ix[#1]($\aleph$)\fi \csname#1\endcsname}
```

```
\def\ipno[#1]{\ifundefined{#1p}\ix[#1p](1)\fi}
\def\pno[#1]{\immediate\write\ref{\string\ix[#1p](\number\pageno)}}
\input ref \newwrite\ref \immediate\openout\ref=ref
\newdimen\dtmp \newdimen\dtmpi \newdimen\dtmpii \newcount\tmp
\def\Bf#1:{{\bf #1}} \def\ct{\hfil\par} \def\cut{\par\vfill\break}
\def\der[#1]{{\bf #1},} \def\hd#1:{\hfil{\elevenrm #1}\hfil}
\def\It#1:{{\it#1}} \def\pc{\hbox{Pc}}  \def\Sl#1:{{\sl#1}}
\def\w#1:{\dtmpi=\hsize \advance\dtmpi-.3em \dtmpii=\dtmpi
  \advance\dtmpii-.3em \parshape=2 0pt \dtmpi .3em \dtmpii {\tenpoint #1.~}}
\def\vws[#1]#2:#3:{{\bf #1} ({\sl #2} in {\sl #3})}  % explain vowel sounds
\def\Lift#1#2{\raise#1em\hbox{#2}}
\def\beginindex{\begingroup \parindent=1em \maxdepth=\maxdimen
 \def\par{\endgraf \futurelet\next\inxentry} \obeylines
 \everypar={\hangindent 2\parindent} \exhyphenpenalty=10000 \raggedright}
\def\inxentry{\ifx\next\sub \let\next=\subentry \else\ifx\next\endindex
 \let\next=\vfill \else\let\next=\mainentry \fi\fi \next}
\def\endindex{\mark{}\break\endgroup}
\let\sub=\indent \newtoks\maintoks \newtoks\subtoks
\def\mainentry#1,{\mark{}\noindent \maintoks={#1}\mark{\the\maintoks}#1,}
\def\subentry\sub#1,{\mark{\the\maintoks}\indent
 \subtoks={#1}\mark{\the\maintoks\sub\the\subtoks}#1,}
\def\boxit#1{\vbox{\hrule\hbox{\vrule\kern3pt
  \vbox{\kern3pt#1\kern3pt}\kern3pt\vrule}\hrule}}
\def\bxt<#1>{\vbox{\hrule\hbox{\vrule\kern.7pt
 \vbox{\kern.7pt#1\kern.7pt}\kern.7pt\vrule}\hrule}}
\def\square#1#2{{\vcenter{\vbox{\hrule height.#2pt \hbox{\vrule width.#2pt
 height#1pt \kern#1pt \vrule width.#2pt \hrule height.#2pt}}}}
\def\sqr{\mathchoice\square34\square34\square{2.1}3\square{1.5}3}
\def\Le{\;{\raise.3em\hbox{$<$}}{\kern-.8em\lower.3em\hbox{$=$}}\;}
\def\Ge{\;{\raise.3em\hbox{$>$}}{\kern-.8em\lower.3em\hbox{$=$}}\;}
\def\matrix#1{\null\,\vcenter{\normalbaselines\m@th
    \ialign{\hfil$##$\hfil&&\quad\hfil$##$\hfil\crcr
      \mathstrut\crcr\noalign{\kern-\baselineskip}
      #1\crcr\mathstrut\crcr\noalign{\kern-\baselineskip}}}\,}
\def\pmatrix#1{\left(\matrix{#1}\right)}
\def\Cgy{\mathop{\cal G}\nolimits}
\def\Cov[#1]{\hbox{${\rm C}\sp{#1}({\cal V})$}}
\def\Gl{\mathop{\raise.23em\hbox{$>$}\kern-.71em\lower.23em\hbox{$<$}}}
\def\Mod{\mathop{\rm mod}} \def\Lift#1#2{\raise#1em\hbox{#2}}
\def\uncatcodespecials{\def\do##1{\catcode`##1=12 }\dospecials}
\newcount\lineno
\def\setupverbatim{\tt \lineno=0 \def\par{\leavevmode\endgraf}
 \catcode`\`=\active \obeylines \uncatcodespecials \obeyspaces
 \everypar{\advance\lineno by1 \llap{\sevenrm\the\lineno\ \ }}}
 {\catcode`\`=\active \gdef|{\char'174}\gdef`{\relax\lq}\obeyspaces
 \global\let =\ }
\def\listing#1{\par\smallskip\begingroup\setupverbatim\ninett\baselineskip.9em
 \input#1 \endgroup}
\def\tran#1{\begingroup\ninepoint\baselineskip.8em\parindent=1em\input#1
 \endgroup}
\def\picx#1:#2:{\epsfxsize=#1 \epsffile{#2.eps}}
\def\picy#1:#2:{\epsfysize=#1 \epsffile{#2.eps}}
\def\nin#1:{\begingroup\baselineskip1em\leftskip1em\parindent-1em
  \def\net##1:##2:##3:{{\tt ##1}{\it ##2}##3\par}\input #1\endgroup}
\def\bib#1:{\begingroup\baselineskip1em\leftskip1em\parindent-1em\hbox{}
 \input #1\endgroup}
\def\art#1:#2:#3:#4:{{\rm #1}~#2~{\it #3} #4\par\smallskip}
 \newcount\kpc \newcount\kpd \kpd=0 \hxdc(\romannumeral\the\kpc)}})
 {\rm #1}~#3~{\it #4} #5 \x[#2].\par}\x[#2].\par}
\def\kart#1[#2]#3:#4:#5:{\ifnum\kpc>0\else\global\advance\kpd by 1\fi
 \global\advance\kpc by 1 \immediate\write\ref{\string\ix[#2](
 {{\sevenrm KNT\number\kpd(\romannumeral\the\kpc)}})} {\rm #1}~#3~{\it #4} #5
 \x[#2].\par}
\def\bbib{\begingroup\baselineskip1em\leftskip1em\parindent-1em}
\def\ebib{\endgroup}
\def\hxdc{{\count0=\count1 \divide\count1 by16
  \ifnum\count1>0 \hxdc\fi \count2=\count1 \multiply\count2 by-16
  \advance\count0 by\count2 \hexdigit}}
\def\hexdigit{\ifnum\count0<10 \number\count0
  \else\advance\count0 by-10 \advance\count0 by`A \char\count0 \fi}
% \pictri <file1>:<file2>:<file3>:<num1>:<num2>:<num3>:
\def\pictri#1:#2:#3:#4:#5:#6:{\smallskip
  \centerline{\vbox{\dtmp=\hsize \advance\dtmp-1em
  \dtmpi=\hsize \advance\dtmpi-2em \divide\dtmpi by3
  \halign to\dtmp{\hfil##\hfil&\hfil##\hfil&\hfil##\hfil\cr
   \epsfxsize=\dtmpi \epsffile{#1.eps} &\epsfxsize=\dtmpi \epsffile{#2.eps} &
   \epsfxsize=\dtmpi \epsffile{#3.eps} \cr (#4) &(#5) &(#6)\cr}}}}
% \picqua <file1>:<file2>:<file3>:<file4>:<num1>:<num2>:<num3>:<num4>:
\def\picqua#1:#2:#3:#4:#5:#6:#7:#8:{\smallskip
 \centerline{\vbox{\dtmp=\hsize \advance\dtmp-1em
  \dtmpi=\hsize \advance\dtmpi-2em \divide\dtmpi by4
  \halign to\dtmp{\hfil##\hfil&\hfil##\hfil&\hfil##\hfil&\hfil##\hfil\cr
  \epsfxsize=\dtmpi \epsffile{#1.eps} &\ifx0#6 \else\epsfxsize=\dtmpi
  \epsffile{#2.eps}\fi &\ifx0#7 \else\epsfxsize=\dtmpi \epsffile{#3.eps}\fi &
  \ifx0#8 \else\epsfxsize=\dtmpi \epsffile{#4.eps}\fi\cr
  (#5) &\ifx0#6 \else(#6)\fi &\ifx0#7 \else(#7)\fi &\ifx0#8 \else(#8)\fi\cr}}}}
% \picpent <file1>:<file2>:<file3>:<file4>:<file5>:<num>:
\def\picpent#1:#2:#3:#4:#5:#6:{\smallskip
 \centerline{\vbox{\dtmp=\hsize \advance\dtmp-1em \dtmpi=\hsize
 \advance\dtmpi-2em \divide\dtmpi by5 \halign to\dtmp{\hfil##\hfil&
 \hfil##\hfil&\hfil##\hfil&\hfil##\hfil&\hfil##\hfil\cr
 \epsfxsize=\dtmpi \epsffile{#1.eps} &\ifx0#2 \else\epsfxsize=\dtmpi
 \epsffile{#2.eps}\fi &\ifx0#3 \else\epsfxsize=\dtmpi \epsffile{#3.eps}\fi &
 \ifx0#4 \else\epsfxsize=\dtmpi \epsffile{#4.eps}\fi &\ifx0#5 \else
 \epsfxsize=\dtmpi \epsffile{#5.eps}\fi\cr
 (a#6) &\ifx0#2 \else(b#6)\fi &\ifx0#3 \else(c#6)\fi &\ifx0#4 \else
 (d#6)\fi &\ifx0#5 \else(e#6)\fi\cr}}}}
\def\picsept#1:#2:#3:#4:#5:#6:#7:#8:{\smallskip
 \centerline{\vbox{\dtmp=\hsize \advance\dtmp-1em \dtmpi=\hsize
 \advance\dtmpi-4em \divide\dtmpi by7
 \halign to\dtmp{\hfil##\hfil&\hfil##\hfil&\hfil##\hfil&\hfil##\hfil&
 \hfil##\hfil&\hfil##\hfil&\hfil##\hfil\cr
 \epsfxsize=\dtmpi \epsffile{#8#1.eps} &\epsfxsize=\dtmpi \epsffile{#8#2.eps} &
 \epsfxsize=\dtmpi \epsffile{#8#3.eps} &\epsfxsize=\dtmpi \epsffile{#8#4.eps} &
 \epsfxsize=\dtmpi \epsffile{#8#5.eps} &\epsfxsize=\dtmpi \epsffile{#8#6.eps} &
 \epsfxsize=\dtmpi \epsffile{#8#7.eps} \cr}}}}
\def\picsepu#1:#2:#3:#4:{\smallskip\centerline{\vbox{\dtmp=\hsize
 \advance\dtmp-1em\dtmpi=\hsize \advance\dtmpi-4em \divide\dtmpi by7
 \halign to\dtmp{##\hfil&\hfil##\hfil&\hfil##\cr \epsfxsize=\dtmpi
 \epsffile{#4#1.eps} &\epsfxsize=\dtmpi \epsffile{#4#2.eps} &
 \epsfxsize=\dtmpi \epsffile{#4#3.eps} \cr}}}}
% \pclr <dim_x>:<filename>:<directory>:<index>:<caption>:
\def\pclr#1:#2:#3:#4:#5:{\par\begingroup\dtmp=\hsize \advance\dtmp-1em
 \dtmpi=\dtmp \advance\dtmpi-#1\advance\dtmpi-1em \smallskip \fig[#4]#5:
 \def\pcl{\centerline{\vbox{\hsize=\dtmp \hbox{\epsfxsize=#1
 \epsffile{#3#2.eps}\quad\hfil\vbox{\hsize=\dtmpi\noindent
 {\bf Figure \x[#4]} #5\smallskip}}}}}\centre{}
 \def\pcr{\centerline{\vbox{\hsize=\dtmp \hbox{\vbox{\hsize=\dtmpi\noindent
 {\bf Figure \x[#4]} #5\smallskip} \quad\hfil
 \epsfxsize=#1 \epsffile{#3#2.eps}}}}} \ipno[#4]
 \ifodd\csname#4p\endcsname \pcr \else \pcl \fi\ct\smallskip\endgroup\pno[#4]}
% \pplr <dim_x>:<filename>:<directory>:<index>:<caption>:<paragraph>:
\def\pplr#1:#2:#3:#4:#5:#6:{\par\begingroup\dtmp=\hsize \advance\dtmp-1em
 \dtmpi=\dtmp \advance\dtmpi-#1\advance\dtmpi-1em\smallskip \fig[#4]#5:
 \def\pcl{\centerline{\vbox{\hsize=\dtmp \hbox{\epsfxsize=#1
 \epsffile{#3#2.eps}\quad\hfil \vbox{\hsize=\dtmpi\parindent=0pt#6\hfil
 \smallskip\noindent{\bf Figure \x[#4]} #5.\smallskip}}}}}
 \def\pcr{\centerline{\vbox{\hsize=\dtmp \hbox{\vbox{\hsize=\dtmpi\noindent
 #6\hfil\smallskip\noindent{\bf Figure \x[#4]} #5.\smallskip} \quad\hfil
 \hbox{\epsfxsize=#1 \epsffile{#3#2.eps}}}}}}
 \ipno[#4] \ifodd\csname#4p\endcsname \pcr \else \pcl \fi\ct\smallskip
 \endgroup\pno[#4]}
\def\plr#1:#2:#3:#4:#5:#6:{\par\begingroup\dtmp=\hsize \advance\dtmp-1em
 \dtmpi=\dtmp\advance\dtmpi-#1 \advance\dtmpi-1em \smallskip \fig[#4]#5:
 \def\pcl{\centerline{\vbox{\hsize=\dtmp \hbox{\epsfxsize=#1
 \epsffile{#3#2.eps}\quad\hfil\vbox{\hsize=\dtmpi\parindent=0pt #6\hfil
 \smallskip{\bf Figure \x[#4].} #5\smallskip}}}}}
 \def\pcr{\centerline{\vbox{\hsize=\dtmp\hbox{\vbox{\hsize=\dtmpi\parindent=0pt
 #6\hfil\smallskip{\bf Figure \x[#4].} #5\smallskip} \quad\hfil
 \hbox{\epsfxsize=#1 \epsffile{#3#2.eps}}}}}}\ipno[#2]
 \ifodd\csname#2p\endcsname \pcr \else \pcl \fi\ct\smallskip\endgroup\pno[#2]}
\def\plrx#1:#2:#3:#4:#5:#6:{\begingroup\dtmp=\hsize \advance\dtmp-1em
 \dtmpi=\dtmp\advance\dtmpi-#1 \advance\dtmpi-1em \smallskip
 \def\pcl{\centerline{\vbox{\hsize=\dtmp \hbox{\epsfxsize=#1
 \epsffile{#3#2.eps}\quad\hfil\vbox{\hsize=\dtmpi\noindent{#6\hfil\smallskip
 \bf #4.} #5\smallskip}}}}} \def\pcr{\centerline{\vbox{\hsize=\dtmp
 \hbox{\vbox{\hsize=\dtmpi\noindent#6\hfil\smallskip{\bf #4.} #5\smallskip}
 \quad\hfil\hbox{\epsfxsize=#1 \epsffile{#3#2.eps}}}}}} \ipno[#2]
 \ifodd\csname#2p\endcsname \pcr \else \pcl \fi\ct\smallskip\endgroup\pno[#2]}
\def\plrs#1:#2:#3:#4:#5:#6:{\begingroup\dtmp=\hsize \advance\dtmp-1em
 \dtmpi=\dtmp\advance\dtmpi-#1 \advance\dtmpi-1em \smallskip \fig[#4]#5:
 \def\pcl{\centerline{\vbox{\hsize=\dtmp \hbox{\epsfxsize=#1 \epsffile{#3#2.ps}
 \quad\hfil \vbox{\hsize=\dtmpi\noindent{#6\hfil\smallskip\bf #4.}
 #5\smallskip}}}}}\def\pcr{\centerline{\vbox{\hsize=\dtmp
 \hbox{\vbox{\hsize=\dtmpi\noindent#6\hfil\smallskip{\bf #4.} #5\smallskip}
 \quad\hfil\hbox{\epsfxsize=#1 \epsffile{#3#2.ps}}}}}} \ipno[#2]
 \ifodd\csname#2p\endcsname \pcr \else \pcl \fi\ct\smallskip\endgroup\pno[#2]}
% \twop:<xsize1>:<file1>:<label1>:<xsize2>:<file2>:<label2>:
\def\twop:#1:#2:#3:#4:#5:#6:{\centerline{\hfil\epsfxsize=#1 \epsffile{#2.eps}
 \hfil\epsfxsize=#4 \epsffile{#5.eps}\hfil}
 \centerline{\hfil #3\hfil\hfil\hfil #6\hfil}}
\def\tps:#1:#2:#3:#4:#5:#6:{\centerline{\hfil\epsfxsize=#1 \epsffile{#2.ps}
 \hfil\epsfxsize=#4 \epsffile{#5.ps}\hfil}\centerline{\hfil #3\hfil\hfil\hfil
 #6\hfil}}
```

```
\def\twap:#1:#2:#3:#4:{\begingroup\dtmp=\hsize \divide\dtmp by2
 \advance\dtmp-1em \centerline{\hfil\epsfxsize=\dtmp \epsffile{#1.eps}\q
 \epsfxsize=\dtmp \epsffile{#3.eps}\hfil}
 \centerline{\hfil #2\hfil\hfil\hfil #4\hfil}\endgroup}
% \ppr<directory>:<fig1>:<fig1d>:<fig2>:<fig2d>{<capt1>}{<capt2>}
\def\ppr#1:#2:#3:#4:#5:#6#7{\begingroup\dtmp=\hsize \advance\dtmp-3em
 \dtmpi=\dtmp\divide\dtmpi by 4 \smallskip\parindent=0pt \dtmpii=0pt
 \setbox1=\vbox{\hsize=2\dtmpi #6} \setbox2=\vbox{\hsize=2\dtmpi #7}
 \ifdim\dtmpii<\ht1 \dtmpii=\ht1\fi \ifdim\dtmpii<\ht2 \dtmpii=\ht2\fi
 \centerline{\epsfxsize=\dtmpi \epsffile{#1#2.eps}~\epsfxsize=\dtmpi
 \epsffile{#1#3.eps}\hfil\epsfxsize=\dtmpi \epsffile{#1#4.eps}~
 \epsfxsize=\dtmpi \epsffile{#1#5.eps}}\centerline{\vbox to\dtmpii
 {\hsize=\dtmpi\unvbox1\vfil}\hfil\vbox to\dtmpii{\hsize=\dtmpi\unvbox2\vfil}}
 \smallskip\endgroup}
\def\prlr#1:#2:#3:#4{\begingroup\dtmp=\hsize \advance\dtmp-3em \dtmpi=\dtmp
 \divide\dtmpi by 4\dtmp=\dtmpi \advance\dtmp1ex \dtmpii=\dtmpi
 \advance\dtmpii-1ex \smallskip\parindent=0pt \setbox1=\vbox{\hsize=2\dtmp
 \hfil\epsfxsize=\dtmpi \epsffile{#1#2.eps}~\epsfxsize=\dtmpi
 \epsffile{#1#3.eps}\hfil}\setbox2=\vbox{\hsize=2\dtmpii #4\smallskip}
 \setbox3=\vbox{\hsize=2\dtmpii \hfill#4\smallskip}
 \def\pcl{\centerline{\copy1\hfil\copy3}} \def\pcr{\centerline{\copy2\hfil
 \copy1}} \ipno[#3] \ifodd\csname#3p\endcsname \pcr \else \pcl \fi\ct\endgroup
 \pno[#3]}
\def\ptlr#1:#2:#3:#4:#5{\begingroup\dtmp=\hsize \advance\dtmp-3em \dtmpi=\dtmp
 \divide\dtmpi by 4\dtmp=\dtmpi \advance\dtmp1ex \dtmpii=\dtmpi
 \advance\dtmpii-1ex\smallskip\parindent=0pt \setbox1=\vbox{\hsize=3\dtmp
 \epsfxsize=\dtmpi \epsffile{#1#2.eps}~\epsfxsize=\dtmpi \epsffile{#1#3.eps}
 ~\epsfxsize=\dtmpi \epsffile{#1#4.eps}}\setbox2=\vbox{\hsize=\dtmpii #5
 \smallskip}\setbox3=\vbox{\hsize=\dtmpii \hfill#5\smallskip}
 \def\pcl{\centerline{\copy1\hfil\copy3}} \def\pcr{\centerline{\copy2\hfil
 \copy1}}\ipno[#4] \ifodd\csname#4p\endcsname \pcr \else \pcl
 \fi\ct\endgroup\pno[#4]}
\def\capt#1:#2:#3:{\nobreak\smallskip\centerline{\dtmp=\hsize \advance\dtmp-2em
 \vbox{\hsize \dtmp \noindent{\bf #1 #2} {\it #3}}}\smallskip}
\def\balg{\begingroup\smallskip\begingroup\obeylines \sfcode`;=3000
 \def\b[##1]{{\bf ##1}} \def\i[##1]{{\it ##1}} \def\r[##1]{{\rm ##1}}}
\def\ealg{\par\endgroup\endgroup\smallskip}
\def\q{\quad} \def\qq{\qquad} \def\qqq{\qq\q} \def\qqqq{\qq\qq}
 \def\qqqqq{\qq\qq\q} \def\qqqqqq{\qq\qq\qq}
\def\btoc{\begingroup\medskip\begingroup\parindent=0pt\obeylines
 \def\atpg{\dotfill}}
\def\etoc{\endgroup\endgroup\medskip}
\def\bdsc#1\edsc{\begingroup\smallskip\dtmp=\hsize \advance\dtmp-3em
  \centerline{\hfil\boxit{\vbox{\hsize=\dtmp #1}}\hfil}\par\endgroup\smallskip}
\def\picl#1:#2:#3:#4:{\dtmp=\hsize \advance\dtmp-1em \dtmpi=\dtmp
 \advance\dtmpi-#1 \smallskip \centerline{\vbox{\halign to\dtmp{##\hfil&\hfil##
 \cr\epsfxsize=#1 \epsffile{#2.eps}&\vbox{\hsize=\dtmpi\noindent
 {\bf Figure \x[#3]} #4\smallskip}\cr}}}\smallskip}
\def\tbv#1:#2:#3:#4:#5:{\begingroup\smallskip\parindent0pt\
 centerline{\vbox{\dtmp=\hsize \advance\dtmp-2em \dtmpi=\dtmp \divide\dtmpi by5
 \advance\dtmpi-.5em \def\h[##1]{\hskip.3em\sl##1}
 \setbox1=\vbox{\hsize=\dtmpi#1} \setbox2=\vbox{\hsize=\dtmpi#2}
 \setbox3=\vbox{\hsize=\dtmpi#3} \setbox4=\vbox{\hsize=\dtmpi#4}
 \setbox5=\vbox{\hsize=\dtmpi#5} \dtmpii=0pt\ifdim\dtmpii<\ht1 \dtmpii=\ht1\fi
 \ifdim\dtmpii<\ht2 \dtmpii=\ht2\fi \ifdim\dtmpii<\ht3 \dtmpii=\ht3\fi
 \ifdim\dtmpii<\ht4 \dtmpii=\ht4\fi \ifdim\dtmpii<\ht5 \dtmpii=\ht5\fi
 \halign to\dtmp{##\hfil&##\hfil&##\hfil&##\hfil&##\hfil\cr
 \vbox to\dtmpii{\hsize=\dtmpi\unvbox1\vfil} &\vbox to\dtmpii{\hsize=\dtmpi
 \unvbox2\vfil} &\vbox to\dtmpii{\hsize=\dtmpi\unvbox3\vfil} &\vbox to
 \dtmpii{\hsize=\dtmpi\unvbox4\vfil} &\vbox to\dtmpii{\hsize=\dtmpi\unvbox5
 \vfil}\cr}}}\endgroup}
\def\tbiv#1:#2:#3:#4:{\begingroup\smallskip\parindent0pt
 \centerline{\vbox{\dtmp=\hsize \advance\dtmp-2em \dtmpi=\dtmp
 \divide\dtmpi by4 \advance\dtmpi-.5em \def\h[##1]{\hskip.3em\sl##1}
 \setbox1=\vbox{\hsize=\dtmpi#1} \setbox2=\vbox{\hsize=\dtmpi#2}
 \setbox3=\vbox{\hsize=\dtmpi#3} \setbox4=\vbox{\hsize=\dtmpi#4} \dtmpii=0pt
 \ifdim\dtmpii<\ht1 \dtmpii=\ht1\fi \ifdim\dtmpii<\ht2 \dtmpii=\ht2\fi
 \ifdim\dtmpii<\ht3 \dtmpii=\ht3\fi \ifdim\dtmpii<\ht4 \dtmpii=\ht4\fi
 \ifdim\dtmpii<\ht5 \dtmpii=\ht5\fi \halign to\dtmp{##\hfil&##\hfil&##\hfil&##
 \hfil\cr\vbox to\dtmpii{\hsize=\dtmpi\unvbox1\vfil} &
 \vbox to\dtmpii{\hsize=\dtmpi\unvbox2\vfil} &
 \vbox to\dtmpii{\hsize=\dtmpi\unvbox3\vfil} &
 \vbox to\dtmpii{\hsize=\dtmpi\unvbox4\vfil}\cr}}}\endgroup}
\def\tabhead{\def\erule{\vskip.2em\hrule\vskip.2em}
 \def\ptl{\quad} \def\ptr{\hfil\quad}}
\def\trule{\noalign{\vskip.2em\hrule\vskip.2em}}
\def\tbl#1\lbt{\begingroup\dtmp=\hsize \advance\dtmp-3em \smallskip
 \centerline{\vbox{\hsize=\dtmp \halign{##\hfil&&\quad##\hfil&\quad##\hfil\cr
 #1}}}\endgroup}
\def\fgr[#1](#2)#3\rgf{\begingroup\dtmp=\hsize \divide\dtmp by#1
 \advance\dtmp-1em\def\pic[##1]{\epsfxsize=\dtmp \epsffile{#2##1.eps}}
 \centerline{\vbox{\halign{##&&\hfil##\hfil&##\cr #3\cr}}}\endgroup}
```

```
\def\fgl[#1]#2\lgf{\begingroup\dtmp=\hsize \divide\dtmp by#1 \advance\dtmp-1em
  \def\lab[##1]{\hbox to\dtmp{\hfil##1\hfil}}
 \centerline{\vbox{\halign{##&&\hfil##\hfil&##\cr #2\cr}}}\endgroup}
% \Pp<lattice>:<Pc type>:<number\(<order>),...>:<Pc prob.>:<std. error>:
\def\Pp#1:#2:#3:#4:#5:{\begingroup \def\(##1){({\bf##1})}
 \def\Vii{Vn 2d} \def\Viii{Vn 3d} \def\Viis{Vn (2,3)s}
 ${}^{\hbox{\sixit #1}}_{\scriptscriptstyle #3}p_#2$ $=$ $#4\pm #5$\endgroup}
\def\X#1:#2:#3:#4:#5:{\begingroup \def\(##1){({\bf##1})} \def\Vii{Vn 2d}
 \def\Viii{Vn 3d} \def\Viis{Vn (2,3)s} \ifx0#5
 ${}^{\hbox{\sixit #1}}_{\scriptscriptstyle #3}x_#2$ $=$ $#4$
 \else${}^{\hbox{\sixit #1}}_{\scriptscriptstyle #3}x_#2$ $=$ $#4\pm #5$\fi
 \endgroup}
\def\din#1.#2.#3.#4.{\begingroup\def\der[##1]{{\bf ##1},} % #1 \latin, #2
 \tenbf , #3 \ninepoint\def\w##1:{\dtmpi=\hsize \advance\dtmpi-.3em
 \dtmpii=\dtmpi \advance\dtmpii-.3em\parshape=2 0pt \dtmpi .3em \dtmpii
 {#2 ##1.~}} \def\Bf[##1]{{\bf##1}} \def\(##1){{\it##1}} \def\Sl[##1]{{\sl##1}}
 \def\[##1]{{\tt##1}} #1\dtmp=\hsize \begindoublecolumns\hsize=\dtmp
 \divide\hsize by2 \advance\hsize by-2em \baselineskip.9em\parindent0pt #3
 \input #4 \enddoublecolumns\endgroup}
\long\def\t[#1]{\def\next{#1}{\tt\frenchspacing\expandafter\strip
 \meaning\next}} \def\strip#1>{}
\def\idx[#1]{\beginindex\input #1\endindex}
\def\tchd{\def\leftheadline{\hbox to \pagewidth{\vbox to 10pt{}
 \llap{\tenbf\romannumeral\folio\kern1pc} \tenit\rhead\hfil}}
 \def\rightheadline{\hbox to \pagewidth{\vbox to 10pt{} \hfil\tenit\rhead\/
 \rlap{\kern1pc\tenbf\romannumeral\folio}}}}
\newbox\partialpage
\def\begindoublecolumns{\dtmp=\hsize \dtmpi=\vsize\begingroup
 \output={\global\setbox\partialpage=\vbox{\unvbox255\bigskip}}\eject
 \output={\doublecolumnout} \hsize=\dtmp \divide\hsize by2 \advance\hsize-.4em
 \vsize=\dtmpi \multiply\vsize by2 \advance\vsize by.4em}
\def\enddoublecolumns{\output={\balancecolumns}\eject
 \endgroup \pagegoal=\vsize}
\def\doublecolumnout{\splittopskip=\topskip \splitmaxdepth=\maxdepth
 \dtmp=\dtmpi \dimen@=\dtmp \advance\dimen@ by-\ht\partialpage
 \setbox0=\vsplit255 to\dimen@ \setbox2=\vsplit255 to\dimen@
 \onepageout\pagesofar\unvbox255 \penalty\outputpenalty}
\def\pagesofar{\unvbox\partialpage
 \wd0=\hsize \wd2=\hsize \hbox to\pagewidth{\box0\hfil\box2}}
\def\balancecolumns{\setbox0=\vbox{\unvbox255} \dimen@=\ht0
 \advance\dimen@ by\topskip \advance\dimen@ by-\baselineskip
 \divide\dimen@ by2 \splittopskip=\topskip{\vbadness=10000 \loop
 \global\setbox3=\copy0 \global\setbox1=\vsplit3 to\dimen@
 \ifdim\ht3>\dimen@ \global\advance\dimen@ by1pt \repeat}\setbox0=\vbox to
 \dimen@{\unvbox1}\setbox2=\vbox to\dimen@{\unvbox3}\pagesofar}
\def\begindoublecols{\dtmp=\hsize \dtmpi=\vsize\begingroup
 \output={\global\setbox\partialpage=\vbox{\unvbox255\bigskip}}\eject
 \output={\doublecolout} \divide\dtmp by2 \advance\dtmp-2em \hsize=\dtmp
 \multiply\dtmpi by2 \advance\dtmpi by 2em \vsize=\dtmpi }
\def\enddoublecols{\output={\balancecols}\eject
 \endgroup \pagegoal=\vsize}
\def\doublecolout{\splittopskip=\topskip \splitmaxdepth=\maxdepth
 \dtmp=\dtmpi \dimen@=\dtmp \advance\dimen@ by-\ht\partialpage
 \setbox0=\vsplit255 to\dimen@ \setbox2=\vsplit255 to\dimen@
 \onepageout\pagesofar \unvbox255 \penalty\outputpenalty}
\def\pagesofar{\unvbox\partialpage\wd0=\hsize \wd2=\hsize \hbox to\pagewidth
 {\box0\hfil\box2}}
\def\balancecols{\setbox0=\vbox{\unvbox255} \dimen@=\ht0 \advance\dimen@ by
 \topskip \advance\dimen@ by-\baselineskip\divide\dimen@ by2
 \splittopskip=\topskip{\vbadness=10000 \loop \global\setbox3=\copy0
 \global\setbox1=\vsplit3 to\dimen@\ifdim\ht3>\dimen@
 \global\advance\dimen@ by1pt \repeat}\setbox0=\vbox to\dimen@{\unvbox1}
 \setbox2=\vbox to\dimen@{\unvbox3}\pagesofar}
\def\begintriplecolumns{\dtmp=\hsize \dtmpi=\vsize \begingroup
 \output={\global\setbox\partialpage=\vbox{\unvbox255\bigskip}}\eject
 \output={\triplecolumnout} \hsize=\dtmp \divide\hsize by3
 \advance\hsize by-.4em\vsize=\dtmpi \multiply\vsize by3 \advance\vsize by.4em}
\def\endtriplecolumns{\output={\balancethreecolumns}\eject
 \endgroup \pagegoal=\vsize}
\def\triplecolumnout{\splittopskip=\topskip \splitmaxdepth=\maxdepth
 \dtmpii=\dtmpi \dimen@=\dtmpii \advance\dimen@ by-\ht\partialpage
 \setbox0=\vsplit255 to\dimen@ \setbox2=\vsplit255 to\dimen@
 \setbox4=\vsplit255 to\dimen@ \onepageout\pagesofar
 \unvbox255 \penalty\outputpenalty}
\def\pagesofar{\unvbox\partialpage \wd0=\hsize \wd2=\hsize \wd4=\hsize
 \hbox to\pagewidth{\box0\hfil\box2\hfil\box4}}
\def\balancethreecolumns{\setbox0=\vbox{\unvbox255} \dimen@=\ht0
 \advance\dimen@ by\topskip \advance\dimen@ by-\baselineskip
 \divide\dimen@ by3 \splittopskip=\topskip{\vbadness=10000 \loop
 \global\setbox5=\copy0 \global\setbox1=\vsplit5 to\dimen@
 \global\setbox3=\vsplit5 to\dimen@ \ifdim\ht5>\dimen@
 \global\advance\dimen@ by1pt \repeat} \setbox0=\vbox to\dimen@{\unvbox1}
 \setbox2=\vbox to\dimen@{\unvbox3}\setbox4=\vbox to\dimen@{\unvbox5}
 \pagesofar}
% cf. manmac.tex
\def\onepageout#1{\shipout\vbox{\offinterlineskip\vbox to 3pc{\iftitle
 \global\titlefalse\setcornerrules\else\ifodd\pageno \rightheadline\else
 \leftheadline\fi\fi\vfill} \vbox to \pageheight{\dtmp=\hsize
 \advance\dtmp by1pc \ifvoid\margin\else \rlap{\kern\dtmp\vbox to\z@{\kern4pt
 \box\margin \vss}}\fi #1 \ifvoid\footins\else \vskip\skip\footins \kern-3pt
 \hrule height\ruleht width\pagewidth \kern-\ruleht \kern3pt \unvbox\footins\fi
 \boxmaxdepth=\maxdepth}}\advancepageno}
\def\pap#1{\begingroup\parindent=1em \ninepoint\input #1\endgroup}
\def\lhd[#1]{\leftline{\(\tenrm\bf #1)}}\def\chd[#1]{\centerline{\(\tenrm #1)}}
\def\qte[#1][#2]{\begingroup\dtmp=\hsize \advance\dtmp-3em
 \setbox1=\vbox{\hsize=\dtmp \parindent=0pt #1\smallskip\rightline{#2}}
 \smallskip\centerline{\copy1}\smallskip\endgroup}
\def\quo[#1]{\begingroup \def\|{\par} \dtmp=\hsize \advance\dtmp-5em
 \setbox1=\vbox{\hsize=\dtmp \parindent=0pt #1}
 \smallskip\centerline{\copy1}\smallskip\endgroup}
\def\bitm[#1]{\smallskip\begingroup \def\itm{\item{#1}}}
\def\eitm{\endgroup\smallskip}
\def\btmn#1\etmn{\smallskip\begingroup \tmp=0
  \def\itm{\advance\tmp by 1 \item{\the\tmp.}} #1\endgroup\smallskip}
\def\btma#1\etma{\smallskip\begingroup \tmp=96
  \def\itm{\advance\tmp by 1 \item{\char\tmp.}} #1\endgroup\smallskip}
\def\btab[#1]#2\etab{\centerline{\setbox1=\vbox{#2} \vbox{\halign{#1\let|=&
 \let*=\cr \copy1}}}}
\def\tdim{\dtmp=\hsize \advance\dtmp by-2em \hsize=\dtmp}
\newcount\pam \pam=96
\def\lox(#1)[#2]{\centerline{{\twelvebf #1}}\global\advance\pam by 1
 \begingroup\medskip\begingroup\parindent=0pt\obeylines
 \immediate\write\ref{\string\ix[#2](\char\the\pam)}
 \immediate\write\ref{\string\ix[#2n](#1)}
 \immediate\write\ref{\string\ix[#2p](\romannumeral\the\pageno)}
 \ifundefined{toc} \else\immediate\write\toc{\x[#2]. \x[#2n]\hfil\dotfill
 \x[#2p]}\fi\input #2 \endgroup\endgroup\bigskip}
\def\loxx(#1)[#2]{\centerline{{\twelvebf #1}}
 \begingroup\medskip\parindent=0pt\obeylines \input #2 \endgroup}
\def\loy(#1)[#2]{\centerline{{\twelvebf #1}}\global\advance\pam by 1
 \begingroup\medskip\begingroup\parindent=0pt\obeylines
 \immediate\write\ref{\string\ix[#2](\char\the\pam)}
 \immediate\write\ref{\string\ix[#2n](#1)}
 \immediate\write\ref{\string\ix[#2p](\number\pageno)}
 \ifundefined{toc} \else\immediate\write\toc{\x[#2]. \x[#2n]\hfil\dotfill
 \x[#2p]}\fi\endgroup\endgroup\bigskip}
\def\loz(#1)[#2]{\centerline{{\twelvebf #1}}\par\bskp\input #2 }
\newcount\apc \apc=64
\def\app[#1]#2:#3:{\par\vfill\break
 \global\advance\apc by 1 \immediate\write\ref{\string\ix[#1](\char\the\apc)}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 \beginchapter {\Title\ Appendix} {}{\char\apc}. {\x[#1n]}\par
 \ifundefined{toc} \else\immediate\write\toc{\x[#1]. \x[#1n]\hfil\dotfill
 \x[#1p]}\fi\stn=0 \alc=0 \fgc=0 \tbc=0 \cac=0 \tmc=0 \asm=0 \dfc=0
 \S\ \Bf \x[#1]. \x[#1n]:\par \begingroup\input #3 \endgroup\par}
\newcount\cch
\def\chp[#1]#2:#3:{\par\vfill\break\global\advance\cch by 1 \ifodd\pageno
 \else\advance\pageno by 1 \fi \immediate\write\ref{\string\ix[#1](\the\cch)}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 \beginchapter {\Title\ Chapter} {}{\x[#1]}. {\x[#1n]}\par
 \ifundefined{toc} \else\immediate\write\toc{\x[#1]. \x[#1n]\dotfill\x[#1p]}\fi
 \stn=0 \alc=0 \fgc=0 \tbc=0 \cac=0 \tmc=0 \asm=0 \dfc=0
 {\elevenpoint\S\ \Bf \x[#1]. \x[#1n]:}\par \begingroup\input #3 \endgroup\par}
\newcount\stn
\def\sap[#1]#2:#3:{\global\advance\stn by 1\beginsection
 \S\ {\bf\x[#1] \x[#1n]}\par
 \ifundefined{toc} \else\immediate\write\toc{\q\x[#1].
 \x[#1n]\dotfill\x[#1p]}\fi \immediate\write\ref{\string\ix[#1](\char\the\apc.
 \the\stn)}\immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}\begingroup\input #3
 \endgroup\par}
\def\sct[#1]#2:#3:{\global\advance\stn by 1
 \immediate\write\ref{\string\ix[#1](\the\cch.\the\stn)}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}\beginsection
 \ifundefined{toc} \else\immediate\write\toc{\q\x[#1]. \x[#1n]\dotfill\x[#1p]}
 \fi \S\ {\bf\x[#1] \x[#1n]}\par \begingroup\input #3 \endgroup\par}
\newcount\alc
\def\alg[#1]#2:{\par\global\advance\alc by 1
 \immediate\write\ref{\string\ix[#1](\the\cch.\the\alc)}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 \ifundefined{loq}\else\immediate\write\loq{\x[#1]. \x[#1n]\dotfill\x[#1p]}\fi}
\newcount\eqc
\def\eqn{\global\advance\eqc by 1
 \eqno{(\bf\the\eqc)_{\hbox{\sevenrm\romannumeral\the\cch}}}}
\def\eql{\global\advance\eqc by 1
 (\bf\the\eqc)_{\hbox{\sevenrm\romannumeral\the\cch}}}
\def\eqa[#1]#2:{\global\advance\eqc by 1
 \immediate\write\ref{\string\ix[#1](\the\eqc{(\sevenrm\romannumeral\the\cch)})}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 \eqno{(\bf\the\eqc)_{\hbox{\sevenrm\romannumeral\the\cch}}}}
\def\eqal[#1]#2:{\global\advance\eqc by 1
 \immediate\write\ref{\string\ix[#1](\the\eqc{(\sevenrm\romannumeral\the\cch)})}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 (\bf\the\eqc)_{\hbox{\sevenrm\romannumeral\the\cch}}}
\newcount\fgc
\def\fig[#1]#2:{\par\global\advance\fgc by 1
 \immediate\write\ref{\string\ix[#1](\the\cch.\the\fgc)}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 \ifundefined{lof}\else\immediate\write\lof{\x[#1]. \x[#1n]\dotfill\x[#1p]}\fi}
\newcount\tbc
\def\tab[#1]#2:{\par\global\advance\tbc by 1
 \immediate\write\ref{\string\ix[#1](\the\cch.\the\tbc)}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 \ifundefined{lot} \else\immediate\write\lot{\x[#1]. \x[#1n]\dotfill
 \x[#1p]}\fi}
\newcount\cac
\def\proclaim #1. #2\par{\medbreak\noindent{\bf#1.\enspace}{\sl#2\par}
  \ifdim\lastskip<\medskipamount \removelastskip\penalty55\medskip\fi}
\def\cly[#1]#2:#3\par{\par\global\advance\cac  by 1
 \immediate\write\ref{\string\ix[#1](\the\cch.\the\tmc [\the\cac])}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 \proclaim Corollary \x[#1]. #3\par
 \ifundefined{lom}\else\immediate\write\lom{\x[#1]. \x[#1n]\dotfill\x[#1p]}\fi}
\newcount\asm
\def\asu[#1]#2:#3\par{\par\global\advance\asm  by 1
 \immediate\write\ref{\string\ix[#1](\the\cch.\the\asm)}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 \proclaim Assumption \x[#1]. #3\par\ifundefined{los} \else
 \immediate\write\los{\x[#1]. \x[#1n]\dotfill\x[#1p]}\fi}
\newcount\dfc
\def\dfn[#1]#2:#3\par{\par\global\advance\dfc  by 1
 \immediate\write\ref{\string\ix[#1](\the\cch.\the\dfc)}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 \proclaim Definition \x[#1]. #3\par
 \ifundefined{lod}\else\immediate\write\lod{\x[#1]. \x[#1n]\dotfill\x[#1p]}\fi}
\newcount\tmc
\def\thm[#1]#2:#3\par{\par\global\advance\tmc  by 1
 \immediate\write\ref{\string\ix[#1](\the\cch.\the\tmc)}
 \immediate\write\ref{\string\ix[#1n](#2)}
 \immediate\write\ref{\string\ix[#1p](\number\pageno)}
 \proclaim Theorem \x[#1]. #3\par
 \ifundefined{lom}\else\immediate\write\lom{\x[#1]. \x[#1n]\dotfill\x[#1p]}\fi}
\def\xp[#1]{\immediate\write\ref{\string\ix[#1](\number\pageno)}
 \ix[#1](\number\pageno)}
\def\mctb[#1:#2:#3:#4:#5:#6:#7:#8]{\smallskip\begingroup\dtmp=\hsize
 \advance\dtmp2em \divide\dtmp by7 \dtmpi=0pt \setbox1=\vbox{\hsize=\dtmp #3}
 \dtmpii=\ht1 \ifdim\dtmpii>\dtmpi \dtmpi=\dtmpii\fi
 \setbox1=\vbox{\hsize=\dtmp#4} \dtmpii=\ht1 \ifdim\dtmpii>\dtmpi
 \dtmpi=\dtmpii\fi \setbox1=\vbox{\hsize=\dtmp #5} \dtmpii=\ht1
 \ifdim\dtmpii>\dtmpi \dtmpi=\dtmpii\fi \setbox1=\vbox{\hsize=\dtmp #6}
 \dtmpii=\ht1 \ifdim\dtmpii>\dtmpi \dtmpi=\dtmpii\fi
 \setbox1=\vbox{\hsize=\dtmp #7} \dtmpii=\ht1 \ifdim\dtmpii>\dtmpi
 \dtmpi=\dtmpii\fi \setbox1=\vbox{\hsize=\dtmp #8} \dtmpii=\ht1
 \ifdim\dtmpii>\dtmpi \dtmpi=\dtmpii\fi \centerline{\vbox{\halign{##\hfil&
 ##\hfil&##\hfil&##\hfil&##\hfil&##\hfil\cr \multispan{6}{\hfill#1\hfill}\cr
 \trule\vbox{\hsize=\dtmp\parindent=0pt~\[Cell]} &
 \vbox{\hsize=\dtmp\parindent=0pt~\[Bond]}&\vbox{\hsize=\dtmp\parindent=0pt
 ~\[cell]} &\vbox{\hsize=\dtmp\parindent=0pt~\[bond]}
 &\vbox{\hsize=\dtmp\parindent=0pt~\[vertex]} &\vbox{\hsize=\dtmp\parindent=0pt
 ~\[edge]} \cr #2\cr \vbox to\dtmpi{\hsize=\dtmp\parindent=0pt #3\hfil\vfill}
 &\vbox to\dtmpi{\hsize=\dtmp\parindent=0pt #4\hfil\vfill}
 &\vbox to\dtmpi{\hsize=\dtmp\parindent=0pt #5\hfil\vfill}
 &\vbox to\dtmpi{\hsize=\dtmp\parindent=0pt #6\hfil\vfill}
 &\vbox to\dtmpi{\hsize=\dtmp\parindent=0pt #7\hfil\vfill}
 &\vbox to\dtmpi{\hsize=\dtmp\parindent=0pt #8\hfil\vfill}\cr}}}\endgroup}
```

```
\def\mvmm(#1;#2;#3,#4;#5,#6){\begingroup\sevenrm\setbox1=\hbox{#1$\pm$#2}
 \ifx0#4\def\tmpa{#3}\else\def\tmpa{#3\times 10^{#4}}\fi
 \ifx0#6\def\tmpb{#5}\else\def\tmpb{#5\times 10^{#6}}\fi
 $\mathrel{\mathop{\copy1}\limits_{\tmpa\atop\tmpb}}$\endgroup}
\def\cn(#1,#2/#3){\begingroup\sevenrm \setbox1=\hbox{#1[#2]} \dtmp=\wd1
 \setbox2=\hbox{\fiverm#3} \ifdim\wd2>\dtmp \dtmp=\wd2\fi
 \setbox2=\hbox{\fiverm#3}$\mathop{\copy1}\limits_{\copy2}$\endgroup}
% 2-homohedral tilings
\def\hm#1.{\begingroup\def\(##1){{\rm ##1}} \ifcase#1\or
 $3_3[3^3]7_1[3^7]$\or$3_3[3^3]9_3[3^9]_\(II)$\or$4_4[3^4]8_4[3^8]$\or
 $3_3[3^3]8_2[3^8]$\or$3_3[3^3]9_3[3^9]_\(III)$\or$4_3[3^4]10_6[3^{10}]_\(I)$
 \or$3_3[3^3]9_3[3^9]_\(I)$\or$4_2[3^4]10_4[3^{10}]$\or
 $4_3[3^4]10_6[3^{10}]_\(II)$\or$4_2[3^4]8_2[3^8]$\or$4_3[3^4]8_3[3^8]_\(I)$\or
 $4_3[3^4]8_3[3^8]_\(II)$\or$4_3[3^4]8_3[3^8]_\(III)$\or
 $4_4[3^4]7_2[3^7]_\(II)$\or$3_3[3^3]12_6[3^{12}]$\or$4_4[3^4]7_2[3^7]_\(I)$
 \or$5_4[3^5]7_4[3^7]_\(I)$\or$3_1[4^3]5_1[4^5]_\(I)$\or
 $3_1[4^3]5_1[4^5]_\(II)$\or$5_3[3^5]8_6[3^8]_\(II)$\or
 $5_3[3^5]8_6[3^8]_\(III)$\or$5_2[3^5]12_{12}[3^{12}]$\or
 $5_4[3^5]7_4[3^7]_\(II)$\or$5_3[3^5]8_6[3^8]_\(I)$\or
 $4_2[3^4]12_6[3^{12}]$\or$4_2[3^4]18_{12}[3^{18}]$\or$5_3[3^5]7_3[3^7]_\(I)$
 \or$5_3[3^5]7_3[3^7]_\(II)$\or$3_2[4^3]5_2[4^5]_\(I)$\or
 $3_2[4^3]5_2[4^5]_\(II)$\or$3_3[4^3]5_3[4^5]$\or$3_1[4^3]6_2[4^6]_\(I)$\or
 $3_1[4^3]6_2[4^6]_\(II)$\or$3_2[4^3]6_4[4^6]$\or$3_3[4^3]6_6[4^6]$
 \or$3_1[4^3]8_4[4^8]$\or$3_1[5^3]4_2[5^4]_\(I)$\or$3_1[5^3]4_2[5^4]$\or
 $3_2[5^3]4_4[5^4]$\fi\endgroup}
\def\kittix{$\raise.1em\hbox{$>$}$\kern-.2em\raise.118em\hbox{:}Kittix}
\def\blr{\begingroup \dtmp=\hsize \advance\dtmp-1em \dtmpi=\dtmp
 \advance\dtmpi-1em \divide\dtmpi by2 \parindent0pt \def\stn{\smallskip}
 \def\b[##1]{\bf ##1} \def\[##1]{\it ##1}
 \def\l##1:##2:{\setbox0=\vbox{\hsize=\dtmpi\sl##1}
 \setbox1=\vbox{\hsize=\dtmpi##2}\ifnum\ht0>\ht1 \dtmpii=\ht0 \else
 \dtmpii=\ht1 \fi\centerline{\vbox to\dtmpii{\hsize=\dtmpi\sl##1}\hfil\vbox to
 \dtmpii{\hsize=\dtmpi##2}}}}
\def\elr{\endgroup}
\def\wra[#1][#2][#3]{\begingroup\baselineskip=2em
 \setbox1=\hbox{#3}$\aleph(\hbox{#1})=$
  $\bigl\{$[#2], (\unhbox1)$\bigr\}$\endgroup}
\def\wrs[#1][#2][#3][#4]{$\aleph(\hbox{#1})=$ $\bigl\{$[#2], (#3), $\langle$#4
 $\rangle$ $\bigr\}$}
\def\head#1{\hbox{} \vskip#1} \def\setcornerrules{} \def\inpt[#1]{\input #1}
\def\beginchapter#1 #2#3. #4\par{\global\exno=0 \subsecno=0 \def\chapno{#2#3}
 \titlepage\def\\{ }\message{#1 #2#3:} \xdef\rhead{#1 #2#3: #4\unskip}
 \vskp1em{\def\TeX{T\kern-.2em\lower.5ex\hbox{E}\kern-.06em X}
 \def\MF{{\vbox to30pt{}\manual ()*+,-.*}}\def\\{#3}
 \ifx\empty\\ \ifodd\pageno \rightline{\inchhigh \kern-.04em}\else
 \leftline{\inchhigh \kern-.04em}\fi\else\ifodd\pageno
 \rightline{\inchhigh #3\kern-.04em}\else\leftline{\inchhigh #3\kern-.04em}
 \fi\fi\vskip .75pc\baselineskip 16pt \lineskiplimit \titlelsl \lineskip 3pt
 \let\\=\cr} \tenpoint\noindent\ignorespaces}
\def\dct:#1:#2:#3:{\dtmp=\hsize \begindoublecolumns\hsize=\dtmp
 \divide\hsize by2\def\rhead{\firstmark\hfil{\tenit #1--#2 dictionary}\hfil
 \botmark}\def\leftheadline{\hbox to \pagewidth{\vbox to 10pt{}
 \llap{\tenbf\folio\kern1pc}\tenbf\rhead}}
 \def\rightheadline{\hbox to \pagewidth{\vbox to 10pt{}\tenbf\rhead
 \rlap{\kern1pc\tenbf\folio}}}\baselineskip.9em\parindent=0pt\eightpoint
 \input #3\enddoublecolumns\cut}
\def\rightharpoonupfill{$\m@th\smash-\mkern-7mu \cleaders\hbox{$\mkern-2mu
 \smash-\mkern-2mu$}\hfill\mkern-7mu\mathord\rightharpoonup$}
\def\overrightharpoonup#1{\vbox{\m@th\ialign{##\crcr \rightharpoonupfill\crcr
 \noalign{\kern-\p@\nointerlineskip}$\hfil\displaystyle{#1}\hfil$\crcr}}}
\def\pet#1\tep{\begingroup\dtmp=\hsize \advance\dtmp by 3em
 \centerline{\hfill\vbox{\parindent=0pt\leftskip=5em #1}\hfill}\endgroup}
\def\another(#1)[#2]{\global\advance#1 by 1
 \immediate\write\ref{\string\ix[#2](\the#1)}}
% \input grammar
\input language
\asl\bengali\chem\czech\daiy\deutsch\espanlol\francjais\gaelic\grammar\hindi
 \hungarian\lanna\latin\lating\math\money\nihongo\norge\pali\physics\polish
 \russian\sanskrit\serbo\slovak\vietnamese\zhongwen
% rm dstidx.tex
% sort -f index.tex > dstidx.tex
% edit dstidx.tex, replace ‘_!0’ by ‘,’
% then append  \parindent=-3em \leftskip3em   to top
%% 1. replace _! by ,
%% 2. append \begindoublecolumns\beginindex at top
%% and \endindex\enddoublecolumns at end
```

§ A.12 Language macros for TEX

```
% language.tex  (c) 2002 (2003) Kit Tiyapan, September 2003.
```

```
\def\asl{\def\uq.{\d u} \def\oq.{\d o} \def\uoq.{\d uo}
  \def\iq.{\d i} \def\eq.{\d e} \def\ieq.{\d ie}  \def\aq.{\d a}
  \def\aoq.{\d ao} \def\oaq.{\d oa} \def\uiq.{\d ui} \def\iuq.{\d iu}
  \def\ouq.{\d ou} \def\ueq.{\d ue} \def\oeq.{\d oe} \def\aeq.{\d ae}
  \def\euq.{\d eu} \def\iaq.{\d ia} \def\aqch.{\d ach} \def\aqct.{\d act}
  \def\aqsp.{\d asp} \def\oqss.{\d oss} \def\eqnt.{\d ent} \def\iqde.{\d ide}
  \def\iqnt.{\d int}
 \def\th.{\`t} \def\Th.{\`T} \def\tsh.{\`ts} \def\Tsh.{\`Ts} \def\tzh.{\`tz} \def\Tzh.{\`Tz}
  \def\bh.{\`b} \def\Bh.{\`B}
  \def\bsh.{\`bs} \def\Bsh.{\`Bs} \def\sh.{\`s} \def\Sh.{\`S}
  \def\ch.{\`c} \def\Ch.{\`C} \def\dh.{\`s} \def\Dh.{\`D}
  \def\fh.{\`f} \def\Fh.{\`F} \def\gh.{\`g} \def\Gh.{\`G}
  \def\gdh.{\`gd} \def\Gdh.{\`Gd} \def\hh.{\`h} \def\Hh.{\`H}
  \def\jh.{\`\j} \def\Jh.{\`J} \def\kh.{\`k} \def\Kh.{\`K}
  \def\lh.{\`l} \def\Lh.{\`L} \def\zh.{\`z} \def\Zh.{\`Z}
  \def\ph.{\`p} \def\Ph.{\`P} \def\rh.{\`r} \def\Rh.{\`R}
  \def\vh.{\`v} \def\Vh.{\`V} \def\wh.{\`w} \def\Wh.{\`W}
  \def\xh.{\`x} \def\Xh.{\`X} \def\yh.{\`y} \def\Yh.{\`Y}
  \def\ysh.{\`ys} \def\Ysh.{\`Ys} \def\qh.{\`q} \def\Qh.{\`Q}
 \def\tz.{\'t} \def\Tz.{\'T} \def\tsz.{\'ts} \def\Tsz.{\'Ts}
  \def\tzz.{\'tz} \def\Tzz.{\'Tz} \def\bz.{\'b} \def\Bz.{\'B}
  \def\bsz.{\'bs} \def\Bsz.{\'Bs} \def\sz.{\'s} \def\Sz.{\'S}
  \def\cz.{\'c} \def\Cz.{\'C} \def\dz.{\'s} \def\Dz.{\'D}
  \def\fz.{\'f} \def\Fz.{\'F} \def\gz.{\'g} \def\Gz.{\'G}
  \def\gdz.{\'gd} \def\Gdz.{\'Gd} \def\hz.{\'h} \def\Hz.{\'H}
  \def\jz.{\'\j} \def\Jz.{\'J} \def\kz.{\'k} \def\Kz.{\'K}
  \def\lz.{\'l} \def\Lz.{\'L} \def\zz.{\'z} \def\Zz.{\'Z}
  \def\pz.{\'p} \def\Pz.{\'P} \def\rz.{\'r} \def\Rz.{\'R}
  \def\vz.{\'v} \def\Vz.{\'V} \def\wz.{\'w} \def\Wz.{\'W}
  \def\xz.{\'x} \def\Xz.{\'X} \def\yz.{\'y} \def\Yz.{\'Y}
  \def\ysz.{\'ys} \def\Ysz.{\'Ys} \def\qz.{\'q} \def\Qz.{\'Q}
 \def\ty.{\^t} \def\Ty.{\^T} \def\tsy.{\^ts} \def\Tsy.{\^Ts}
  \def\tyy.{\^ty} \def\Tyy.{\^Ty} \def\by.{\^b} \def\By.{\^B}
  \def\bsy.{\^bs} \def\Bsy.{\^Bs} \def\sy.{\^s} \def\Sy.{\^S}
  \def\cy.{\^c} \def\Cy.{\^C} \def\dy.{\^s} \def\Dy.{\^D}
  \def\fy.{\^f} \def\Fy.{\^F} \def\gy.{\^g} \def\Gy.{\^G}
  \def\gdy.{\^gd} \def\Gdy.{\^Gd} \def\hy.{\^h} \def\Hy.{\^H}
  \def\jy.{\^\j} \def\Jy.{\^J} \def\ky.{\^k} \def\Ky.{\^K}
  \def\ly.{\^l} \def\Ly.{\^L} \def\yy.{\^y} \def\Yy.{\^Y}
  \def\py.{\^p} \def\Py.{\^P} \def\ry.{\^r} \def\Ry.{\^R}
  \def\vy.{\^v} \def\Vy.{\^V} \def\wy.{\^w} \def\Wy.{\^W}
  \def\xy.{\^x} \def\Xy.{\^X} \def\yy.{\^y} \def\Yy.{\^Y}
  \def\ysy.{\^ys} \def\Ysy.{\^Ys} \def\qy.{\^q} \def\Qy.{\^Q}
 \def\tq.{\d t} \def\Tq.{\d T}
  \def\tsq.{\d ts} \def\Tsq.{\d Ts} \def\tqq.{\d tq} \def\Tqq.{\d Tq}
  \def\bq.{\d b} \def\Bq.{\d B} \def\bsq.{\d bs} \def\Bsq.{\d Bs}
  \def\sq.{\d s} \def\Sq.{\d S} \def\cq.{\d c} \def\Cq.{\d C}
  \def\dq.{\d s} \def\Dq.{\d D} \def\fq.{\d f} \def\Fq.{\d F}
  \def\gq.{\d g} \def\Gq.{\d G} \def\gdq.{\d gd} \def\Gdq.{\d Gd}
  \def\hq.{\d h} \def\Hq.{\d H} \def\jq.{\d \j} \def\Jq.{\d J}
  \def\kq.{\d k} \def\Kq.{\d K} \def\lqq.{\d l} \def\Lq.{\d L}
  \def\qq.{\d q} \def\Qq.{\d Q} \def\pq.{\d p} \def\Pq.{\d P}
  \def\rq.{\d r} \def\Rq.{\d R} \def\vq.{\d v} \def\Vq.{\d V}
  \def\wq.{\d w} \def\Wq.{\d W} \def\xq.{\d x} \def\Xq.{\d X}
  \def\wq.{\d w} \def\yq.{\d y} \def\Yq.{\d Y} \def\ysq.{\d ys} \def\Ysq.{\d Ys}
  \def\qq.{\d q} \def\Qq.{\d Q}
 \def\byh.{\`by} \def\Byh.{\`By} \def\tqy.{\d{\^t} \def\Tqy.{\d{\^T}}}
 \def\cniz.{cn\'\i} \def\Cniz.{Cn\'I}
  \def\mfiz.{mf\'\i} \def\Mfiz.{Mf\'I} \def\myah.{my\`a} \def\Myah.{My\`a}
  \def\myaz.{my\'a} \def\Myaz.{My\'a}
 \def\iqde.{\d ide}\def\vzl.{\'vl} \def\Vzl.{\'Vl}}

\def\bengali{\def\aa.{\=a} \def\Aa.{\=A} \def\ii.{\=\i} \def\Ii.{\=I}
 \def\uu.{\=u} \def\Uu.{\=U} \def\rq.{\d r} \def\Rq.{\d R}
 \def\mq.{\d m} \def\Mq.{\d M} \def\hq.{\d h} \def\Hq.{\d H}
 \def\nx.{\.n} \def\Nx.{\.N} \def\nl.{\~n} \def\Nl.{\~N}
 \def\tq.{\d t} \def\Tq.{\d T} \def\tqh.{\d th} \def\Tqh.{\d Th}
 \def\dq.{\d d} \def\Dq.{\d D} \def\dqh.{\d dh} \def\Dqh.{\d Dh}
 \def\nq.{\d n} \def\Nq.{\d N} \def\sz.{\'s} \def\Sz.{\'S}
 \def\sq.{\d s} \def\Sq.{\d S}}

\def\chem{\def\fmr[##1]{\hbox{$\rm ##1$}} \def\dgc{\hbox{$^\circ\hbox{C}$}}}

\def\czech{\def\cz.{\'c} \def\Cz.{\'C} \def\yz.{\'y} \def\Yz.{\'Y}}

\def\daiy{\def\av.{\v a} \def\Av.{\v A} \def\ah.{\`a} \def\Ah.{\`A}
  \def\af.{\=a} \def\Af.{\=A} \def\az.{\'a} \def\Az.{\'A}
 \def\aq.{\d a} \def\Aq.{\d A} \def\aqv.{\d{\v a}} \def\Aqv.{\d{\v A}}
  \def\aqh.{\d{\`a}} \def\Aqh.{\d{\`A}} \def\aqf.{\d{\=a}} \def\Aqf.{\d{\=A}}
  \def\aqz.{\d{\'a}} \def\Aqz.{\d{\'A}}
 \def\iv.{\v\i} \def\Iv.{\v I} \def\ih.{\`\i} \def\Ih.{\`I}
  \def\if.{\=\i} \def\If.{\=I} \def\iz.{\'\i} \def\Iz.{\'I}
```

```
\def\iq.{\d i} \def\Iq.{\d I} \def\iqv.{\d{\v\i}} \def\Iqv.{\d{\v I}}
 \def\iqh.{\d{\‘\i}} \def\Iqh.{\d{\‘I}} \def\iqf.{\d{\=\i}} \def\Iqf.{\d{\=I}}
 \def\iqz.{\d{\’\i}} \def\Iqz.{\d{\’I}}
\def\uev.{\v ue} \def\Uev.{\v Ue} \def\ueh.{\‘ue} \def\Ueh.{\‘Ue}
 \def\uef.{\=ue} \def\Uef.{\=Ue} \def\uez.{\’ue} \def\Uez.{\’Ue}
\def\uqe.{\d ue} \def\Uqe.{\d Ue} \def\uqev.{\d{\v u}e} \def\Uqev.{\d{\v U}e}
 \def\uqeh.{\d{\‘u}e} \def\Uqeh.{\d{\‘U}e} \def\uqef.{\d{\=u}e} \def\Uqef.{\d{\=U}e}
 \def\uqez.{\d{\’u}e} \def\Uqez.{\d{\’U}e}
\def\uv.{\v u} \def\Uv.{\v U} \def\uh.{\‘u} \def\Uh.{\‘U}
 \def\uf.{\=u} \def\Uf.{\=U} \def\uz.{\’u} \def\Uz.{\’U}
\def\uq.{\d u} \def\Uq.{\d U}
 \def\uqv.{\d{\v u}} \def\Uqv.{\d{\v U}} \def\uqh.{\d{\‘u}} \def\Uqh.{\d{\‘U}}
 \def\uqf.{\d{\=u}} \def\Uqf.{\d{\=U}} \def\uqz.{\d{\’u}} \def\Uqz.{\d{\’U}}
\def\ev.{\v e} \def\Ev.{\v E} \def\eh.{\‘e} \def\Eh.{\‘E}
 \def\ef.{\=e} \def\Ef.{\=E} \def\ez.{\’e} \def\Ez.{\’E}
\def\eq.{\d e} \def\Eq.{\d E} \def\eqv.{\d{\v e}} \def\Eqv.{\d{\v E}}
 \def\eqh.{\d{\‘e}} \def\Eqh.{\d{\‘E}} \def\eqf.{\d{\=e}} \def\Eqf.{\d{\=E}}
 \def\eqz.{\d{\’e}} \def\Eqz.{\d{\’E}}
\def\aev.{\v ae} \def\Aev.{\v Ae} \def\aeh.{\‘ae} \def\Aeh.{\‘Ae}
 \def\aef.{\=ae} \def\Aef.{\=Ae} \def\aez.{\’ae} \def\Aez.{\’Ae}
\def\aqe.{\d ae} \def\Aqe.{\d Ae} \def\aqev.{\d{\v a}e} \def\Aqev.{\d{\v A}e}
 \def\aqeh.{\d{\‘a}e} \def\Aqeh.{\d{\‘A}e} \def\aqef.{\d{\=a}e} \def\Aqef.{\d{\=A}e}
 \def\aqez.{\d{\’a}e} \def\Aqez.{\d{\’A}e}
\def\ov.{\v o} \def\Ov.{\v O} \def\oh.{\‘o} \def\Oh.{\‘O}
 \def\of.{\=o} \def\Of.{\=O}  \def\oz.{\’o} \def\Oz.{\’O}
\def\oq.{\d o} \def\Oq.{\d O} \def\oqv.{\d{\v o}} \def\Oqv.{\d{\v O}}
 \def\oqh.{\d{\‘o}} \def\Oqh.{\d{\‘O}}  \def\oqf.{\d{\=o}} \def\Oqf.{\d{\=O}}
 \def\oqz.{\d{\’o}} \def\oqz.{\d{\’o}}
\def\auv.{\v au} \def\Auv.{\v Au} \def\auh.{\‘au} \def\Auh.{\‘Au}
 \def\auf.{\=au} \def\Auf.{\=Au} \def\auz.{\’au} \def\Auz.{\’Au}
\def\aqu.{\d au} \def\Aqu.{\d Au} \def\aquv.{\d{\v a}u} \def\Aquv.{\d{\v A}u}
 \def\aquh.{\d{\‘a}u} \def\Aquh.{\d{\‘A}u} \def\aquf.{\d{\=a}u} \def\Aquf.{\d{\=A}u}
 \def\aquz.{\d{\’a}u} \def\Aquz.{\d{\’A}u}
\def\oev.{\v oe} \def\Oev.{\v Oe} \def\oeh.{\‘oe} \def\Oeh.{\‘Oe}
 \def\oef.{\=oe} \def\Oef.{\=Oe} \def\oez.{\’oe} \def\Oez.{\’Oe}
\def\oqe.{\d oe} \def\Oqe.{\d Oe}
 \def\oqev.{\d{\v o}e} \def\Oqev.{\d{\v O}e} \def\oqeh.{\d{\‘o}e} \def\Oqeh.{\d{\‘O}e}
 \def\oqef.{\d{\=o}e} \def\Oqef.{\d{\=O}e} \def\oqez.{\d{\’o}e}
 \def\Oqez.{\d{\’O}e} \def\oqeez.{\d{\’o}ee}
\def\iav.{\v\i a} \def\Iav.{\v Ia} \def\iah.{\‘\i a} \def\Iah.{\‘Ia}
 \def\iaf.{\=\i a} \def\Iaf.{\=Ia} \def\iaz.{\’\i a} \def\iaz.{\’\i a}
\def\iqa.{\d ia} \def\Iqa.{\d Ia} \def\iqav.{\d{\v\i}a} \def\Iqav.{\d{\v I}a}
 \def\iqah.{\d{\‘\i}a} \def\Iqah.{\d{\‘I}a} \def\iqaf.{\d{\=\i}a} \def\Iqaf.{\d{\=I}a}
 \def\iqaz.{\d{\’\i}a} \def\Iqaz.{\d{\’I}a}
\def\uav.{\v ua} \def\Uav.{\v Ua} \def\uah.{\‘ua} \def\Uah.{\‘Ua}
 \def\uaf.{\=ua} \def\Uaf.{\=Ua} \def\uaz.{\’ua} \def\Uaz.{\’Ua}
\def\uqa.{\d ua} \def\Uqa.{\d Ua}
 \def\uqav.{\d{\v u}a} \def\Uqav.{\d{\v U}a} \def\uqah.{\d{\‘u}a} \def\Uqah.{\d{\‘U}a}
 \def\uqaf.{\d{\=u}a} \def\Uqaf.{\d{\=U}a} \def\uqaz.{\d{\’u}a} \def\Uqaz.{\d{\’U}a}
\def\oav.{\v oa} \def\Oav.{\v Oa} \def\oah.{\‘oa} \def\Oah.{\‘Oa}
 \def\oaf.{\=oa} \def\Oaf.{\=Oa} \def\oaz.{\’oa} \def\Oaz.{\’Oa}
\def\oqa.{\d oa} \def\Oqa.{\d Oa} \def\oqav.{\d{\v o}a} \def\Oqav.{\d{\v O}a}
 \def\oqah.{\d{\‘o}a} \def\Oqah.{\d{\‘O}a} \def\oqaf.{\d{\=o}a} \def\Oqah.{\d{\=O}a}
 \def\oqaz.{\d{\’o}a} \def\Oqaz.{\d{\’O}a}
\def\amq.{a\d m} \def\Amq.{A\d m} \def\amqv.{\v a\d m} \def\Amqv.{\v A\d m}
 \def\amqh.{\‘a\d m} \def\Amqh.{\‘A\d m} \def\ammqh.{\‘a\d m\d{\"m}}
 \def\amqf.{\=a\d m} \def\Amqf.{\=A\d m} \def\amqz.{\’a\d m} \def\Amqz.{\’A\d m}
\def\aiv.{\v ai} \def\aiv.{\v ai} \def\aih.{\‘ai} \def\aih.{\‘ai} \def\aif.{\=ai}
 \def\aif.{\=ai} \def\aiz.{\’ai} \def\aiz.{\’ai}
\def\aiq.{a\d i} \def\Aqi.{\d Ai} \def\aiqv.{\v a\d i} \def\Aiqv.{\v A\d i}
 \def\aiqh.{\‘a\d i} \def\Aiqh.{\‘A\d i} \def\aiqf.{\=a\d i} \def\Aiqf.{\=A\d i}
 \def\aiqz.{\’a\d i} \def\Aiqz.{\’A\d i}
\def\aov.{\v ao} \def\Aov.{\v Ao} \def\aoh.{\‘ao} \def\Aoh.{\‘Ao}
 \def\aof.{\=ao} \def\Aof.{\=Ao} \def\aoz.{\’ao} \def\Aoz.{\’Ao}
\def\auqh.{\‘a\d u} \def\Auqh.{\‘A\d u}
\def\tq.{\d t} \def\Tq.{\d T}  \def\tqh.{\d th} \def\Tqh.{\d Th}
 \def\kqh.{\d kh} \def\Kqh.{\d Kh}  \def\gq.{\d g} \def\Gq.{\d G}
 \def\dq.{\d d} \def\Dq.{\d D}  \def\dz.{\’d} \def\Dz.{\’D}
 \def\dqz.{\d{\’d}} \def\Dqz.{\d{\’D}}  \def\dqh.{\d dh} \def\Dqh.{\d Dh}
 \def\nq.{\d n} \def\Nq.{\d N}  \def\bq.{\d b} \def\Bq.{\d B}
 \def\fq.{\d f} \def\Fq.{\d F}  \def\hq.{\d h} \def\Hq.{\d H}
 \def\lqq.{\d l} \def\Lq.{\d L}  \def\rq.{\d r} \def\Rq.{\d R}
 \def\sq.{\d s} \def\Sq.{\d S}  \def\yq.{\d y} \def\Yq.{\d Y}
 \def\lz.{\’l} \def\Lz.{\’L}  \def\sz.{\’s} \def\Sz.{\’S}
\def\ae.{\"a} \def\bqe.{\d{\"b}} \def\eee.{\"e} \def\nge.{\"ng} \def\ke.{\"k}
 \def\me.{\"m} \def\ne.{\"n} \def\ye.{\"y}
\def\ax.{\.a}\def\Ax.{\.A}\def\aux.{\.au}\def\aux.{\d{\.a}u}\def\Aux.{\d{\.A}u}
 \def\bx.{\.b}\def\bhx.{\.bh} \def\cx.{\.c} \def\dx.{\.d} \def\dhx.{\.dh}
 \def\dqhx.{\d{\.dh}} \def\dzx.{\’dx} \def\fx.{\.f} \def\fqx.{\d{\.f}}
 \def\gx.{\.g} \def\ghx.{\.gh} \def\hx.{\.h} \def\hqx.{\d{\.h}} \def\kx.{\.k}
 \def\khx.{\.kh} \def\lx.{\.l} \def\mx.{\.m} \def\nx.{\.n}
 \def\nqx.{\d{\.n}} \def\px.{\.p} \def\rx.{\.r} \def\sx.{\.s} \def\sqx.{\d{\.s}}
```

```
  \def\szx.{\d{\'s}} \def\tx.{\.t} \def\thx.{\.th} \def\tqx.{\d{\.t}} \def\tqhx.{\d{\.th}}
  \def\vx.{\.v} \def\yx.{\.y} \def\yqx.{\d{\.y}} \def\zx.{\.z}}
\def\deutsch{\def\acc{{\it acc}} \def\datv{{\it dat}} \def\eie{{\it e-e}}
  \def\eim{{\it e-m}} \def\eir{{\it e-r}} \def\etw{{\it et}} \def\gen{{\it gen}}
  \def\jm{{\it jm}} \def\jn{{\it jn}} \def\ohn{{\it ohn.}} \def\Ae.{\"A} \def\ae.{\"a}
  \def\Oe.{\"O} \def\oe.{\"o} \def\Ue.{\"U} \def\ue.{\"u} \def\Ss.{\ss }}
\def\espanlol{\def\nl.{\~n} \def\Nl.{\~N}}
\def\francjais{\def\qch{{\it q.ch.}} \def\Az.{\'A} \def\az.{\'a}
 \def\Ez.{\'E} \def\ez.{\'e} \def\Ah.{\`A} \def\ah.{\`a}  \def\Eh.{\`E} \def\eh.{\`e}
 \def\Ey.{\^E} \def\ey.{\^e} \def\Iy.{\^I} \def\iy.{\^\i} \def\Ie.{\"I} \def\ie.{\"\i}}
\def\gaelic{\def\ah.{\`a} \def\Ah.{\'A} \def\eh.{\`e} \def\Eh.{\'E}
 \def\ih.{\`\i} \def\Ih.{\`I} \def\oh.{\`o} \def\Oh.{\`O} \def\uh.{\`u} \def\Uh.{\`U}}
\def\grammar{\def\ab{{\it abrv}} \def\adv{{\it adv}} \def\adj{{\it adj}}
 \def\ant{{\it ant}} \def\arcc{{\it archaic}} \def\atb{{\it atrib}}
 \def\auxi{{\it aux}} \def\brit{{\it Brit.}} \def\cca{{\it circa}}
 \def\cf{{\it cf}} \def\chem{{\it chem}} \def\chss{{\it chess}}
 \def\cnj{{\it conj}} \def\cnst{{\it const}} \def\clfr{{\it classf}}
 \def\col{{\it col}} \def\cor{{\it corrupt}} \def\dd{$\ldots$}
 \def\de{{\it derog}} \def\dtm{{\it det}} \def\dto{{\it ditto}}
 \def\dy{{\it dy}} \def\eg{{\it eg}} \def\esp{{\it esp}}
 \def\etc{{\it etc}} \def\etl{{\it et al}} \def\etal{{\it et al}}
 \def\expr{{\it expr}} \def\f{{\it f}} \def\figu{{\it fig}}  \def\fml{{\it fml}}
 \def\Fr{{\it Fr.}} \def\ger{{\it Ger.}} \def\gram{{\it gram}}
 \def\grk{{\it Gr.}} \def\humou{{\it humour.}} \def\ibid{{\it ibid.}}
 \def\idst{{\it i.e.}} \def\ifm{{\it infml}} \def\imp{{\it imp}}
 \def\ind{{\it ind}} \def\inj{{\it intj}}  \def\joc{{\it joc}}
 \def\jpn{{\it Jpn.}} \def\lan{{\it Lan.}}  \def\lat{{\it Lat}.}
 \def\law{{\it law}} \def\lo{{\it loc}} \def\lt{{\it lit}}
 \def\m{{\it m}} \def\mat{{\it math}} \def\matl{{\it mat}}
 \def\med{{\it med}} \def\mil{{\it mil}} \def\mo{{\it mod}}
 \def\mt{{\it m\d oay}} \def\n{{\it n}} \def\nom{{\it nom}}
 \def\nt{{\it nt}} \def\nym{{\it acronym}} \def\ob{{\it obs}}
 \def\oen{{\it O.E.}} \def\ofr{{\it O.Fr.}} \def\on{{\it ono}}
 \def\os{{\it o.s.}} \def\pa{{\it Pa.}} \def\part{{\it part}}
 \def\pl{{\it pl}} \def\pn{{\it pn}} \def\ppt{{\it pres. part.}}
 \def\pres{{\it pres}} \def\prp{{\it prp}} \def\prs{{\it prs}}
 \def\prf{{\it prf}} \def\qed{{\it q.e.d.}} \def\qntf{{\it quantf}}
 \def\rad{{\it rad}} \def\refl{{\it refl}} \def\rh{{\it rhe}}
 \def\rp{\lower4pt\hbox{{\tenrm\~{}}}\raise4pt\hbox{}} \def\smb{{\it sb}}
 \def\sfx{{\it sfx}}\def\sic{{\ninepoint [sic]}} \def\sk{{\it Sk.}}
 \def\slng{{\it sl}} \def\sng{{\it sing}} \def\st{{\it sth}}
 \def\subj{{\it subj.}} \def\dai{{\it Th.}} \def\theor{{\it theor}}
 \def\us{{\it usu}} \def\vb{{\it v}} \def\viz{{\it viz.}}
 \def\vs{{\it vs}} \def\vul{{\it vul}}}
\def\hindi{\def\aa.{\=a} \def\Aa.{\=A} \def\ii.{\=\i} \def\Ii.{\=I}
 \def\uu.{\=u} \def\Uu.{\=U} \def\rq.{\d r} \def\Rq.{\d R}
 \def\nx.{\.n} \def\Nx.{\.N} \def\nl.{\~n} \def\Nl.{\~N}
 \def\nq.{\d n} \def\Nq.{\d N} \def\sz.{\'s} \def\Sz.{\'S}
 \def\sq.{\d s} \def\Sq.{\d S} \def\kq.{\d k} \def\Kq.{\d K}
 \def\kqh.{\d kh} \def\Kqh.{\d Kh} \def\gq.{\d g} \def\Gq.{\d G}
 \def\rq.{\d r} \def\Rq.{\d R} \def\rqh.{\d rh} \def\Rqh.{\d Rh}}

\def\hungarian{\def\az.{\'a} \def\Az.{\'A} \def\ez.{\'e} \def\Ez.{\'E}
 \def\iz.{\'\i} \def\Iz.{\'I} \def\oe.{\"o} \def\Oe.{\"O}
 \def\ow.{\H o} \def\Ow.{\H O} \def\uz.{\'u} \def\Uz.{\'U}
 \def\ue.{\"u} \def\Ue.{\"U} \def\uw.{\H u} \def\Uw.{\H U}}

\def\lanna{\def\av.{\v a} \def\Av.{\v A} \def\ah.{\`a} \def\Ah.{\`A}
  \def\ay.{\^a} \def\Ay.{\^A} \def\af.{\=a} \def\Af.{\=A}
  \def\az.{\'a} \def\Az.{\'A}
 \def\aq.{\d a} \def\Aq.{\d A} \def\aqv.{\d{\v a}} \def\Aqv.{\d{\v A}}
  \def\aqh.{\d{\`a}} \def\Aqh.{\d{\`A}} \def\aqy.{\d{\^a}} \def\Aqy.{\d{\^A}}
  \def\aqf.{\d{\=a}} \def\Aqf.{\d{\=A}} \def\aqz.{\d{\'a}} \def\Aqz.{\d{\'A}}
 \def\iv.{\v\i} \def\Iv.{\v I}
  \def\ih.{\`\i} \def\Ih.{\`I} \def\iy.{\^\i} \def\Iy.{\^I}
  \def\if.{\=\i} \def\If.{\=I} \def\iz.{\'\i} \def\Iz.{\'I}
 \def\iq.{\d i} \def\Iq.{\d I} \def\iqv.{\d{\v\i}} \def\Iqv.{\d{\v I}}
  \def\iqh.{\d{\`\i}} \def\Iqh.{\d{\`I}} \def\iqy.{\d{\^\i}} \def\Iqy.{\d{\^I}}
  \def\iqf.{\d{\=\i}} \def\Iqf.{\d{\=I}} \def\iqz.{\d{\'\i}} \def\Iqz.{\d{\'I}}
 \def\uev.{\v ue} \def\Uev.{\v Ue} \def\ueh.{\`ue} \def\Ueh.{\`Ue}
  \def\uey.{\^ue} \def\Uey.{\^Ue} \def\uef.{\=ue} \def\Uef.{\=Ue}
  \def\uez.{\'ue} \def\Uez.{\'Ue}
 \def\uqe.{\d ue} \def\Uqe.{\d Ue} \def\uqev.{\d{\v u}e} \def\Uqev.{\d{\v U}e}
  \def\uqeh.{\d{\`u}e} \def\Uqeh.{\d{\`U}e} \def\uqey.{\d{\^u}e} \def\Uqey.{\d{\^U}e}
  \def\uqef.{\d{\=u}e} \def\Uqef.{\d{\=U}e} \def\uqez.{\d{\'u}e} \def\Uqez.{\d{\'U}e}
 \def\uv.{\v u} \def\Uv.{\v U}
  \def\uh.{\`u} \def\Uh.{\`U} \def\uy.{\^u} \def\Uy.{\^U}
  \def\uf.{\=u} \def\Uf.{\=U} \def\uz.{\'u} \def\Uz.{\'U}
 \def\uq.{\d u} \def\Uq.{\d U} \def\uqv.{\d{\v u}} \def\Uqv.{\d{\v U}}
  \def\uqh.{\d{\`u}} \def\Uqh.{\d{\`U}} \def\uqy.{\d{\^u}} \def\Uqy.{\d{\^U}}
  \def\uqf.{\d{\=u}} \def\Uqf.{\d{\=U}} \def\uqz.{\d{\'u}} \def\Uqz.{\d{\'U}}
 \def\ev.{\v e} \def\Ev.{\v E} \def\eh.{\`e} \def\Eh.{\`E} \def\ey.{\^e} \def\Ey.{\^E}
 \def\ef.{\=e} \def\Ef.{\=E} \def\ez.{\'e} \def\Ez.{\'E}
\def\eq.{\d e} \def\Eq.{\d E} \def\eqv.{\d{\v e}} \def\Eqv.{\d{\v E}}
 \def\eqh.{\d{\`e}} \def\Eqh.{\d{\`E}} \def\eqy.{\d{\^e}} \def\Eqy.{\d{\^E}}
 \def\eqf.{\d{\=e}} \def\Eqf.{\d{\=E}} \def\eqz.{\d{\'e}} \def\Eqz.{\d{\'E}}
\def\aev.{\v ae} \def\Aev.{\v Ae}
 \def\aeh.{\`ae} \def\Aeh.{\`Ae} \def\aey.{\^ae} \def\Aey.{\^Ae}
 \def\aef.{\=ae} \def\Aef.{\=Ae} \def\aez.{\'ae} \def\Aez.{\'Ae}
\def\aqe.{\d ae} \def\Aqe.{\d Ae} \def\aqev.{\d{\v a}e} \def\Aqev.{\d{\v A}e}
 \def\aqeh.{\d{\`a}e} \def\Aqeh.{\d{\`A}e} \def\aqey.{\d{\^a}e} \def\Aqey.{\d{\^A}e}
 \def\aqef.{\d{\=a}e} \def\Aqef.{\d{\=A}e} \def\aqez.{\d{\'a}e} \def\Aqez.{\d{\'A}e}
\def\ov.{\v o} \def\Ov.{\v O} \def\oh.{\`o} \def\Oh.{\`O} \def\oy.{\^o} \def\Oy.{\^O}
 \def\of.{\=o} \def\Of.{\=O}  \def\oz.{\'o} \def\Oz.{\'O}
\def\oq.{\d o} \def\Oq.{\d O} \def\oqv.{\d{\v o}} \def\Oqv.{\d{\v O}}
 \def\oqh.{\d{\`o}} \def\Oqh.{\d{\`O}}  \def\oqy.{\d{\^o}} \def\Oqy.{\d{\^O}}
 \def\oqf.{\d{\=o}} \def\Oqf.{\d{\=O}} \def\oqz.{\d{\'o}} \def\oqz.{\d{\'o}}
\def\ouv.{\v ou} \def\Ouv.{\v Ou} \def\ouh.{\`ou} \def\Ouh.{\`Ou}
 \def\ouy.{\^ou} \def\Ouy.{\^Ou} \def\ouf.{\=ou} \def\Ouf.{\=Ou}
 \def\ouz.{\'ou} \def\Ouz.{\'Ou}
\def\ouq.{o\d u} \def\Ouq.{O\d u} \def\ouqv.{\v o\d u} \def\Ouqv.{\v O\d u}
 \def\ouqh.{\`o\d u} \def\Ouqh.{\`O\d u}  \def\ouqy.{\^o\d u} \def\Ouqy.{\^O\d u}
 \def\ouqf.{\=o\d u} \def\Ouqf.{\=O\d u} \def\ouqz.{\'o\d u} \def\ouqz.{\'o\d u}
\def\oqu.{\d ou} \def\Oqu.{\d Ou} \def\oquv.{\d{\v o}u} \def\Oquv.{\d{\v O}u}
 \def\oquh.{\d{\`o}u} \def\Oquh.{\d{\`O}u}  \def\oquy.{\d{\^o}u} \def\Oquy.{\d{\^O}u}
 \def\oquf.{\d{\=o}u} \def\Oquf.{\d{\=O}u} \def\oquz.{\d{\'o}} \def\oquz.{\d{\'o}u}
\def\auv.{\v au} \def\Auv.{\v Au} \def\auh.{\`au} \def\Auh.{\`Au}
 \def\auy.{\^au} \def\Auy.{\^Au} \def\auf.{\=au} \def\Auf.{\=Au}
 \def\auz.{\'au} \def\Auz.{\'Au}
\def\auq.{a\d u} \def\Auq.{A\d u} \def\auqv.{\v a\d u} \def\Auqv.{\v A\d u}
 \def\auqh.{\`a\d u} \def\Auqh.{\`A\d u} \def\auqy.{\^a\d u} \def\Auqy.{\^A\d u}
 \def\auqf.{\=a\d u} \def\Auqf.{\=A\d u} \def\auqz.{\'a\d u} \def\Auqz.{\'A\d u}
\def\aqu.{\d au} \def\Aqu.{\d Au}
 \def\aquv.{\d{\v a}u} \def\Aquv.{\d{\v A}u} \def\aquh.{\d{\`a}u} \def\Aquh.{\d{\`A}u}
 \def\aquy.{\d{\^a}u} \def\Aquy.{\d{\^A}u} \def\aquf.{\d{\=a}u} \def\Aquf.{\d{\=A}u}
 \def\aquz.{\d{\'a}u} \def\Aquz.{\d{\'A}u}
\def\aiv.{\v ai} \def\aiv.{\v ai} \def\aih.{\`ai} \def\aih.{\`ai}
 \def\aiy.{\^ai} \def\aiy.{\^ai} \def\aif.{\=ai} \def\aif.{\=ai}
 \def\aiz.{\'ai} \def\aiz.{\'ai}
\def\aov.{\v ao} \def\Aov.{\v Ao} \def\aoh.{\`ao} \def\Aoh.{\`Ao}
 \def\aoy.{\^ao} \def\Aoy.{\^Ao} \def\aof.{\=ao} \def\Aof.{\=Ao}
 \def\aoz.{\'ao} \def\Aoz.{\'Ao}
\def\amq.{a\d m} \def\Amq.{A\d m} \def\amqv.{\v a\d m} \def\Amqv.{\v A\d m}
 \def\amqh.{\`a\d m} \def\Amqh.{\`A\d m} \def\amqy.{\^a\d m} \def\Amqy.{\^A\d m}
 \def\amqf.{\=a\d m} \def\Amqf.{\=A\d m} \def\amqz.{\'a\d m} \def\Amqz.{\'A\d m}
\def\oav.{\v oa} \def\Oav.{\v Oa} \def\oah.{\`oa} \def\Oah.{\`Oa}
 \def\oay.{\^oa} \def\Oay.{\^Oa} \def\oaf.{\=oa} \def\Oaf.{\=Oa}
 \def\oaz.{\'oa} \def\Oaz.{\'Oa}
\def\oqa.{\d oa} \def\Oqa.{\d Oa} \def\oqav.{\d{\v o}a} \def\Oqav.{\d{\v O}a}
 \def\oqah.{\d{\`o}a} \def\Oqah.{\d{\`O}a} \def\oqay.{\d{\^o}a} \def\Oqay.{\d{\^O}a}
 \def\oqaf.{\d{\=o}a} \def\Oqah.{\d{\=O}a} \def\oqaz.{\d{\'o}a} \def\Oqaz.{\d{\'O}a}
\def\iav.{\v\i a} \def\Iav.{\v Ia} \def\iah.{\`\i a} \def\Iah.{\`Ia}
 \def\iay.{\^\i a} \def\Iay.{\^Ia} \def\iaf.{\=\i a} \def\Iaf.{\=Ia}
 \def\iaz.{\'\i a} \def\iaz.{\'\i a}
\def\iqa.{\d ia} \def\Iqa.{\d Ia} \def\iqav.{\d{\v\i}a} \def\Iqav.{\d{\v I}a}
 \def\iqah.{\d{\`\i}a} \def\Iqah.{\d{\`I}a} \def\iqay.{\d{\^\i}a} \def\Iqay.{\d{\^I}a}
 \def\iqaf.{\d{\=\i}a} \def\Iqaf.{\d{\=I}a} \def\iqaz.{\d{\'\i}a} \def\Iqaz.{\d{\'I}a}
\def\iaq.{i\d a} \def\Iaq.{I\d a} \def\iaqv.{\v\i\d a} \def\Iaqv.{\v I\d a}
 \def\iaqh.{\`\i\d a} \def\Iaqh.{\`I\d a} \def\iaqy.{\^\i\d a} \def\Iaqy.{\^I\d a}
 \def\iaqf.{\=\i\d a} \def\Iaqf.{\=I\d a} \def\iaqz.{\'\i\d a} \def\Iaqz.{\'I\d a}
\def\iqaq.{\d i\d a} \def\Iqaq.{\d I\d a}
\def\iqaqv.{\d{\v\i}\d a} \def\Iqaqv.{\d{\v I}\d a}
 \def\iqaqh.{\d{\`\i}\d a} \def\Iqaqh.{\d{\`I}\d a} \def\iqaqy.{\d{\^\i}\d a}
 \def\Iqaqy.{\d{\^I}\d a} \def\iqaqf.{\d{\=\i}\d a} \def\Iqaqf.{\d{\=I}\d a}
 \def\iqaqz.{\d{\'\i}\d a} \def\Iqaqz.{\d{\'I}\d a}
\def\oev.{\v oe} \def\Oev.{\v Oe} \def\oeh.{\`oe} \def\Oeh.{\`Oe}
 \def\oey.{\^oe} \def\Oey.{\^Oe} \def\oef.{\=oe} \def\Oef.{\=Oe}
 \def\oez.{\'oe} \def\Oez.{\'Oe}
\def\oqe.{\d oe} \def\Oqe.{\d Oe} \def\oqev.{\d{\v o}e} \def\Oqev.{\d{\v O}e}
 \def\oqeh.{\d{\`o}e} \def\Oqeh.{\d{\`O}e} \def\oqey.{\d{\^o}e} \def\Oqey.{\d{\^O}e}
 \def\oqef.{\d{\=o}e} \def\Oqef.{\d{\=O}e} \def\oqez.{\d{\'o}e} \def\Oqez.{\d{\'O}e}
\def\uav.{\v ua} \def\Uav.{\v Ua} \def\uah.{\`ua} \def\Uah.{\`Ua}
 \def\uay.{\^ua} \def\Uay.{\^Ua} \def\uaf.{\=ua} \def\Uaf.{\=Ua}
 \def\uaz.{\'ua} \def\Uaz.{\'Ua}
\def\uqa.{\d ua} \def\Uqa.{\d Ua} \def\uqav.{\d{\v u}a} \def\Uqav.{\d{\v U}a}
 \def\uqah.{\d{\`u}a} \def\Uqah.{\d{\`U}a} \def\uqay.{\d{\^u}a} \def\Uqay.{\d{\^U}a}
 \def\uqaf.{\d{\=u}a} \def\Uqaf.{\d{\=U}a} \def\uqaz.{\d{\'u}a} \def\Uqaz.{\d{\'U}a}
\def\tq.{\d t} \def\Tq.{\d T} \def\tqh.{\d th} \def\Tqh.{\d Th}
 \def\dq.{\d d} \def\Dq.{\d D} \def\dqh.{\d dh} \def\Dqh.{\d Dh}
 \def\nq.{\d n} \def\Nq.{\d N} \def\yq.{\d y} \def\Yq.{\d Y}
 \def\sz.{\'s} \def\Sz.{\'S} \def\lqq.{\d l} \def\Lq.{\d L}
 \def\anqg.{a\d ng} \def\Anqg.{A\d ng} \def\angq.{an\d g} \def\Angq.{An\d g}
 \def\hq.{\d h} \def\Hq.{\d H} \def\yz.{\'y} \def\Yz.{\'Y}
 \def\yqb.{\d{\`y}} \def\Yqb.{\d{\`Y}} \def\jbh.{\`\j h} \def\Jbh.{\`Jh}
```

```
  \def\tqbh.{\d{\`t}h} \def\Tqbh.{\d{\`T}h} \def\nb.{\`n} \def\Nb.{\`N}
  \def\pb.{\`p} \def\Pb.{\`P} \def\pbh.{\`ph} \def\Pbh.{\`Ph}
  \def\bb.{\`b} \def\Bb.{\`B} \def\mb.{\`m} \def\Mb.{\`M}
  \def\iqa.{\d ia} \def\Iqa.{\d Ia} \def\lb.{\`l} \def\Lb.{\`L}
  \def\sb.{\`s} \def\Sb.{\`S} \def\na.{na} \def\Na.{Na}
  \def\yh.{\`y} \def\Yh.{\`Y} \def\pq.{\d p} \def\Pq.{\d P}
  \def\fq.{\d f} \def\Fq.{\d F} \def\ax.{\.a} \def\Ax.{\.A}}
\def\latin{\def\Af.{\=A \def\af.{\=a}\def\Ef.{\=E}\def\ef.{\=e}\def\If.{\=I}\def\if.{\=\i}
  \def\Of.{\=O} \def\of.{\=o} \def\aa.{\=a} \def\Aa.{\=A}
 \def\ee.{\=e} \def\Ee.{\=E} \def\ii.{\=\i} \def\Ii.{\=I}
 \def\oo.{\=o} \def\Oo.{\=O} \chardef\aE="1A \chardef\oE="1B
 \def\Aee.{\AE } \def\aee.{\aE } \def\Oee.{\OE } \def\oee.{\oE }}
\def\lating{\def\[##1]{{\it ##1}} \def\(##1){{\bf ##1}} \def\und{{\eightrm~and~}}
 \def\nm{{\eightit nom.}} \def\vo{{\eightit voc.}} \def\ac{{\eightit acc.}}
 \def\gn{{\eightit gen.}} \def\datv{{\eightit dat.}} \def\ab{{\eightit abl.}}
 \def\sg{{\eightit sg.}} \def\pl{{\eightit pl.}}
 \def\nmsg##1:{{\eightit nom.sg.} \(##1)} \def\vosg##1:{{\eightit voc.sg.} \(##1)}
  \def\acsg##1:{{\eightit acc.sg.} \(##1)} \def\gnsg##1:{{\eightit gen.sg.} \(##1)}
  \def\dtsg##1:{{\eightit dat.sg.} \(##1)} \def\absg##1:{{\eightit abl.sg.} \(##1)}
  \def\nmpl##1:{{\eightit nom.pl.} \(##1)} \def\vopl##1:{{\eightit voc.pl.} \(##1)}
  \def\acpl##1:{{\eightit acc.pl.} \(##1)} \def\gnpl##1:{{\eightit gen.pl.} \(##1)}
  \def\dtpl##1:{{\eightit dat.pl.} \(##1)} \def\abpl##1:{{\eightit abl.pl.} \(##1)}
 \def\dif[##1]{##1\[a], ##1\[am], ##1\[ae], ##1\[ae], ##1\[\af.],
  ##1\(ae), ##1\(\af.s), ##1\(\af.rum), ##1\(\if.s), ##1\(\if.s)}
 \def\diim[##1]{##1\[us], ##1\[um], ##1\[\if.], ##1\[\of.], ##1\[\of.],
  ##1\(\if.), ##1\(\of.s), ##1\(\of.rum), ##1\(\if.s), ##1\(\if.s)}
 \def\diima[##1]{\sg\nm\ ##1\[us], \vo\ ##1\[e], \ac\ ##1\[um], \gn\ ##1\[\if.],
  \dt\ ##1\[\of.], \ab\ ##1\[\of.], \pl\nm\und\vo\ ##1\(\if.), \ac\ ##1\(\of.s),
  \gn\ ##1\(\of.rum), \dt\und\ab\ ##1\(\if.s)}
 \def\diin[##1]{##1\[um], ##1\[um], ##1\[\if.], ##1\[\of.], ##1\[\of.],
  ##1\(a), ##1\(a), ##1\(\of.rum), ##1\(\if.s), ##1\(\if.s)}
 \def\diif[##1]{##1\[er], ##1\[erum], ##1\[r\if.], ##1\[r\of.], ##1\[r\of.],
  ##1\(r\if.), ##1\(r\of.s), ##1\(r\of.rum), ##1\(r\if.s), ##1\(r\if.s)}
 \def\diiia:##1[##2]{\sg\nm\und\vo\ ##1, \ac\ ##2\[em], \gn\ ##2\[is], \dt\ ##2\[\if.],
  \ab\ ##2\[e], \pl\nm, \vo\und\ac\ ##2\[\ef.s], \gn\ ##2\[um], \dt\und\ab\ ##2\[ibus]}}
\def\math{\def\bap{\bigcap} \def\bcp{\bigcup} \def\I{\hbox{I}} \def\rar{\rightarrow}
 \def\e:##1x##2[##3]{\hbox{$##1\times ##2^{##3}$}}
  \def\Er[##1,##2]{$##1\pm ##2$} \def\C(##1,##2){{}^{##1}\hbox{C}_{##2}}
  \def\Cc(##1,##2){\pmatrix{##1\cr##2}} \def\Det|##1|{\left\vert\matrix{##1}\right\vert}
  \def\eno(##1){\eqno{(##1)}} \def\eln{\eqalign} \def\elno{\eqalignno}
  \def\frc[##1/##2]{{##1\over##2}} \def\B[##1]{\hbox{##1}}
  \def\C[##1,##2]{\hbox{${}^{##1}{\rm C}_{##2}$}}
  \def\Cm[##1,##2]{\hbox{${}_{##1}{\rm C}_{##2}$}}
  \def\Pmu[##1,##2]{\hbox{${}^{##1}{\rm P}_{##2}$}}
 \def\bmap{\begingroup \def\normalbaselines{\baselineskip20pt
   \lineskip3pt \lineskiplimit3pt}}
 \def\mprt##1{\smash{\mathop{\hbox{\rightarrowfill}}\limits^{~##1~}}}
 \def\mprg##1{\smash{\mathop{\longrightarrow}\limits^{~##1~}}}
 \def\mpdn##1{\Big\downarrow\rlap{$\vcenter{\hbox{$\scriptstyle##1$}}$}}
 \def\emap{\endgroup} \def\mtc[##1]{\matrix{##1}}
  \def\mtx[##1]{\left[\matrix{##1}\right]}
  \def\vtr[##1:##2:##3]{##1_{##2},\ldots,##1_{##3}}
 \outer\def\plm ##1.##2\par{\medbreak \noindent{\it##1.\enspace}##2\par
  \ifdim\lastskip<\medskipamount \removelastskip\penalty55\medskip\fi}
 \def\V[##1]{\overrightharpoonup{##1}} \def\epnr{\bar{\epsilon}}
 \def\Ar{\overline{A}} \def\Vr{{\overline{V}}}
 \def\dr{\bar{d}} \def\fr{\bar{f}} \def\nr{\bar{n}} \def\pr{\bar{p}} \def\ur{\bar{u}}
 \def\wrr{{\bar{w}}} \def\xr{\bar{x}} \def\yr{\bar{y}} \def\zr{\bar{z}}
 \def\xld{\tilde{x}} \def\yld{\tilde{x}} \def\Ald{\widetilde{A}} \def\Xld{\widetilde{X}}
 \def\Ab{{\bf A}} \def\bb{{\bf b}} \def\Bb{{\bf B}} \def\Cb{{\bf C}} \def\Db{{\bf S}}
  \def\Eb{{\bf E}} \def\Ib{{\bf I}} \def\Pb{{\bf P}} \def\rb{{\bf r}} \def\Tb{{\bf T}}
  \def\ub{{\bf u}} \def\xb{{\bf x}} \def\yb{{\bf y}} \def\xbh{\hat{\bf x}}
 \def\Ed{\dot{E}} \def\Fd{\dot{F}} \def\Md{\dot{M}} \def\Vd{\dot{V}}
 \def\md{\dot{m}} \def\sd{\dot{s}} \def\xd{\dot{x}} \def\yd{\dot{y}} \def\zd{\dot{z}}
  \def\xbd{\dot{\xb}} \def\aht{\hat{a}} \def\bht{\hat{b}} \def\cht{\hat{c}}
  \def\thb{{\bf\theta}} \def\thh{\hat{\theta}}
  \def\thbh{\hat{\bf\theta}} \def\xibh{\hat{\bf\xi}}
 \def\Bl{{\cal B}} \def\Cl{{\cal C}} \def\Fl{{\cal F}} \def\Hl{{\cal H}}
 \def\NL{{\cal N}} \def\Ol{{\cal O}} \def\Pl{{\cal P}} \def\Tl{{\cal T}}
 \def\Vl{{\cal V}} \def\Rl{{\cal R}} \def\dm{{\rm d}} \def\fm{{\rm f}}
 \def\Am{{\rm A}} \def\Dm{{\rm D}} \def\Em{{\rm E}} \def\Im{{\rm I}} \def\Nm{{\rm N}}
  \def\Om{{\rm O}} \def\Pm{{\rm P}} \def\Rm{{\rm R}} \def\Tm{{\rm T}} \def\Vm{{\rm V}}
  \def\Xm{{\rm X}} \def\Ym{{\rm Y}} \def\Zm{{\rm Z}}
 \def\p{\prime} \def\pp{{\prime\prime}} \def\ppp{{\prime\prime\prime}}
 \def\alp{\aleph} \def\and{\hbox{and}}\def\ang{\angle} \def\aph{\alpha}\def\Aph{\Alpha}
  \def\apx{\approx} \def\bap{\bigcap} \def\bdbl{\begindoublecolumns}
  \def\bkh{\hbox{$\backslash$}} \def\bo{\cal O} \def\bop{\bigoplus}
  \def\bskp{\bigskip} \def\bta{\beta} \def\bup{\bigcup} \def\cdt{\cdot} \def\cds{\cdots}
  \def\css{\times} \def\cir{\circ \def\dg{\hbox{$^\circ$}}\def\dlt{\delta} \def\Dlt{\Delta}
  \def\dtr{\bigtriangledown} \def\edbl{\enddoublecolumns} \def\eightp{\eightpoint}
  \def\ems{\emptyset} \def\epn{\epsilon} \def\Epn{\Epsilon} \def\fll{\forall}
  \def\fnt{\footnote} \def\gmm{\gamma} \def\Gmm{\Gamma}
  \def\har{\hbar} \def\hfl{\hfil} \def\hll{\hfill} \def\icl{\supset}\def\imt{\imath}
  \def\iny{\infty} \def\itg{\intop\nolimits} \def\itm{\item} \def\itmm{\itemitem}
  \def\ity{\infty} \def\jmt{\jmath} \def\kpa{\kappa} \def\lar{\leftarrow}
  \def\lds{\ldots} \def\lfl{\lfloor} \def\llr{\longleftrightarrow}
  \def\Llr{\Longleftrightarrow} \def\lmd{\lambda} \def\Lmd{\Lambda} \def\lng{\langle}
  \def\lra{\leftrightarrow} \def\Lra{\Leftrightarrow}
  \def\lrw{\longrightarrow} \def\Lrw{\Longrightarrow} \def\mpt{\mapsto}
  \def\nba{\nabla} \def\nind{\noindent} \def\ninep{\ninepoint} \def\ols{\obeylines}
  \def\opl{\oplus} \def\ovr{\over}\def\omg{\omega} \def\Omg{\Omega} \def\oss{\obeyspaces}
  \def\pint{\parindent} \def\prc{\prec} \def\prd{\prod}\def\ptl{\partial}\def\prt{\propto}
  \def\qiv{\equiv} \def\rar{\rightarrow} \def\rfl{\rfloor} \def\rng{\rangle}
  \def\scc{\succ} \def\sgm{\sigma} \def\Sgm{\Sigma} \def\som{\asymp} \def\sqt{\sqrt}
  \def\sst{\subset} \def\stt{\hbox{ s.t. }} \def\sub{\subset} \def\tms{\times}
  \def\trg{\triangle} \def\tta{\theta} \def\Tta{\Theta} \def\twbf{\twelvebf}
  \def\twrm{\twelverm} \def\utr{\bigtriangleup} \def\vep{\varepsilon} \def\vph{\varphi}
  \def\vrt{\vert}\def\vrtm{$\vert$}\def\vskp{\vskip}\def\wdg{\wedge}\def\wtld{\widetilde}
  \def\xst{\exists} \def\zta{\zeta}
 \def\q{\quad} \def\qq{\qquad} \def\lf{\left} \def\rt{\right}
 \def\Abs{\mathop{\rm abs}\nolimits} \def\abs{\mathop{\rm abs}\nolimits}
  \def\arc{\mathop{\rm arc}\nolimits} \def\cosec{\mathop{\rm cosec}\nolimits}
  \def\D{{\rm d}} \def\det{\mathop{\rm det}\nolimits}
  \def\Div{\mathop{\rm div}\nolimits} \def\Em{\mathop{\rm E}\nolimits}
  \def\erf{\mathop{\rm erf}\nolimits} \def\Grad{\mathop{\rm grad}\nolimits}
  \def\inf{\mathop{\rm inf}\nolimits} \def\Kur{\mathop{\cal K}\nolimits}
  \def\min{\mathop{\rm min}\nolimits} \def\Min{\mathop{\rm min}\nolimits}
  \def\max{\mathop{\rm max}\nolimits} \def\Max{\mathop{\rm max}\nolimits}
  \def\mod{\mathop{\rm mod}\nolimits} \def\Mom{\mathop{\cal M}\nolimits}
  \def\Med{\mathop{\rm med}\nolimits} \let\med=\Med
  \def\Mad{\mathop{\rm mad}\nolimits} \let\mad=\Mad
  \def\Neb{\mathop{\aleph}\nolimits} \def\Nor{\mathop{{\cal N}^c_v}\nolimits}
  \def\Oc{\mathop{\rm O}\nolimits} \def\Per{\mathop{\cal P}\nolimits}
  \def\Rl{\mathop{\rm Re}\nolimits} \def\sgn{\mathop{\rm sgn}\nolimits}
  \def\sup{\mathop{\rm sup}\nolimits} \def\Voy{\mathop{\cal V}\nolimits}
 \def\centre##1{\line{\hss##1\hss}} \def\haln{\halign} \def\ovln{\overline}
  \def\lline##1{\line{##1\hss}} \def\mspn{\multispan} \def\pshp{\parshape}
  \def\rline##1{\line{\hss##1}} \def\sskp{\smallskip}
 \def\nxs{\raise1pt\hbox{$/$}\kern-5.1pt\xst}
 \def\proof{\noindent{\bf Proof.} } \def\endprf{\hfill$\sqr$}
 \def\qedprf{\par\hfill{q.e.d.}}}
\def\money{\def\pound{{\it \$}}}
\def\nihongo{\def\kanji(##1)[##2]{\setbox1=\hbox{##1} \dtmp=\wd1
 \setbox2=\vbox{\hsize=\dtmp\noindent\sevenrm--\hfil##2\hfil--}
 \hbox{}\kern-1em$\mathrel{\mathop{\kern\z@\copy1}
 \limits_{\sevenrm\kern.5em\raise.5em\copy2}}$}
 \def\radcal(##1)[##2]{\setbox1=\hbox{##1} \dtmp=\wd1
  \setbox2=\vbox{\hsize=\dtmp\noindent\sevenbf--\hfil##2\hfil--}
  \hbox{}\kern-1em$\mathrel{\mathop{\kern\z@\copy1}
 \limits_{\sevenrm\kern.5em\raise.5em\copy2}}$}
 \def\Ou.{\=O} \def\ou.{\=o} \def\Uu.{\=U} \def\uu.{\=u} \def\Arak.{\kanji(Ara)[k]}
 \def\Hiroshis.{\kanji(Hiroshi)[s]} \def\kawa.{\radcal(kawa)[b]}
 \def\Kawa.{\radcal(Kawa)[b]}\def\Suzug.{\kanji(Suzu)[g]}}
\def\norge{\def\aA{\accent23a} \def\axx.{\aA } \def\Axx.{\AA }
 \def\oz.{\'o} \def\Oz.{\'O} \chardef\oO="1C \def\ozz.{\oO } \def\Ozz.{\O }}
\def\pali{\def\aa.{\=a} \def\Aa.{\=A} \def\ii.{\=\i} \def\Ii.{\=I}
  \def\uu.{\=u} \def\Uu.{\=U}
 \def\dq.{\d d} \def\Dq.{\d D}  \def\dqh.{\d dh} \def\Dqh.{\d Dh}
  \def\tq.{\d t} \def\Tq.{\d T}  \def\tqh.{\d th} \def\Tqh.{\d Th}
  \def\nq.{\d n}\def\Nq.{\d N} \def\nl.{\~n}\def\Nl.{\~N} \def\nx.{\.n}\def\Nx.{\.N}}
\def\physics{\def\unt[##1]{$\rm ##1$} \def\mum{$\mu$m} \def\Rey{{\rm Re}}}
\def\polish{\def\aj.{\c a} \def\Aj.{\c A} \def\cz.{\'c} \def\Cz.{\'C}
 \def\ej.{\c e} \def\Ej.{\c E} \def\lz.{\'l} \def\Lz.{\'L}
 \def\lzz.{\l } \def\Lzz.{\L } \def\nz.{\'n} \def\Nz.{\'N}
 \def\oz.{\'o} \def\Oz.{\'O} \def\sz.{\'s} \def\Sz.{\'S}
 \def\zz.{\'z} \def\Zz.{\'Z} \def\zx.{\.z} \def\Zx.{\.Z}
 \def\dzz.{d\'z} \def\Dzz.{D\'z} \def\dzx.{d\.z} \def\Dzx.{D\.z}}
\def\russian{\def\iv.{\v\i} \def\Iv.{\v I} \def\eee.{\"e} \def\Ee.{\"E}}
\def\sanskrit{\def\aa.{\=a} \def\Aa.{\=A} \def\ii.{\=\i} \def\Ii.{\=I}
 \def\uu.{\=u} \def\Uu.{\=U} \def\dq.{\d d} \def\Dq.{\d D}
  \def\dqh.{\d dh} \def\Dqh.{\d Dh}  \def\tq.{\d t} \def\Tq.{\d T}
  \def\tqh.{\d th} \def\Tqh.{\d Th} \def\nx.{\.n} \def\Nx.{\.N}
  \def\nl.{\~n} \def\Nl.{\~N} \def\rq.{\d r} \def\Rq.{\d R} \def\rqh.{\d rh}
  \def\Rqh.{\d Rh} \def\lqq.{\d l} \def\Lq.{\d L} \def\mq.{\d m}
  \def\Mq.{\d M} \def\hq.{\d h} \def\Hq.{\d H}
 \def\sz.{\'s} \def\Sz.{\'S} \def\sq.{\d s} \def\Sq.{\d S}}
\def\serbo{\def\cv.{\v c} \def\Cv.{\v C} \def\cz.{\'c} \def\Cz.{\'C}
 \def\dz.{\'d} \def\Dz.{\'D} \def\dzv.{d\v z} \def\Dzv.{D\v z}
 \def\sv.{\v s} \def\Sv.{\v S} \def\zv.{\v z} \def\Zv.{\v Z}}
\def\slovak{\def\az.{\'a} \def\Az.{\'A} \def\iz.{\'\i} \def\Iz.{\'I}
 \def\yz.{\'y} \def\Yz.{\'Y} \def\oz.{\'o} \def\Oz.{\'O}
 \def\uz.{\'u} \def\Uz.{\'U} \def\ae.{\"a} \def\Ae.{\"A}
 \def\oy.{\^o} \def\Oy.{\^O} \def\rz.{\'r} \def\Rz.{\'R}
 \def\lz.{\'l} \def\Lz.{\'L} \def\cv.{\v c} \def\Cv.{\v C}
 \def\sv.{\v s} \def\Sv.{\v S} \def\zv.{\v z} \def\Zv.{\v Z}
 \def\dzv.{d\v z} \def\Dzv.{D\v z} \def\dv.{\v d} \def\Dv.{\v D}
 \def\tv.{\v t} \def\Tv.{\v T} \def\nv.{\v n} \def\Nv.{\v N}
 \def\lv.{\v l} \def\Lv.{\v L}}
\def\tex{\long\def\veb##1{\def\next{##1}{\tt\frenchspacing\expandafter\strip\meaning\next}}
 \def\strip##1>{} \def\bx{{\it bx}} \def\kx{{\it kx}}  \def\mx{{\it mx}}
 \def\plx{{\it plx}} \def\px{{\it px}} %TeX book, KiteX, manmac, plain TeX, primitive TeX}
\def\vietnamese{\def\dt.{d\kern-.5em\raise.25em\hbox{--}}
  \def\Dt.{{D\kern-.75em\raise.1em\hbox{--}\kern.25em}}
 \def\ap.{\u a} \def\Ap.{\u A} \def\ay.{\^a} \def\Ay.{\^A}
 \def\ey.{\^e} \def\Ey.{\^E} \def\oy.{\^o} \def\Oy.{\^O}
 \def\az.{\'a} \def\Az.{\'A}
 \def\as.{a\kern-.5em\raise.4em\hbox{?}}
 \def\As.{{A\kern-.6em\raise.59em\hbox{?}\kern.1em}}
 \def\al.{\~a} \def\Al.{\~A} \def\aq.{\d a} \def\Aq.{\d A}
 \def\ez.{\'e} \def\Ez.{\'E} \def\eh.{\`e} \def\Eh.{\`E}
 \def\es.{e\kern-.5em\raise.39em\hbox{?}} \def\Es.{E\kern-.5em\raise.62em\hbox{?}}
 \def\el.{\~e} \def\El.{\~E} \def\eq.{\d e} \def\Eq.{\d E}}
\def\zhongwen{\def\af.{\=a} \def\az.{\'a} \def\av.{\v a} \def\ah.{\`a}
  \def\eif.{\=ei} \def\eiz.{\'ei} \def\eiv.{\vei} \def\eih.{\`ei}
  \def\if.{\=\i} \def\iz.{\'\i} \def\iv.{\v\i} \def\ih.{\`\i}
  \def\of.{\=o} \def\oz.{\'o} \def\ov.{\v o} \def\oh.{\`o}
  \def\ouf.{\=ou} \def\ouz.{\'ou} \def\ouv.{\v ou} \def\ouh.{\`ou}
  \def\uf.{\=u} \def\uz.{\'u} \def\uv.{\v u} \def\uh.{\`u}
  \def\uof.{u\=o} \def\uoz.{u\'o} \def\uov.{u\v o} \def\uoh.{u\`o}
 \def\putji(##1)[##2]{\begingroup\setbox1=\hbox{##1} \dtmp=\wd1
  \setbox2=\vbox{\hsize=\dtmp\noindent\sevenrm--\hfil##2\hfil--}
  \hbox{}\kern-1em$\mathrel{\mathop{\kern\z@\copy1}
  \limits_{\sevenrm\kern.5em\raise.5em\copy2}}$\endgroup}
 \def\putbs(##1){\begingroup\setbox1=\hbox{##1} \dtmp=\wd1
  \setbox2=\vbox{\hsize=\dtmp\noindent\sevenrm$\cdot$\hfil} \hbox{}\kern-1em
  $\mathrel{\mathop{\kern\z@\copy1}\limits_{\sevenrm\kern.5em\raise.7em\copy2}}$
  \lineskip-.5em\endgroup}
 \def\aifc.{\putji(\=ai)[c]}\def\Aifc.{\putji(\=Ai)[c]}\def\aifcm.{{\it grief or sorrow}}
 \def\baofc.{\putji(b\=ao)[c]} \def\Baofc.{\putji(B\=ao)[c]
 \def\baofcm.{{\it praise or commend}} \def\biaovc.{\putji(bi\v ao)[c]}
 \def\Biaovc.{\putji(Bi\v ao)[c]} \def\biaovcm.{{\it surface}}
 \def\caizc.{\putji(c\'ai)[c]} \def\Caizc.{\putji(C\'ai)[c]}
 \def\caizcm.{{\it cut paper or cloth}} \def\chazng.{\putbs(ch\'ang)}
 \def\Chazng.{\putbs(Ch\'ang)} \def\daihc.{\putji(d\`ai)[c]}\def\Daihc.{\putji(D\`ai)[c]}
 \def\daihcm.{{\it a sack or bag}} \def\gofng.{\putbs(g\=ong)}\def\Gofng.{\putbs(G\=ong)}
 \def\guovc.{\putji(gu\v o)[c]}\def\Guovc.{\putji(Gu\v o)[c]}
 \def\guovcm.{{\it wrap or bind}} \def\livc.{\putji(l\v\i)[c]}
 \def\Livc.{\putji(L\v\i)[c]} \def\livcm.{{\it lining or inside}}
 \def\lozng.{\putbs(l\'ong)} \def\Lozng.{\putbs(L\'ong)}
 \def\mizngry.{\putji(m\'\i ng)[ry]} \def\Mizngry.{\putji(M\'\i ng)[ry]}
 \def\niaovc.{\putji(ni\v ao)[c]} \def\Niaovc.{\putji(Ni\v ao)[c]}
 \def\niaovcm.{{\it slender and delicate}}
 \def\qiuzc.{\putji(qi\'u)[c]} \def\Qiuzc.{\putji(Qi\'u)[c]} \def\qiuzcm.{{\it fur coat}}
 \def\rih.{\putbs(r\`\i)} \def\Rih.{\putbs(R\`\i)} \def\rihm.{{\it the Sun}}
 \def\shuaifc.{\putji(shu\=ai)[c]} \def\Shuaifc.{\putji(Shu\=ai)[c]}
 \def\shuaifcm.{{\it decline}} \def\xiafngc.{\putji(xi\=ang)[c]}
 \def\Xiafngc.{\putji(Xi\=ang)[c]} \def\xiafngcm.{{\it assist}}
  \def\xiehc.{\putji(xi\`e)[c]} \def\Xiehc.{\putji(Xi\`e)[c]} \def\xiehcm.{{\it blaspheme}}
  \def\xizc.{\putji(x\'\i)[c]} \def\Xizc.{\putji(X\'\i)[c]} \def\xizcm.{{\it raid}}
 \def\yif.{\putbs(y\=\i)} \def\Yif.{\putbs(Y\=\i)} \def\yifm.{{\it clothes}}
  \def\yihc.{\putji(y\`\i)[c]} \def\Yihc.{\putji(Y\`\i)[c]} \def\yihcm.{{\it descendants}}
  \def\yueh.{\putbs(yu\`e)} \def\Yueh.{\putbs(Yu\`e)} \def\yuehm.{{\it the moon}}
 \def\zhafngg.{\putji(zh\=ang)[g]} \def\Zhafngg.{\putji(Zh\=ang)[g]}
  \def\zhihc.{\putji(zh\`\i)[c]} \def\Zhihc.{\putji(Zh\`\i)[c]}
  \def\zhihcm.{{\it control, make, or system}} \def\zhofngc.{\putji(zh\=ong)[c]}
  \def\Zhofngc.{\putji(Zh\=ong)[c]} \def\zhofngcm.{{\it inner feelings}}
  \def\zhuafngc.{\putji(zhu\=ang)[c]} \def\Zhuafngc.{\putji(Zhu\=ang)[c]}
  \def\zhuafngcm.{{\it clothing}}}
```

§ A.13 Program for the dissertation

```
% dst.tex   December 2002
\input thshead\head1in
\def\kart#1[#2]#3:#4:#5:{\advance\kpc by 1 \global\count1=\kpc
 \ix[#2]({{\sevenrm KNT\hxdc(\romannumeral\the\kpc)}})\par}\input thskbb
\def\kart#1[#2]#3:#4:#5:{\advance\kpc by 1 \global\count1=\kpc
 \ix[#2]({{\sevenrm KNT\hxdc(\romannumeral\the\kpc)}})
 {\rm #1}~#3~{\it #4} #5 \x[#2].\par}
\begingroup\tchd\titlepage
\centre{\titlefont Division of space by Voronoi graphs,}\bskp
\centre{\titlefont application to the models of porous membranes,}\vskp.7in
\centre{\epsfxsize=1in \epsffile{umist.ps}}\vskp.9in
\centre{\twrm A thesis submitted to the}\vskp.3in
\centre{\twrm University of Manchester Institute of Science and Technlogy}\vskp.3in
```

```
\centre{\twrm For the degree of Doctor of Philosophy}\vskp.9in
\centre{\titlefont Kittisak Nui Tiyapan}\vskp.3in
\centre{\twrm BEng, BSc, BEng, MSc}\vskp.3in
\centre{\twrm Department of Chemical Engineering}\vskp.3in
\centre{\twrm \date}\cut
\head1in\titlepage\parindent=0in\(Declaration)\ct
The work that \S\ \x[phyd] in this thesis is based on has been submitted in
support of an application for the degree of B.Eng. in Mineral Engineering at the
Chulalongkorn University, Bangkok, Thailand, in the year 1991 AD.\ct
\rline{Kit Tiyapan}\cut
\head1in\titlepage
\rline{To my father and my mother Niwat and Somjit Tiyapan,}
\rline{and my chemistry and sword teacher Siddhiponr Songsataya.}
\rline{To Emeritus Professor David J. Bell}
\rline{and Emeritus Professor Graham Arthur Davies,}
\rline{both of whom have started this project,}
\rline{one of whom has powered it.}
\rline{To these people, then, is my dedication of this thesis,}
\rline{but the thesis may dedicate itself to whomever it wishes.}\cut
\head.3in \newwrite\toc
\lox(Table of contents)[dsttoc] \immediate\openout\toc=dsttoc\newwrite
\lof\lox(List of figures)[dstlof] \immediate\openout\lof=dstlof\newwrite\lot
\lox(List of tables)[dstlot] \immediate\openout\lot=dstlot\newwrite\lom
\lox(List of theorems)[dstlom] \immediate\openout\lom=dstlom\newwrite\loq
\lox(List of algorithms)[dstlog] \immediate\openout\loq=dstlog\newwrite\lod
\lox(List of definitions)[dstlod] \immediate\openout\lod=dstlod\newwrite\los
\lox(List of assumptions)[dstlos] \immediate\openout\los=dstlos
\loy(Acknowledgements)[thsack]
\loy(Notation)[thsnot]\din.\tenbf.\tenpoint.thsnot.\endgroup\cut\head0in
\chp[cfnd]Introduction:thstro:\sct[smth]Mathematics:dstmth:\sct[sgem]Geometry:thsgem:
 \sct[sphy]Physics:thsphy:\sct[scsm]Cosmological structure:thscsm:
 \sct[sflt]Filtration:dstflt:\sct[sstt]Statistics:thsstt:
 \sct[spoi]Poisson process:dstpoi:\sct[sptr]Phase transition:thsptr:
 \sct[sran]Random processes:thsran:\sct[snat]Structures in nature:thsnat:
 \sct[scpg]Computational geometry:dstcpg:\sct[salg]Geometric algorithms:dstalg:
\chp[cdiv]Division of space:dstdiv:\sct[sste]Stereographic projection:thsste:
\sct[scov]Covering lattices:thscov:
 \sct[sfng]Viscous fingers:thsfng:\sct[scrs]Crystals, quasicrystals, and polycrystals:thscrs:
 \sct[srdl]Random lines:dstrdl:\sct[scxh]Convex hull:dstcxh:
 \sct[sprv]From convex hull to the Voronoi tessellation:dstprv:
\sct[stlg]Tilings and patterns:thstlg:
\chp[cvor]Voronoi tessellation:thsvn:\sct[sqeq]Quadratic equations:thsqeq:
\sct[sqfm]Quadratic forms:thsqfm:
 \sct[sval]Voronoi algorithms:dstval:\sct[svst]Voronoi statistics in literature:dstvst:
 \sct[ssta]Voronoi statistics:thssta:\sct[svst]Voronoi section:thsvst:
 \sct[snmv]Number of vertices and edges:thsnumv:
 \sct[svfc]Faces in different dimensions:thsvfc:
 \sct[spcb]Beam intersection study:thspcb:\sct[svov]Voronoi of Voronoi:dstvov:
 \sct[strv]Transformations of Voronoi:dsttrv:\sct[scxv]Compressed Voronoi:dstcxv:
 \sct[svhg]Voronoi tessellation in higher dimensions:dstvhg:
\chp[cplt]Percolation:thspcl:\sct[sscv]S-curves and the percolative phenomena:dstscv:
 \sct[sv2d]Voronoi percolation in two dimensions:thsv2d:
 \sct[sv3d]Voronoi percolation in three dimensions:thsv3d:
 \sct[sv2s]Percolation of 2-dimensional Voronoi sections:thsv2s:
 \sct[sntp]Network percolation:thsnpc:
 \sct[sscl]Percolation statistics in literature:fypscl:
 \sct[scnm]Percolation of $n$-gons in continuum:thscnm:
 \sct[spgt]Polygon percolation threshold:dstpgt:\sct[shmd]2-homohedral tilings:thshmd:
 \sct[scgy]Cosmology:thscgy:\sct[sphl]Philosophy:thsphl:\sct[sdcv]Enlightenment:thsdcv:
 \sct[scrm]Forest fire fighting:thsfir:\sct[sfrc]Fractals and percolation:thsfrc:
\chp[cmmb]Porous media:dstprm:\sct[smdc]Medical science:thsmdc:
 \sct[szeo]Zeolites:thszeo:\sct[scry]Crystalisation:dstcry:
 \sct[sflw]Fluid flow within networks:thsflw:\sct[smat]Material science:thsmat:
 \sct[sclm]Forces between particles:thsclm:\sct[sire]Arbitrarily shaped particles:dstarb:
 \sct[snnp]Non Poisson number distributions of particles:dstnnp:
\chp[cfmb]Filtering membranes:dstfmb:\sct[ssls]Separation processes:dstsls:
 \sct[sdef]Dead-end filtration:dstdef:\sct[scgt]The centre of gravity:dstcgt:
 \sct[smdm]Molecular dynamics:dstmdm:
 \sct[spad]Problem definition and algorithms:dstpad:
 \sct[sfti]Simplified algorithm for filtration:dstfti:
 \sct[ssam]Filtering problem when physical blockage is prominent:dstsam:
 \sct[spfg]Percolative filtering with very small particles:dstpfg:
 \sct[spwp]Percolation within percolation:dstpwp:
 \sct[stlg]The first part, suspended particles:dsttlg:
 \sct[sftc]The second part, flow through the cells:dstftc:
\chp[cptm]Percolation in traffic modelling:thstrf:\sct[strf]Traffic networks:dsttnt:
 \sct[sprn]Percolation of road networks:dstprn:
 \sct[sgph]Theory of graphs and its applications:dstgph:
\chp[cpem]Percolation theory as economic models:dstcem:
 \sct[sobj]Object-location:thsobj:sct[seco]Economic modelling:thseco:
\chp[cadd]Addenda::\sct[ssoc]Sociology:thssoc:\sct[scon]Control systems:thscon:
\sct[stex]\TeX nicalities:thstex:
```

```
\chp[ccon]Conclusion:dstcon:
\app[aprg]Programs::\sap[sobp]Object location:\listing{pgmobj}:
 \sap[stwv]Network percolation, two dimensions:\listing{pgmtwv}:
 \sap[sthv]Network percolation, three dimensions:\listing{pgmthv}:
 \sap[stws]Network percolation, 2--d section:\listing{pgmtws}:
 \sap[scnn]Continuum percolation of $n$-gons:\listing{pgmcnn}:
 \sap[sptl]Tilings:\listing{pgmptl}:\sap[spcv]Covering lattices:\listing{pgmpcv}:
 \sap[sgxy]Covering contour:\listing{pgmgxy}:\sap[snov]Number of vertices:\listing{pgmnov}:
 \sap[svea]Vertices per cell and cell ratio:\listing{pgmvea}:
 \sap[stex]\TeX's macros:\listing{pgmtex}:
 \sap[stxl]Language macros for \TeX:\listing{pgmlng}:
\sap[sden]Degenerate test and other programs:\listing{pgmden}:
 \sap[ssgn]Face statistics in $n$ dimensions:\listing{pgmsgn}:
\sap[ppcb]Beam intersection program:\listing{pgmpcb}:
 \sap[pvhd]Number of vertices in high dimensions:\listing{pgmvhd}:
 \sap[pbnv]Example batch program, simulation and data extraction:\listing{pgmbnv}:
 \sap[pvov]Voronoi operator of various order:\listing{pgmvov}:
 \sap[pvff]Voronoi data structure for filtering membrane study:\listing{pgmvff}:
 \sap[pctc]Centroid process on 2-d VT:\listing{pgmctc}:
 \sap[pcgi]Centroid process on 3-d VT:\listing{pgmcgi}:
 \sap[parb]Defining irregular objects:\listing{pgmarb}:
 \sap[prnx]Investigating the distribution observed from within the network:\listing{pgmrnx}:
 \sap[pftn]Miscellaneous functions:\listing{pgmftn}:
 \sap[ptrp]Percolated traffic networks:\listing{pgmtrf}:
 \sap[pvsp]Volume, surface area, cell- and face perimeters:\listing{pgmvsp}\listing{pgmvsq}:
 \sap[pvhg]Volume in higher dimensions:\listing{pgmvhg}:
 \sap[ptrr]Regular lattices in three dimensions:\listing{pgmtrr}:
\sap[pchl]Effects of channelling:\listing{pgmchl}:
 \sap[pstp]Stereographic projection:\listing{pgmstp}:
\app[atrm]Terminology::\begingroup\parindent=0pt
 \ninepoint \baselineskip.9em \inpt[thstrm] \endgroup
 \sap[sabv]Abbreviation:\din.\tenbf.\tenpoint.dstabv.:
\app[arsc]Resources::\sap[sbio]Biographies:thsbio:
 \sap[acmp]Computation and softwares:thssft:
 \sap[sweb]Internet resources:{\nin thsnet:}:
\app[abib]Bibliography:{\bib thsbib:}:
 \sap[akbb]My writings, Kittisak Nui Tiyapan:{\bib thskbb:}:
\app[apub]Essays and relevant publications::
\sap[phyd]Antimonytrioxide extraction from ore by hydrometallurgy:\pap{phyd}:
 \sap[pcyb]Cyberspace:\pap{pcyb}:\sap[pend]The End of Stars:\pap{pend}:
 \sap[pext]Self-tuning Extremum Control:\pap{pext}:
 \sap[pand]The story of Andromedra:\pap{pand}:\sap[plet]The names of the notes:\pap{plet}:
 \sap[pkmp]To be unkempt:\pap{pkmp}:
 \sap[pcrt]Critical probability of 2-d tessellation:\pap{pcrt}:
 \sap[psom]Some properties of stochastic optimal control:\pap{psom}:
 \sap[pfrc]Fractals in traffic control:\pap{pfrc}:
 \sap[pdis]Distributed parameter systems:\pap{pdis}:
 \sap[pvis]Vision robots:\pap{pvis}:\sap[psng]Singular perturbation:\pap{psng}:
 \sap[poex]Object-location using Extremum Control:\pap{poex}:
 \sap[prag]On pragmatists and idealists:\pap{prag}:\sap[pmor]The Morris Worm:\pap{pmor}:
 \sap[psim]RLS algorithm for object-location problem:\pap{psim}:
\sap[pmoc]Modelling the economics:\pap{pmod}:
 \sap[pflw]Money flow in networks:\pap{pflw}:
 \sap[ptrf]Modelling of traffic congestion:\pap{ptrf}:
 \sap[pctm]Critical probability in traffic modelling and control:\pap{pctm}:
 \sap[pvar]Variable structure control and singular perturbation:\pap{pvar}:
\sap[ptrn]Projects on translation:\pap{ptrn}:
 \sap[pabs]Abstracts of books I wrote:\pap{pabs}:\cut
\app[atrn]Translation::\sap[adir]G. L. Dirichlet, 1848:\tran{trndirich}:
 \sap[avni]G. F. Voronoi, 1908 {(I)}:\tran{trnvoroni}:
 \sap[avnj]G. F. Voronoi, 1908 {(II)}:\tran{trnvoronii}:
 \sap[avnk]G. F. Voronoi, 1909:\tran{trnvoroniii}:\cut
\app[aidx]Index:dstidx:\bye
```

§ A.14 Program for the thesis

```
% tss.tex  September 2003
\input thshead
\head1in\begingroup\tchd\titlepage
\centre{\titlefont Division of space by Voronoi graphs,}\bskp
\centre{\titlefont Percolation within percolation and}\bskp
\centre{\titlefont application to the models of porous membranes}\vskp.7in
\centre{\epsfxsize=1in \epsffile{umist.ps}}\vskp.9in
\centre{\twrm A thesis submitted to the}\vskp.3in
\centre{\twrm University of Manchester Institute of Science and Technology}\vskp.3in
\centre{\twrm for the degree of Doctor of Philosophy}\vskp.9in
\centre{\titlefont Kittisak Nui Tiyapan}\vskp.3in
\centre{\twrm BEng, BSc, BEng, MSc}\vskp.6in
\centre{\twrm \date}\cut\head1in
{\elevenrm\baselineskip=17pt\noindent
No portion of the work referred to in this thesis has been submitted in support of
an application for another degree or qualification of this or any other university, or other
institute of learning.\vskip.7in \rline{K N Tiyapan}}\cut\head1in
\titlepage{\baselineskip=15pt
\rline{To my father and my mother Niwat and Somjit Tiyapan (\[Nivat, Somcit\tx. T\iq.y\ax.ban\dhx.a)],}
\rline{my chemistry and sword teacher Siddhiponr Songsataya (\[Sidhibaur Drongsa\tx.\yx.a]),
and}
\rline{my dad and mum Allister and Patricia Johnstone.}
\rline{To Emeritus Professor David John Bell and}
\rline{Emeritus Professor Graham Arthur Davies,}
\rline{both of whom had started this project,}\rline{one of whom had powered it.}
\rline{To these people, then, is my dedication of this thesis,}
\rline{but the thesis may dedicate itself to whomever it wishes.}}\cut\head1.2in
\lox(Abstract)[tssabs]\cut\head.3in\newwrite\toc
\lox(Table of contents)[dsttoc] \immediate\openout\toc=dsttoc\newwrite\lof
\lox(List of figures)[dstlof] \immediate\openout\lof=dstlof\newwrite\lot
\lox(List of tables)[dstlot] \immediate\openout\lot=dstlot\newwrite\lom
\lox(List of theorems)[dstlom] \immediate\openout\lom=dstlom\newwrite\loq
\lox(List of algorithms)[dstlog] \immediate\openout\loq=dstlog\newwrite\lod
\lox(List of definitions)[dstlod] \immediate\openout\lod=dstlod\newwrite\los
\lox(List of assumptions)[dstlos] \immediate\openout\los=dstlos\cut
\loy(Acknowledgements)[thsack]\cut
\loy(Notation)[thsnot]\din.\elevenpoint.\tenpoint.thsnot.\endgroup\cut\head0in
\chp[cfnd]Preface and introductions:thstroii:\sct[smth]Mathematics:dstmth:
 \sct[sgem]Geometry:thsgem:\sct[sphy]Physics:thsphy:
 \sct[scsm]Cosmological structure:thscsm:
 \sct[sflt]Filtration:dstflt:\sct[sstt]Statistics:thsstt:\sct[spoi]Poisson process:dstpoi:
 \sct[sptr]Phase transition:thsptr:\sct[sran]Random processes:thsran:
 \sct[snat]Structures in nature:thsnat:\sct[scpg]Computational geometry:dstcpg:
 \sct[salg]Geometric algorithms:dstalg:\sct[stex]\TeX nicalities:thstex:
\chp[cdiv]Division of space:dstdiv:\sct[sste]Stereographic projection:thssste:
 \sct[scov]Covering lattices:thscov:\sct[sfng]Viscous fingers:thsfng:
 \sct[scrs]Crystals, quasicrystals, and polycrystals:thscrs:
 \sct[srdl]Random lines and percolation:dstrdl:\sct[scxh]Convex hull:dstcxh:
 \sct[sprv]From convex hull to the Voronoi tessellation:dstprv:
 \sct[stlg]Tilings and patterns:thstlg:
\chp[cvor]Voronoi tessellation:thsvn:\sct[sqeq]Quadratic equations:thsqeq:
 \sct[sqfm]Quadratic forms:thsqfm:\sct[sval]Voronoi algorithms:dstval:
 \sct[svst]Voronoi statistics in literature:dstvst:
 \sct[ssta]Voronoi statistics:thsssta:\sct[svst]Voronoi section:thsvst:
 \sct[snmv]Number of vertices and edges:thsnumv:
 \sct[svfc]Faces in different dimensions:thsvfc:
 \sct[spcb]Beam intersection study:thspcb:\sct[svov]Voronoi of a Voronoi:dstvov:
 \sct[strv]Transformations of a Voronoi:dsttrv:\sct[scxv]Compressed Voronoi:dstcxv:
 \sct[svhg]Voronoi tessellation in higher dimensions:dstvhg:
\chp[cplt]Percolation:thspcl:\sct[sscv]S-curves and the percolative phenomena:dstscv:
 \sct[sv2d]Voronoi percolation in two dimensions:thsv2d:
 \sct[sv3d]Voronoi percolation in three dimensions:thsv3d:
 \sct[sv2s]Percolation of 2-dimensional Voronoi sections:thsv2s:
 \sct[sntp]Network percolation:thsnpc:
 \sct[sscl]Percolation statistics in literature:fypscl:
 \sct[scnm]Percolation of $n$-gons in continuum:thscnm:
 \sct[spgt]Polygon percolation threshold:dstpgt:
 \sct[shmd]2-homohedral tilings:thshmd:\sct[scgy]Cosmology:thscgy:
 \sct[scrm]CCTV, forest fire, the navy and porcupines:thsfir:
 \sct[sfrc]Fractals:thsfrc:
\chp[cmmb]Porous media:dstprm:\sct[szeo]Zeolites:thszeo:
 \sct[scry]Crystalisation:dstcry:
 \sct[sflw]Fluid flow within networks:thsflw:\sct[smat]Material science:thsmat:
 \sct[sclm]Forces between particles:thsclm:\sct[sire]Arbitrarily shaped particles:dstarb:
 \sct[snnp]Non Poisson number distributions of particles:dstnnp:
\chp[cfmb]Filtering membranes:dstfmb:\sct[ssls]Separation processes:dstsls:
 \sct[sdef]Dead-end filtration:dstdef:\sct[scgt]The centre of gravity:dstcgt:
 \sct[smdm]Molecular dynamics:dstmdm:\sct[spad]Problem definition and algorithms:dstpad:
 \sct[sfti]Simplified algorithm for filtration:dstfti:
 \sct[ssam]Filtering problem when physical blockage is prominent:dstsam:
 \sct[spfg]Percolative filtering with very small particles:dstpfg:
 \sct[spwp]Percolation within percolation:dstpwp:
 \sct[stlg]The first part, suspended particles:dsttlg:
 \sct[sftc]The second part, flow through the cells:dstftc:
\chp[cptm]Percolation in traffic modelling:thstrf:
 \sct[sprn]Percolation of road networks:dstprn:
 \sct[sgph]Graphs theory and its applications:dstgph:
\chp[ccon]Conclusion:dstcon:\sct[sftw]Suggestions for future work:tssftw:
\app[aprg]Programs::\sap[sobp]Object location:\listing{pgmobj}:
 \sap[stwv]Network percolation, two dimensions:\listing{pgmtwv}:
 \sap[sthv]Network percolation, three dimensions:\listing{pgmthv}:
 \sap[stws]Network percolation, 2--d section:\listing{pgmtws}:
 \sap[scnn]Continuum percolation of $n$-gons:\listing{pgmcnn}:
 \sap[sptl]Tilings:\listing{pgmptl}:\sap[spcv]Covering lattices:\listing{pgmpcv}:
 \sap[sgxy]Covering contour:\listing{pgmgxy}:
 \sap[snov]Number of vertices:\listing{pgmnov}:
```

```
 \sap[svea]Vertices per cell and cell ratio:\listing{pgmvea}:
 \sap[stex]\TeX's macros:\listing{pgmtex}:
 \sap[stxl]Language macros for \TeX:\listing{pgmlng}:
 \sap[sdst]Program for the dissertation:\listing{tssdst}:
 \sap[stss]Program for the thesis:\listing{tsstss}:
 \sap[sden]Degeneracy test and other programs:\listing{pgmden}:
 \sap[ssgn]Face statistics in $n$ dimensions:\listing{pgmsgn}:
 \sap[ppcb]Beam intersection program:\listing{pgmpcb}:
 \sap[pvhd]Number of vertices in high dimensions:\listing{pgmvhd}:
 \sap[pbnv]Example batch program, simulation and data extraction:\listing{pgmbnv}:
 \sap[pvov]Voronoi operator of various order:\listing{pgmvov}:
 \sap[pvff]Voronoi data structure for filtering membrane study:\listing{pgmvff}:
 \sap[pctc]Centroid process on 2-d VT:\listing{pgmctc}:
 \sap[pcgi]Centroid process on 3-d VT:\listing{pgmcgi}:
 \sap[parb]Defining irregular objects:\listing{pgmarb}:
 \sap[prnx]Investigating the distribution observed from within the network:\listing{pgmrnx}:
 \sap[pftn]Miscellaneous functions:\listing{pgmftn}:
 \sap[ptrp]Percolated traffic networks:\listing{pgmtrf}:
 \sap[pvsp]Volume, surface area, cell- and face perimeters:\listing{pgmvsp}\listing{pgmvsq}:
 \sap[pvhg]Volume in higher dimensions:\listing{pgmvhg}:
 \sap[ptrr]Regular lattices in three dimensions:\listing{pgmtrr}:
 \sap[pchl]Effects of channelling:\listing{pgmchl}:
 \sap[pstp]Stereographic projection:\listing{pgmstp}:
 \sap[ppgt]Percolation of regular polygons in two dimensions:\listing{pgmpgt}:
\app[atrm]Terminology and other resources::\begingroup\parindent=0pt
 \ninepoint\baselineskip.9em\inpt[thstrm] \endgroup
 \sap[sabv]Abbreviation:\din.\tenbf.\tenpoint.dstabv.:\sap[sbio]Biographies:thsbio:
 \sap[acmp]Computation and softwares:thssft:
 \sap[sweb]Internet resources:{\nin thsnet:}:\sct[stex]\TeX nicalities:thstex:
\app[abib]Bibliography:{\bib thsbib:}:
 \sap[akbb]My writings, Kittisak Nui Tiyapan:{\bib thskbb:}:
\app[apub]Publications and submissions of papers::
 \sap[pcrt]Critical probability of 2-d tessellation:\pap{pcrt}:
 \sap[pabs]Abstracts of books I wrote:\pap{pabs}:
 \sap[polc]On object-location problems:\pap{polc}:
 \sap[pant]Antimonytrioxide extraction from ore by hydrometallurgy:\pap{pant}:
 \sap[pcop]Continuum percolation of polygons:\pap{pcop}:
 \sap[pppc]An algorithm for the percolation of polygons in continuum:\pap{pppc}:
 \sap[ppcq]Quantum mechanics within percolation within percolation:\pap{ppcq}:\cut
\app[atrn]Translation::\sap[adir]G. L. Dirichlet, 1848:\tran{trndirich}:
 \sap[avni]G. F. Voronoi, 1908 {(I)}:\tran{trnvoroni}:
 \sap[avnj]G. F. Voronoi, 1908 {(II)}:\tran{trnvoronii}:
 \sap[avnk]G. F. Voronoi, 1909:\tran{trnvoroniii}:\cut
\app[aidx]Index:dstidx:\bye
```

§ A.15 Degeneracy test and other programs

```
% degen.m
Dim=40; Xbunch=(0:2:Dim)'; Ycore=ones(size(Xbunch)); X=[]; Y=[];
for j=0:2:Dim,
  X=[X;Xbunch]; Y=[Y;(j*Ycore)];
end
[Xsq,Ysq]=voronoi(X,Y); Tri=delaunay(X,Y); NumTri=size(Tri,1); [XHex,YHex]=voronoi(X,Y);
% honey.m
Sq3=sqrt(3); Dim=40; Shift=1; y=1:Sq3:10; X=[]; Y=[];
Xinit=0; Xbunch=(Shift:6:Dim)'; Ycore=ones(size(Xbunch));
for j=0:(2*Sq3):Dim,
  X=[X;Xbunch]; Y=[Y;j*Ycore];
end
Xinit=3; Xbunch=(3:6:Dim)'; Ycore=ones(size(Xbunch));
for j=Sq3:(2*Sq3):Dim,
  X=[X;Xbunch]; Y=[Y;j*Ycore];
end
Tri=delaunay(X,Y); NumTri=size(Tri,1); [XHex,YHex]=voronoi(X,Y);
% cover.m
Sq3=sqrt(3); Dim=40; y=1:Sq3:10; X=[]; Y=[]; Xinit=0;
Xbunch=(0:6:Dim)'; Ycore=ones(size(Xbunch));
for j=0:(2*Sq3):Dim,
  X=[X;Xbunch]; Y=[Y;j*Ycore];
end
Xinit=3; Xbunch=(3:6:Dim)'; Ycore=ones(size(Xbunch));
for j=Sq3:(2*Sq3):Dim,
  X=[X;Xbunch]; Y=[Y;j*Ycore];
end
[XHex,YHex]=voronoi(X,Y); NumE=size(XHex,2); NeighE=sparse(NumE,NumE);
for i=1:(NumE-1),
  for j=(i+1):NumE,
    if(((XHex(1,i)==XHex(1,j)) & (YHex(1,i)==YHex(1,j))) | ...
     ((XHex(1,i)==XHex(2,j)) & (YHex(1,i)==YHex(2,j))) | ...
     ((XHex(2,i)==XHex(1,j)) & (YHex(2,i)==YHex(1,j))) | ...
     ((XHex(2,i)==XHex(2,j)) & (YHex(2,i)==YHex(2,j))))
      NeighE(i,j)=1; NeighE(j,i)=1;
    end
  end
end
% covers.m
Xmid=[]; Ymid=[];
for i=1:NumE,
  Xmid=[Xmid;((XHex(1,i)+XHex(2,i))/2)]; Ymid=[Ymid;((YHex(1,i)+YHex(2,i))/2)];
end
[I,J]=find(tril(NeighE)); NumV=size(I,1); Xcov=[]; Ycov=[];
for i=1:NumV,
  Xcov=[Xcov;[Xmid(I(i,1),1),Xmid(J(i,1),1)]]; Ycov=[Ycov;[Ymid(I(i,1),1),Ymid(J(i,1),1)]];
end
Xcovt=Xcov'; Ycovt=Ycov'; NumE1=size(Xcov,1); NeighE1=sparse(NumE1,NumE1);
for i=1:(NumE1-1),
  for j=(i+1):NumE1,
    if(((Xcovt(1,i)'==Xcovt(1,j)') & (Ycovt(1,i)'==Ycovt(1,j)')) | ...
     ((Xcovt(1,i)'==Xcovt(2,j)') & (Ycovt(1,i)'==Ycovt(2,j)')) | ...
     ((Xcovt(2,i)'==Xcovt(1,j)') & (Ycovt(2,i)'==Ycovt(1,j)')) | ...
     ((Xcovt(2,i)'==Xcovt(2,j)') & (Ycovt(2,i)'==Ycovt(2,j)')))
      NeighE1(i,j)=1; NeighE1(j,i)=1;
    end
  end
end
Xmid=[]; Ymid=[];
for i=1:NumE1,
  Xmid=[Xmid;((Xcovt(1,i)'+Xcovt(2,i)')/2)]; Ymid=[Ymid;((Ycovt(1,i)'+Ycovt(2,i)')/2)];
end
[I,J]=find(tril(NeighE1)); NumV=size(I,1); Xc1=[]; Yc1=[];
for i=1:NumV,
  Xc1=[Xc1;[Xmid(I(i,1),1),Xmid(J(i,1),1)]]; Yc1=[Yc1;[Ymid(I(i,1),1),Ymid(J(i,1),1)]];
end
% coverss.mXc1t=Xc1'; Yc1t=Yc1';
NumE2=size(Xc1,1); NeighE2=sparse(NumE2,NumE2);
for i=1:(NumE2-1),
  for j=(i+1):NumE2,
    if(((Xc1t(1,i)'==Xc1t(1,j)') & (Yc1t(1,i)'==Yc1t(1,j)')) | ...
     ((Xc1t(1,i)'==Xc1t(2,j)') & (Yc1t(1,i)'==Yc1t(2,j)')) | ...
     ((Xc1t(2,i)'==Xc1t(1,j)') & (Yc1t(2,i)'==Yc1t(1,j)')) | ...
     ((Xc1t(2,i)'==Xc1t(2,j)') & (Yc1t(2,i)'==Yc1t(2,j)')))
      NeighE2(i,j)=1; NeighE2(j,i)=1;
    end
  end
end
Xmid=[]; Ymid=[];
for i=1:NumE2,
  Xmid=[Xmid;((Xc1t(1,i)'+Xc1t(2,i)')/2)]; Ymid=[Ymid;((Yc1t(1,i)'+Yc1t(2,i)')/2)];
end
[I,J]=find(tril(NeighE2)); NumV=size(I,1); Xc2=[]; Yc2=[];
for i=1:NumV,
  Xc2=[Xc2;[Xmid(I(i,1),1),Xmid(J(i,1),1)]]; Yc2=[Yc2;[Ymid(I(i,1),1),Ymid(J(i,1),1)]];
end
% crop.m     by K N J Tiyapan, 15 July 2001
clear all; format long g; format compact; NumCell=1000; rand('state',sum(100*clock));
[XVoro,YVoro]=voronoi(X,Y); SizeV=size(XVoro,2); Xv=[]; Yv=[];
for j=1:SizeV,
  if((((XVoro(1,j)>0) & (XVoro(1,j)<1)) & ((YVoro(1,j)>0) & (YVoro(1,j)<1))) | ...
   (((XVoro(2,j)>0) & (XVoro(2,j)<1)) & ((YVoro(2,j)>0) & (YVoro(2,j)<1))))
    Xv=[Xv,XVoro(:,j)]; Yv=[Yv,YVoro(:,j)];
  end
end
```

§ A.16 Face statistics in n dimensions

```
% statsgenn.m    by K N J Tiyapan, 1st July, 2001
echo off; clear all; format short g; more off;
pt1 =fopen('./v50.dat','r'); sc1 =fscanf(pt1, '%d', 4);
Dimension=sc1(1,1);  NumVAll =sc1(2,1); NumC =sc1(3,1);
sc2 =fscanf(pt1, '%f', [Dimension, NumVAll]); VerticeAll =sc2';
CVMat =sparse(NumC, NumVAll); CFrame =ones(NumC, 1); VCFrame=zeros(NumVAll,1);
VFrame=ones(NumVAll,1);
for i=1:NumC,
  sc1 =fscanf(pt1, '%d', 1);
  for j=1:sc1,
    sc2 =fscanf(pt1, '%d', 1); Num =sc2+1; CVMat(i,Num) =1;
    if ( max(abs(VerticeAll(Num, :))) > 0.5 )
      CFrame(i,1) =0; VFrame(Num,1)=0;
    end
  end
```

```
end
fclose(pt1);
for i=1:NumC,
  VInC=find(CVMat(i,:)'); NumVInC=size(VInC,1);
  if(CFrame(i,1)==1)
    for j=1:NumVInC,
      VCFrame(VInC(j,1),1)=1;
    end
  end
end
CVNiceCMat=[];
for i=1:NumC,
  if(CFrame(i,1)==1)
    CVNiceCMat=[CVNiceCMat;CVMat(i,:)];
  end
end
CNumVNiceCMat=sum(CVNiceCMat,2); NumV=sum(VCFrame);
VVCFrameMat=zeros(NumV,2); Vertice=zeros(NumV,Dimension); Count=0;
for i=1:NumVAll,
  if(VCFrame(i,1)==1)
    Count=Count+1;
    VVCFrameMat(Count,1)=i; VVCFrameMat(Count,2)=Count;
    Vertice(Count,:)=VerticeAll(i,:);
  end
end
pt3=fopen('./n50.dat','w'); pt2=fopen('./c50.dat','r'); line=fgetl(pt2);
sc1 =fscanf(pt2, '%d', 1); sc2=fscanf(pt2,'%f',[Dimension,NumC]); Cell=sc2';
fclose(pt2); CNeighCCMat=sparse(NumC, NumC); t=cputime; FVAllMat=[]; FNumVAllMat=[];
for i=1:(NumC-1),
  for j=(i+1):NumC,
    VShared=and(CVMat(i,:), CVMat(j,:)); NumShared =sum(VShared, 2);
    NumFVAllMat=size(FVAllMat,1);
    if (NumShared >= Dimension)
      CNeighCCMat(i,j) =1; CNeighCCMat(j,i) =1; Exist=0;
      for k=1:NumFVAllMat,
        MatchExistingFV=sum(and(VShared,FVAllMat(k,:)),2);
        if(MatchExistingFV>=Dimension)
          Exist=1; break;
        end
      end
      if(Exist==0)
        FVAllMat=[FVAllMat;VShared]; FNumVAllMat=[FNumVAllMat;NumShared];
      end
    end
  end
end
FVMat=[]; FNumVMat=[]; FVCFMat=[]; FNumVCFMat=[];
for i=1:NumFVAllMat,
  VThisFace=find(FVAllMat(i,:)'); NumVThisFace=size(VThisFace,1);
  IncludeMe=1; IncludeMeToo=1;
  for j=1:NumVThisFace,
    if(VFrame(VThisFace(j,1),1)==0)
      IncludeMe=0;
    end
    if(VCFrame(VThisFace(j,1),1)==0)
      IncludeMeToo=0;
    end
  end
  if(IncludeMe==1)
    FVMat=[FVMat;FVAllMat(i,:)]; FNumVMat=[FNumVMat;FNumVAllMat(i,:)];
  end
  if(IncludeMeToo==1)
    FVCFMat=[FVCFMat;FVAllMat(i,:)]; FNumVCFMat=[FNumVCFMat;FNumVAllMat(i,:)];
  end
end
NumFVMat=size(FVMat,1); NumFVCFMat=size(FVCFMat,1); FDim=Dimension-1;
fprintf(pt3,'Face dimension: %d\n',FDim); fprintf(pt3,'Number of faces: %d\n',NumFVMat);
fprintf(pt3,'Number of vertices: \n [');
for i=1:NumFVMat,
  fprintf(pt3,'%d ',FNumVMat(i,1));
  if(mod(i,10)==0)
    fprintf(pt3,'...\n');
  end
end
fprintf(pt3,']\n'); fprintf(pt3,'Number of faces of nice cells: %d\n',NumFVCFMat);
fprintf(pt3,'Number of vertices: \n [');
for i=1:NumFVCFMat,
  fprintf(pt3,'%d ',FNumVCFMat(i,1));
  if(mod(i,10)==0)
    fprintf(pt3,'...\n');
  end
end
fprintf(pt3,']\n'); DVMat=FVCFMat; NumD=NumFVCFMat;
for d=3:Dimension,
  FaceCond=Dimension-d+2; DNeighDDMat=sparse(NumD,NumD); dVMat=[]; dNumVMat=[];
  for i=1:(NumD-1),
    for j=(i+1):NumD,
      VShared=and(DVMat(i,:), DVMat(j,:)); NumShared =sum(VShared, 2);
      NumdVMat=size(dVMat,1);
      if (NumShared >= FaceCond)
        DNeighDDMat(i,j) =1; DNeighDDMat(j,i) =1; Exist=0;
        for k=1:NumdVMat,
          MatchExistingdV=sum(and(VShared,dVMat(k,:)),2);
          if(MatchExistingdV>=FaceCond)
            Exist=1; break;
          end
        end
        if(Exist==0)
          dVMat=[dVMat;VShared]; dNumVMat=[dNumVMat;NumShared];
        end
      end
    end
  end
  FDim=Dimension-d+1; fprintf(pt3,'Face dimension: %d\n',FDim);
  fprintf(pt3,'Number of faces: %d\n',NumdVMat);
  if(FDim~=1)
    fprintf(pt3,'Number of vertices: \n [');
    for i=1:NumdVMat,
      fprintf(pt3,'%d ',dNumVMat(i,1));
      if(mod(i,10)==0)
        fprintf(pt3,'...\n');
      end
    end
    fprintf(pt3,']\n');
  end
  DVMat=dVMat; NumD=NumdVMat;
  if(FDim==2)
    FVMat=DVMat;
  end
end
Time=cputime-t; NumNiceC=sum(CFrame); NumVBound=sum(VFrame);
```

§ A.17 Beam intersection program

```
% penc.m     by K N J Tiyapan, 15th July, 2001
clear all; format long g; format compact; NumCell=1000; rand('state',sum(100*clock));
X=1.5*rand(NumCell,1)-0.25*ones(NumCell,1); Y=1.5*rand(NumCell,1)-0.25*ones(NumCell,1);
% X=poissrnd(.5,NumCell,1); Y=poissrnd(.5,NumCell,1);
% X=raylrnd([1:NumCell])'; Y=raylrnd([1:NumCell])';
% Max=0.8*max([X;Y]); X=X/Max; Y=Y/Max;
[XVoro,YVoro]=voronoi(X,Y); SizeV=size(XVoro,2); Xv=[]; Yv=[];
for j=1:SizeV,
  if((((XVoro(1,j)>0) & (XVoro(1,j)<1)) & ((YVoro(1,j)>0) & (YVoro(1,j)<1))) | ...
   (((XVoro(2,j)>0) & (XVoro(2,j)<1)) & ((YVoro(2,j)>0) & (YVoro(2,j)<1))))
    Xv=[Xv,XVoro(:,j)]; Yv=[Yv,YVoro(:,j)];
  end
end
clf; plot(Xv,Yv); V1=[Xv(1,:);Yv(1,:)]'; V2=[Xv(2,:);Yv(2,:)]'; NumE=size(V1,1);
axis equal; axis([0 1 0 1]); Slope=2; Const=-.5; Cx=-.1; Dx=1.1;
Cy=Slope*Cx+Const; Dy=Slope*Dx+Const; C=[Cx,Cy]; D=[Dx,Dy]; CD=D-C; DistVect=[];
hold on; plot([C(1,1);D(1,1)],[C(1,2);D(1,2)]);
for i=1:NumE,
  A=V1(i,:); B=V2(i,:); AB=B-A; CA=A-C; Denom=det([AB;CD]);
  RNom=det([CD;CA]); SNom=det([AB;CA]);
  if(Denom~=0)
    r=RNom/Denom; s=SNom/Denom;
    if((r<=1) & (r>=0) & (s>=0) & (s<=1))
      P=A+r*AB; hold on; plot(P(1,1),P(1,2),'.','lineWidth',2);
      CP=P-C; Distance=sqrt(CP(1,1)*CP(1,1)+CP(1,2)*CP(1,2));
      DistVect=[DistVect;Distance];
    end
  end
end
SortDist=sort(DistVect); Dist=[]; NumSortDist=size(SortDist,1);
for i=2:NumSortDist,
  Dist=[Dist;(SortDist(1,1)-SortDist(i,1))];
end
Dist=abs(Dist); NumDist=size(Dist,1); PairDist=[];
for i=2:NumDist,
   PairDist=[PairDist;(Dist(i,1)-Dist((i-1),1))];
end
Xc=[]; Yc=[];
```

```
for i=1:NumCell,
  if((X(i,1)>0) & (X(i,1)<1) & (Y(i,1)>0) & (Y(i,1)<1))
    Xc=[Xc;X(i,1)]; Yc=[Yc;Y(i,1)];
  end
end
NumC=size(Xc,1); NormBase=1/sqrt(NumC); format short g; NumE; NumC;
NumEExcess=size(XVoro,2); MeanPairDist=mean(PairDist); VarPairDist=var(PairDist);
PairDistMNorm=PairDist/MeanPairDist; MeanPairDistMNorm=mean(PairDistMNorm);
VarPairDistMNorm=var(PairDistMNorm); Moment2PairDistMNorm=moment(PairDistMNorm,2);
Moment3PairDistMNorm=moment(PairDistMNorm,3); PairDistNNorm=PairDist/NormBase;
MeanPairDistNNorm=mean(PairDistNNorm); VarPairDistNNorm=var(PairDistNNorm);
Moment2PairDistNNorm=moment(PairDistNNorm,2); Moment3PairDistNNorm=moment(PairDistNNorm,3);
```

§ A.18 Number of vertices in high dimensions

```
% vhd.m ,aka f69.m, (c) K. N. Tiyapan 25th March, 2001
clear; more off; format long g; echo off; TimeStartPreparing =cputime;
pt1 =fopen('/home/mjkvjkt/vn/v69','r'); sc1 =fscanf(pt1, '%d', 4);
Dimension =sc1(1,1); VerticeNum =sc1(2,1); CellNum =sc1(3,1);
sc2 =fscanf(pt1, '%f', [Dimension, VerticeNum]); Vertices =sc2'; Framed =ones(CellNum, 1);
for i=1:CellNum,
  sc3 =fscanf(pt1, '%d', 1);
  for j=1:sc1,
    sc4 =fscanf(pt1, '%d', 1); Num =sc4+1;
    if ( max(abs(Vertices(Num, :))) > 0.5 )
      Framed(i,1) =0;
    end
  end
end
fclose(pt1); pt2 =fopen('/home/mjkvjkt/vn/n69','r');
sc5=fscanf(pt2, '%d', CellNum); VerticesPerCell =sc5; fclose(pt2);
MinVPerCell =min(VerticesPerCell); MaxVPerCell =max(VerticesPerCell);
MeanVPerCell =mean(VerticesPerCell); ScndMVPerCell =moment(VerticesPerCell,2);
ThrdMVPerCell =moment(VerticesPerCell,3); FrthMVPerCell =moment(VerticesPerCell,4);
VarVPerCell =var(VerticesPerCell); StdVPerCell =std(VerticesPerCell);
GMeanVPerCell =geomean(VerticesPerCell); HMeanVPerCell =harmmean(VerticesPerCell);
MedVPerCell =median(VerticesPerCell); MadVPerCell =mad(VerticesPerCell);
KurVPerCell =kurtosis(VerticesPerCell); TabVPerCell =tabulate(VerticesPerCell);
VPerInnerCell =frameit(VerticesPerCell,Framed); InnerVNum =sum(Framed);
MinVPerInnerCell =min(VPerInnerCell); MaxVPerInnerCell =max(VPerInnerCell);
MeanVPerInnerCell =mean(VPerInnerCell); ScndMVPerInnerCell =moment(VPerInnerCell,2);
ThrdMVPerInnerCell =moment(VPerInnerCell,3); FrthMVPerInnerCell =moment(VPerInnerCell,4);
VarVPerInnerCell =var(VPerInnerCell); StdVPerInnerCell =std(VPerInnerCell);
GMeanVPerInnerCell =geomean(VPerInnerCell); HMeanVPerInnerCell =harmmean(VPerInnerCell);
MedVPerInnerCell =median(VPerInnerCell); MadVPerInnerCell =mad(VPerInnerCell);
KurVPerInnerCell =kurtosis(VPerInnerCell); TabVPerInnerCell =tabulate(VPerInnerCell);
more on; clf; bar(TabVPerCell(:,1),TabVPerCell(:,2));
```

§ A.19 Example batch program, simulation and data extraction

```
/home/bin/rbox 1000 t3765098 D6|/home/bin/qhull v o>/home/qhull/wrk/v761
/usr/bin/tail -1000 /home/qhull/wrk/v761|/usr/bin/cut -f1 -d" ">/home/qhull/wrk/n761
/usr/bin/rm -f /home/qhull/wrk/v761
/usr/local/bin/matlab < /home/qhull/wrk/f761.m
```

§ A.20 Voronoi operator of various order

```
% vov.m, voronoi of voronoi, (c) Kit Tiyapan, 2002
clear all; itn=6; can=100;
rand('state',sum(100*clock));
x{1}=rand(can,2);
for m=1:itn,
  [va{m},ca{m}]=voronoin(x{m}); van(m)=size(va{m},1);
  vin{m}=ones(1,van(m)); vin{m}(1)=0;
  for i=2:van(m),
    if((max(va{m}(i,:))>1)|(min(va{m}(i,:))<0))
      vin{m}(i)=0;
    end
  end
  c{m}=[]; cnt=0;
  for i=1:can(m),
    ca{m}{i,2}=size(ca{m}{i,1},2); in=1;
    for j=1:ca{m}{i,2},
      if(~vin{m}(ca{m}{i,1}(j)))
        in=0; break;
      end
    end
    if(in)
      cnt=cnt+1; c{m}{cnt,1}=ca{m}{i,1}; c{m}{cnt,2}=ca{m}{i,2};
    end
  end
  cn(m)=size(c{m},1); cnt=0;
  for i=1:van(m),
    if(vin{m}(i))
      cnt=cnt+1; vin{m}(i)=cnt;
    end
  end
  for i=1:cn(m),
    for j=1:c{m}{i,2},
      c{m}{i,1}(j)=vin{m}(c{m}{i,1}(j));
    end
  end
  v{m}=[];
  for i=1:van(m),
    if(vin{m}(i))
      v{m}=[v{m};va{m}(i,:)];
    end
  end
  vn(m)=size(v{m},1); x{m+1}=v{m}; can(m+1)=vn(m); figure(m); clf; hold on;
  for i=1:cn(m),
    tmp=[c{m}{i,1},c{m}{i,1}(1)];
    for j=1:c{m}{i,2},
        plot([v{m}(tmp(j),1),v{m}(tmp(j+1),1)],[v{m}(tmp(j),2),v{m}(tmp(j+1),2)]);
    end
  end
  axis equal; axis off;
end
```

§ A.21 Voronoi data structure for filtering membrane study

```
% vff.m, Voronoi for filtration, (c) 2002, Kit Tiyapan. 10th Nov.
clear all; can=100; rand('state',sum(100*clock)); ca=rand(can,3);
[va,vca]=voronoin(ca); van=size(va,1); vin=zeros(1,van);
for i=1:van,
  vin(i)=(min(va(i,:))>0) & (max(va(i,:))<1);
end
cin=ones(1,can);
for i=1:can,
  vca{i,2}=vca{i,1}; vca{i,1}=size(vca{i,2},2);
  for j=1:vca{i,1},
    if(~vin(vca{i,2}(j)))
      cin(i)=0; break;
    end
  end
end
c=[]; vc=[]; cnt=0; v=[]; cnu=0; vin=sparse(1,van);
for i=1:can,
  if(cin(i))
    cnt=cnt+1; cin(i)=cnt; c(cnt,:)=ca(i,:); vc{cnt,1}=vca{i,1}; vc{cnt,2}=vca{i,2};
    for j=1:vc{cnt,1},
      if(~vin(vca{i,2}(j)))
        cnu=cnu+1; vin(vca{i,2}(j),1)=cnu; v(cnu,:)=va(vca{i,2}(j),:);
      end
    end
    for j=1:vc{cnt,1},
      vc{cnt,2}(j)=vin(vca{i,2}(j));
    end
  end
end
cn=cnt; vn=cnu;
for i=1:cn,
  tmp=ones(size(vc{i,2})); vc{i,3}=sparse(tmp,vc{i,2},tmp,1,vn);
end
ta=delaunayn(ca); tan=size(ta,1);
t=[]; cnt=0; bdr=[]; cnu=0;
for i=1:tan,
  in=1;
  for j=1:4,
    if(~cin(ta(i,j)))
      in=0; break;
    end
  end
  if(in)
    cnt=cnt+1;
    t(cnt,:)=ta(i,:);
    for j=1:4,
      t(cnt,j)=cin(ta(i,j));
    end
  else
    for j=1:3,
      for k=(j+1):4,
        if((cin(ta(i,j))) | (cin(ta(i,k))))
          cnu=cnu+1; tma=vca{ta(i,j),2}; tmb=vca{ta(i,k),2};
          tmd=ones(1,vca{ta(i,j),1}); tme=ones(1,vca{ta(i,k),1});
          tmp=find(sparse(tmd,tma,tmd,1,van) & sparse(tme,tmb,tme,1,van)); tmq=[];
          if(cin(ta(i,j)))
            bdr{cnu,1}=cin(ta(i,j));
          else
            bdr{cnu,1}=cin(ta(i,k));
          end
          bdr{cnu,2}=size(tmp,2);
          for l=1:bdr{cnu,2},
            tmq=[tmq,vin(tmp(l))];
          end
          bdr{cnu,3}=tmq;
        end
      end
    end
  end
end
tn=cnt; bdrn=size(bdr,1); bcc=sparse(cn,cn);
for i=1:tn,
  for j=1:3,
    for k=(j+1):4,
      bcc(t(i,j),t(i,k))=1; bcc(t(i,k),t(i,j))=1;
    end
  end
end
[tmf,tmg]=find(triu(bcc)); bn=size(tmf,1); b=[];
for i=1:bn,
  b{i,1}(1,:)=[tmf(i),tmg(i)]; b{i,4}=vc{tmf(i),3} & vc{tmg(i),3};
  [nth,b{i,3}]=find(b{i,4}); b{i,2}=size(b{i,3},2); tmp=[];
  for j=1:b{i,2},
    tmp=[tmp;v(b{i,3}(j),:)];
  end
  tma=max(tmp,[],1)-min(tmp,[],1); tmb=sortrows([1,2,3;tma]',2);
  tmb=round(tmb(:,1)); tmq=[tmp(:,tmb(1,1)),tmp(:,tmb(2,1))];
  tmp=delaunay(tmq(:,1),tmq(:,2)); tmq=sparse(b{i,2},b{i,2});
  for j=1:size(tmp,1),
    for k=1:2,
      for l=(k+1):3,
        tmq(tmp(j,k),tmp(j,l))=tmq(tmp(j,k),tmp(j,l))+1;
        tmq(tmp(j,l),tmp(j,k))=tmq(tmp(j,l),tmp(j,k))+1;
      end
    end
  end
  [tma,tmb,tmc]=find(tmq); tmq=sparse(vn,vn);
  for j=1:size(tma,1),
    if(~(tmc(j)-1))
      tmq(b{i,3}(tma(j)),b{i,3}(tmb(j)))=1; tmq(b{i,3}(tmb(j)),b{i,3}(tma(j)))=1;
    end
  end
  tma=[b{i,3}(1)]; tmb=b{i,3}(1); [nth,tmc]=find(tmq(tmb,:));
  tmd=tmc(1); tma=[tma,tmd]; tmq(tmb,tmd)=0; tmq(tmd,tmb)=0; tmb=tmd;
  while(tmb-b{i,3}(1))
    [nth,tmc]=find(tmq(tmb,:)); tmd=tmc(1); tma=[tma,tmd];
    tmq(tmb,tmd)=0; tmq(tmd,tmb)=0; tmb=tmd;
  end
  b{i,6}=tma; b{i,5}=b{i,6}(1,1:b{i,2}); b{i,7}=2;
end
for i=1:bn,
  bcc(b{i,1}(1),b{i,1}(2))=i; bcc(b{i,1}(2),b{i,1}(1))=i;
end
bc=[];
for i=1:cn,
  bc{i,1}=[];
end
for i=1:bn,
  bc{b{i,1}(1),1}=[bc{b{i,1}(1)},i]; bc{b{i,1}(2),1}=[bc{b{i,1}(2)},i];
end
n=bn;
for i=1:bdrn,
  n=n+1; b{n,1}=bdr{i,1}; b{n,2}=bdr{i,2}; b{n,3}=bdr{i,3};
  tmp=ones(1,bdr{i,2}); b{n,4}=sparse(tmp,bdr{i,3},tmp,1,vn); tmp=[];
  for j=1:b{n,2},
    tmp=[tmp;v(b{n,3}(j),:)];
  end
  tma=max(tmp,[],1)-min(tmp,[],1); tmb=sortrows([1,2,3;tma]',2);
  tmb=round(tmb(:,1)); tmq=[tmp(:,tmb(1,1)),tmp(:,tmb(2,1))];
  tmp=delaunay(tmq(:,1),tmq(:,2)); tmq=sparse(b{n,2},b{n,2});
  for j=1:size(tmp,1),
    for k=1:2,
      for l=(k+1):3,
        tmq(tmp(j,k),tmp(j,l))=tmq(tmp(j,k),tmp(j,l))+1;
        tmq(tmp(j,l),tmp(j,k))=tmq(tmp(j,l),tmp(j,k))+1;
      end
    end
  end
  [tma,tmb,tmc]=find(tmq); tmq=sparse(vn,vn);
  for j=1:size(tma,1),
    if(~(tmc(j)-1))
      tmq(b{n,3}(tma(j)),b{n,3}(tmb(j)))=1; tmq(b{n,3}(tmb(j)),b{n,3}(tma(j)))=1;
    end
  end
  tma=[b{n,3}(1)]; tmb=b{n,3}(1); [nth,tmc]=find(tmq(tmb,:)); tmd=tmc(1);
  tma=[tma,tmd]; tmq(tmb,tmd)=0; tmq(tmd,tmb)=0; tmb=tmd;
  while(tmb-b{n,3}(1))
    [nth,tmc]=find(tmq(tmb,:)); tmd=tmc(1); tma=[tma,tmd];
    tmq(tmb,tmd)=0; tmq(tmd,tmb)=0; tmb=tmd;
  end
  b{n,6}=tma; b{n,5}=b{n,6}(1,1:b{n,2}); b{n,7}=1;
end
for i=(bn+1):n,
  bc{b{i,1}}=[];
end
for i=(bn+1):n,
  bc{b{i,1},1}=[bc{b{i,1}},i];
end
% for graphical tests
clf; hold on;
for k=1:cn,
  [nth,ntg,tma]=find(bcc(k,:));
  for i=1:size(tma,2),
    tmp=[];
    for j=1:(b{tma(i),2}+1),
      tmp=[tmp;v(b{tma(i),6}(j),:)];
    end
    plot3(tmp(:,1),tmp(:,2),tmp(:,3));
  end
end
for i=bn:n,
     tmp=[];
    for j=1:(b{i,2}+1),
      tmp=[tmp;v(b{i,6}(j),:)];
    end
    plot3(tmp(:,1),tmp(:,2),tmp(:,3));
end
axis equal; axis off; rotate3d;
```

§ A.22 Centroid process on 2-d VT

```
% ctc.m, test centroid of polygons. (c) Kit Tiyapan, 2002.
clear all; rand('state',sum(100*clock)); can=200; ca=rand(can,2);
for z=1:3,
  [va,vca]=voronoin(ca); van=size(va,1); vcan=size(vca,1); vin=zeros(1,van);
  for i=1:van,
    if((max(va(i,:))<1) & (min(va(i,:)>0)))
      vin(i)=1;
    end
  end
  cin=zeros(1,vcan);
  for i=1:vcan,
    vca{i,2}=vca{i}; vca{i,1}=size(vca{i,2},2); in=1;
    for j=1:vca{i,1},
      if(~vin(vca{i,2}(j)))
        in=0; break;
      end
    end
    if(in)
      cin(i)=1;
    end
  end
  vin=zeros(1,van);
  for i=1:vcan,
    if(cin(i))
      for j=1:vca{i,1},
        vin(vca{i,2}(j))=1;
      end
    end
  end
  v=[]; cnt=0;
  for i=1:van,
    if(vin(i))
      cnt=cnt+1; vin(i)=cnt; v(cnt,:)=va(i,:);
    end
  end
  vn=cnt; c=[]; vc=[]; cnt=0;
  for i=1:vcan,
    if(cin(i))
      cnt=cnt+1; c(cnt,:)=ca(i,:); vc{cnt,1}=vca{i,1}; vc{cnt,2}=vca{i,2};
    end
  end
  cn=cnt;
  for i=1:cn,
    for j=1:vc{i,1},
      vc{i,2}(j)=vin(vc{i,2}(j));
    end
  end
  for i=1:cn,
    tmp=[];
    for j=1:vc{i,1},
      tmp=[tmp;v(vc{i,2}(j),:)];
    end
    d=delaunay(tmp(:,1),tmp(:,2)); dn=size(d,1); tmj=[]; tmk=[]; tmi=0;
    for j=1:dn,
      tma=[v(vc{i,2}(d(j,1)),:);v(vc{i,2}(d(j,2)),:);v(vc{i,2}(d(j,3)),:)];
      tmp=sum(tma)/3; tma=[tma;tma(1,:)]; tmb=[];
      for k=1:3,
        tmb=[tmb,sqrt(sum((diff(tma(k:(k+1),:),1,1)).^2))];
      end
      tmc=sum(tmb)/2; tmq=sqrt(tmc*(tmc-tmb(1))*(tmc-tmb(2))*(tmc-tmb(3)));
      tmk=[tmk;tmq]; tmi=tmi+tmq*tmp;
    end
    vc{i,3}=sum(tmk); vc{i,4}=tmi/sum(tmk);
  end
  vs{z,1}=c; vs{z,2}=v; vs{z,3}=vc; g=[];
  for i=1:cn,
    g=[g;vc{i,4}];
  end
  ca=g; figure(z); clf; hold on;
  for i=1:cn,
    tmp=[];
    for j=1:vc{i,1},
      tmp=[tmp;v(vc{i,2}(j),:)];
    end
    tmp=[tmp;tmp(1,:)]; plot(tmp(:,1),tmp(:,2));
  end
  for i=1:cn,
    plot(vc{i,4}(1),vc{i,4}(2),'o');
  end
  for i=1:cn,
    plot(c(i,1),c(i,2),'.');
  end
  axis equal; axis off;
end
```

§ A.23 Centroid process on 3-d VT

```
% cgi.m, c.g. operator on 3-d Voronoi, Kit Tiyapan (c) 12th November 2002.
clear all; can=400; rand('state',sum(100*clock)); ca=rand(can,3);
for z=1:3,
  cmx=max(max(ca)); cmn=min(min(ca)); [va,vca]=voronoin(ca);
  van=size(va,1); vin=zeros(1,van);
  for i=1:van,
    vin(i)=(min(va(i,:))>cmn) & (max(va(i,:))<cmx);
  end
  cin=ones(1,can);
  for i=1:can,
    vca{i,2}=vca{i,1}; vca{i,1}=size(vca{i,2},2);
    for j=1:vca{i,1},
      if(~vin(vca{i,2}(j)))
        cin(i)=0; break;
      end
    end
  end
  c=[];
  vc=[]; cnt=0; v=[]; cnu=0; vin=sparse(1,van);
  for i=1:can,
  if(cin(i))
    cnt=cnt+1; cin(i)=cnt; c(cnt,:)=ca(i,:); vc{cnt,1}=vca{i,1}; vc{cnt,2}=vca{i,2};
    for j=1:vc{cnt,1},
      if(~vin(vca{i,2}(j)))
        cnu=cnu+1; vin(vca{i,2}(j),1)=cnu; v(cnu,:)=va(vca{i,2}(j),:);
      end
    end
    for j=1:vc{cnt,1},
      vc{cnt,2}(j)=vin(vca{i,2}(j));
    end
  end
end
cn=cnt; vn=cnu;
for i=1:cn,
  tmp=ones(size(vc{i,2})); vc{i,3}=sparse(tmp,vc{i,2},tmp,1,vn);
end
ta=delaunayn(ca); tan=size(ta,1); t=[]; cnt=0; bdr=[]; cnu=0;
tmt=sparse(1,vn); tmm=sparse(can,can);
for i=1:tan,
  in=1;
  for j=1:4,
    if(~cin(ta(i,j)))
      in=0; break;
    end
  end
  if(in)
    cnt=cnt+1; t(cnt,:)=ta(i,:);
    for j=1:4,
      t(cnt,j)=cin(ta(i,j));
    end
  else
    for j=1:3,
      for k=(j+1):4,
        tma=ta(i,j); tmb=ta(i,k);
        if(((cin(tma)) | (cin(tmb))) & ~tmm(tma,tmb))
          tmm(tma,tmb)=1; tmm(tmb,tma)=1; tma=vca{ta(i,j),2}; tmb=vca{ta(i,k),2};
          tmd=ones(1,vca{ta(i,j),1}); tme=ones(1,vca{ta(i,k),1});
          tmp=find(sparse(tmd,tma,tmd,1,van) & sparse(tme,tmb,tme,1,van));
          tmq=[]; tmn=size(tmp,2);
          for l=1:tmn,
            tmq=[tmq,vin(tmp(l))];
          end
          tma=ones(1,tmn); tmb=sparse(tma,tmq,tma,1,vn); xst=0;
          for m=1:cnu,
            if(~(tmn-bdr{m,2}))
              tmc=tmb & tmt(m,:);
              if(min(tmc))
                xst=1; break;
              end
            end
            if(xst)
              break;
            end
          end
          if(~xst)
            cnu=cnu+1;
            if(cin(ta(i,j)))
              bdr{cnu,1}=cin(ta(i,j));
            else
              bdr{cnu,1}=cin(ta(i,k));
            end
            bdr{cnu,2}=tmn; bdr{cnu,3}=tmq; bdr{cnu,4}=tmb; tmt(cnu,:)=tmb;
          end
        end
      end
    end
  end
end
tn=cnt; bdrn=size(bdr,1); bcc=sparse(cn,cn);
for i=1:tn,
  for j=1:3,
    for k=(j+1):4,
      bcc(t(i,j),t(i,k))=1; bcc(t(i,k),t(i,j))=1;
    end
  end
end
[tmf,tmg]=find(triu(bcc)); bn=size(tmf,1); b=[];
for i=1:bn,
  b{i,1}(1,:)=[tmf(i),tmg(i)]; b{i,4}=vc{tmf(i),3} & vc{tmg(i),3};
  [nth,b{i,3}]=find(b{i,4}); b{i,2}=size(b{i,3},2); tmp=[];
  for j=1:b{i,2},
    tmp=[tmp;v(b{i,3}(j),:)];
  end
  tma=max(tmp,[],1)-min(tmp,[],1); tmb=sortrows([1,2,3;tma]',2);
  tmb=round(tmb(:,1)); tmq=[tmp(:,tmb(1,1)),tmp(:,tmb(2,1))];
  tmp=delaunay(tmq(:,1),tmq(:,2)); tmq=sparse(b{i,2},b{i,2});
  for j=1:size(tmp,1),
    for k=1:2,
      for l=(k+1):3,
        tmq(tmp(j,k),tmp(j,l))=tmq(tmp(j,k),tmp(j,l))+1;
        tmq(tmp(j,l),tmp(j,k))=tmq(tmp(j,l),tmp(j,k))+1;
      end
    end
  end
  [tma,tmb,tmc]=find(tmq); tmq=sparse(vn,vn);
  for j=1:size(tma,1),
    if(~(tmc(j)-1))
      tmq(b{i,3}(tma(j)),b{i,3}(tmb(j)))=1; tmq(b{i,3}(tmb(j)),b{i,3}(tma(j)))=1;
    end
  end
  tma=[b{i,3}(1)]; tmb=b{i,3}(1); [nth,tmc]=find(tmq(tmb,:));
  tmd=tmc(1); tma=[tma,tmd]; tmq(tmb,tmd)=0; tmq(tmd,tmb)=0; tmb=tmd;
  while(tmb-b{i,3}(1))
    [nth,tmc]=find(tmq(tmb,:)); tmd=tmc(1); tma=[tma,tmd];
    tmq(tmb,tmd)=0; tmq(tmd,tmb)=0; tmb=tmd;
  end
  b{i,6}=tma; b{i,5}=b{i,6}(1,1:b{i,2}); b{i,7}=2;
end
for i=1:bn,
  bcc(b{i,1}(1),b{i,1}(2))=i; bcc(b{i,1}(2),b{i,1}(1))=i;
end
bc=[];
for i=1:cn,
  bc{i,2}=[];
end
for i=1:bn,
  bc{b{i,1}(1),2}=[bc{b{i,1}(1),2},i]; bc{b{i,1}(2),2}=[bc{b{i,1}(2),2},i];
end
n=bn;
for i=1:bdrn,
  n=n+1; b{n,1}=bdr{i,1}; b{n,2}=bdr{i,2}; b{n,3}=bdr{i,3};
  tmp=ones(1,bdr{i,2}); b{n,4}=sparse(tmp,bdr{i,3},tmp,1,vn); tmp=[];
  for j=1:b{n,2},
    tmp=[tmp;v(b{n,3}(j),:)];
  end
  tma=max(tmp,[],1)-min(tmp,[],1); tmb=sortrows([1,2,3;tma]',2);
  tmb=round(tmb(:,1)); tmq=[tmp(:,tmb(1,1)),tmp(:,tmb(2,1))];
  tmp=delaunay(tmq(:,1),tmq(:,2)); tmq=sparse(b{n,2},b{n,2});
  for j=1:size(tmp,1),
    for k=1:2,
      for l=(k+1):3,
        tmq(tmp(j,k),tmp(j,l))=tmq(tmp(j,k),tmp(j,l))+1;
        tmq(tmp(j,l),tmp(j,k))=tmq(tmp(j,l),tmp(j,k))+1;
      end
    end
  end
  [tma,tmb,tmc]=find(tmq); tmq=sparse(vn,vn);
  for j=1:size(tma,1),
    if(~(tmc(j)-1))
      tmq(b{n,3}(tma(j)),b{n,3}(tmb(j)))=1; tmq(b{n,3}(tmb(j)),b{n,3}(tma(j)))=1;
    end
  end
  tma=[b{n,3}(1)]; tmb=b{n,3}(1); [nth,tmc]=find(tmq(tmb,:)); tmd=tmc(1);
  tma=[tma,tmd]; tmq(tmb,tmd)=0; tmq(tmd,tmb)=0; tmb=tmd;
  while(tmb-b{n,3}(1))
    [nth,tmc]=find(tmq(tmb,:)); tmd=tmc(1); tma=[tma,tmd];
    tmq(tmb,tmd)=0; tmq(tmd,tmb)=0; tmb=tmd;
  end
  b{n,6}=tma; b{n,5}=b{n,6}(1,1:b{n,2}); b{n,7}=1;
end
for i=(bn+1):n,
  bc{b{i,1},2}=[bc{b{i,1},2},i];
end
bn=size(b,1);
for i=1:cn,
  bc{i,1}=size(bc{i,2},2);
end
fc=[];
for i=1:cn,
  fc{i,2}=[];
end
for i=1:bn,
  for j=1:b{i,7},
    fc{b{i,1}(j),2}=[fc{b{i,1}(j),2},i];
    end
  end
  for i=1:cn,
    fc{i,1}=size(fc{i,2},2);
  end
  tma=[]; tmb=[];
  for i=1:cn,
    tmp=[]; tmf=[]; tmg=[];
    for j=1:vc{i,1},
      tmp=[tmp;v(vc{i,2},:)];
    end
    tmd=delaunayn(tmp);
    tmn=size(tmd,1);
    for j=1:tmn,
      tmq=[tmp(tmd(j,1),:);tmp(tmd(j,2),:);tmp(tmd(j,3),:);tmp(tmd(j,4),:)];
      tma=sum(tmq)/4; tmb=abs(det([tmq,ones(4,1)])/6); tmf=[tmf;tma*tmb]; tmg=[tmg,tmb];
    end
    vc{i,5}=sum(tmg); vc{i,4}=sum(tmf)/vc{i,5};
  end
  bb=[];
  for i=1:bn,
    if(~(b{i,7}-1))
      bb=[bb,i];
    end
  end
  vs{z,1}=c; vs{z,2}=v; vs{z,3}=vc; vs{z,4}=b; vs{z,5}=bc; vs{z,6}=fc; vs{z,7}=bb; g=[];
  for i=1:cn,
    g=[g;vc{i,4}];
  end
  gn=size(g,1); ca=g; can=gn;
end
mag=400;   % start statistics
for z=1:3,
  vc=vs{z,3}; cn=size(vc,1); tmp=[];
  for i=1:cn,
    tmp=[tmp;vc{i,5}];
  end
  tmp=mag*tmp; nc(z)=cn; sm(z)=mean(tmp); ss(z)=std(tmp);
  mt(z)=moment(tmp,3); mf(z)=moment(tmp,4);
end
mag=10; mag=mag^3; tmq=[];
for z=1:3,
  tmq{z}=[];
end
for z=1:3,
  for i=1:size(vs{z,1},1),
    tmq{z}=[tmq{z};vs{z,3}{i,5}];
  end
  tmq{z}=mag*tmq{z};
end
tmp=[];
for z=1:3,
  tmp=[tmp;tmq{z}];
end
tma=min(tmp); tmb=max(tmp); n=20; tmc=(tmb-tma)/n; tma=tma:tmc:tmb;
for i=1:z,
  tmb=hist(tmq{i},tma); figure(i); clf; axes('FontSize',13);
  bar(tma,tmb); xlabel('volume','FontSize',15); ylabel('number of cells','FontSize',15);
end
for zi=1:z,
  figure(z+zi); clf; hold on; v=vs{zi,2}; b=vs{zi,4}; bb=vs{zi,7}; bbn=size(bb,2);
  for i=1:bbn,
    tmp=[];
    for j=1:b{bb(i),2},
      tmp=[tmp;v(b{bb(i),5}(j),:)];
    end
    tmp=[tmp;tmp(1,:)]; fill3(tmp(:,1),tmp(:,2),tmp(:,3),b{bb(i),1});
  end
  axis equal; axis off; rotate3d; view(96,0);
end
for zi=1:z,
  figure(z+zi); clf; hold on; v=vs{zi,2}; b=vs{zi,4}; bb=vs{zi,7}; bbn=size(bb,2);
  for i=1:bbn,
    tmp=[];
    for j=1:b{bb(i),2},
      tmp=[tmp;v(b{bb(i),3}(j),:)];
    end
    plot3(tmp(:,1),tmp(:,2),tmp(:,3));
  end
  axis equal; axis off; rotate3d;
end
```

§ A.24 Defining irregular objects

```
% tioa.m, three irregular object algorithms, Kit Tiyapan (c) 12th November 2002
% tioa1.m
clear all;
v{1,1}=[1,1;3,4;3,8;3.5,9;4,8;4,4;6,1;4,3;4,2;3.5,1;3,2;3,3];
v{2,1}=[1,2;1,8;1.5,9;2,8;2,2;1.5,1]; v{3,1}=[1,7;3.5,9;6,7;4,8;4,2;3.5,1;3,2;3,8];
vn=size(v,1); mag=10;
for i=1:vn,
  v{i,2}=size(v{i,1},1);
end
for i=1:vn,
  tmp=max(v{i,1}); v{i,3}=sparse(tmp(1),tmp(2));
  for j=1:v{i,2},
    tmp=mag*[v{i,1};v{i,1}(1,:)]; tmx=tmp((j+1),1)-tmp(j,1);
    tmy=tmp((j+1),2)-tmp(j,2); tms=sign(tmx);
    if(~tmx)
      for k=tmp(j,2):sign(tmy):tmp((j+1),2),
        v{i,3}(tmp(j,1),k)=1;
      end
    else
      if(~tmy)
        for k=tmp(j,1):tms:tmp((j+1),1),
          v{i,3}(k,tmp(j,2))=1;
        end
      else
        tmm=tmy/tmx; tmc=tmp(j,2)-tmm*tmp(j,1);
        for k=tmp(j,1):tms:tmp((j+1),1),
          tmb=round(tmm*k+tmc); v{i,3}(k,tmb)=1;
        end
      end
    end
  end
end
tma=0;
ost(1,:)=[0,0];
for i=1:(vn-1),
  tmp=max(v{i}); tma=tma+tmp(1); ost((i+1),:)=[tma,0];
end
tmx=[]; tmy=[];
for i=1:vn,
  [tma,tmb]=find(v{i,3}); tmn=size(tma,1); tma=mag*ost(i,1)*ones(tmn,1)+tma;
  tmb=mag*ost(i,2)*ones(tmn,1)+tmb; tmx=[tmx;tma]; tmy=[tmy;tmb];
end
tmn=size(tmx,1); clf;
for i=1:tmn,
  tma=tmx(i)-.5; tmb=tmx(i)+.5; tmc=tmy(i)-.5; tmd=tmy(i)+.5;
  fill([tma,tmb,tmb,tma],[tmc,tmc,tmd,tmd],k); hold on;
end
axis equal; axis off;
% tioa2.m
clear all; v{1,2}=[1,1;3,4;3,8;3.5,9;4,8;4,4;6,1;4,3;4,2;3.5,1;3,2;3,3];
v{2,2}=[1,2;1,8;1.5,9;2,8;2,2;1.5,1]; v{3,2}=[1,7;3.5,9;6,7;4,8;4,2;3.5,1;3,2;3,8];
vn=size(v,1); mag=10; iny=1000; tmn=0;
for i=1:vn,
  v{i,1}=size(v{i,2},1); tmm=max(max(v{i,2}));
  if(tmm>tmn)
    tmn=tmm;
  end
end
for i=1:vn,
  tmp=[v{i,2};v{i,2}(1,:)];
  for j=1:v{i,1},
    tma=tmp((j+1),1)-tmp(j,1); tmb=tmp((j+1),2)-tmp(j,2); tmc=max(abs(tma),abs(tmb));
    if(tmc<tmn)
      tmn=tmc;
    end
    if(tma)
      v{i,3}(j,1)=tmb/tma;
    else
      v{i,3}(j,1)=iny;
    end
  end
end
stp=tmn/mag;
for i=1:vn,
  tmp=round(max(v{i,2})/stp); v{i,4}=sparse(tmp(1),tmp(2)); tmp=[v{i,2};v{i,2}(1,:)];
  for j=1:v{i,1},
    if(abs(v{i,3}(j))<0.5)
      tma=round(tmp(j,1)/stp); tmb=round(tmp((j+1),1)/stp);
      for k=tma:sign(tmb-tma):tmb,
        tmc=round(tmp(j,2)/stp+(k-tma)*v{i,3}(j)); v{i,4}(k,tmc)=1;
      end
    else
      tma=round(tmp(j,2)/stp); tmb=round(tmp((j+1),2)/stp);
      for k=tma:sign(tmb-tma):tmb,
        tmc=round(tmp(j,1)/stp+(k-tma)/v{i,3}(j)); v{i,4}(tmc,k)=1;
      end
    end
  end
end
for i=1:vn,
  v{i,5}=max(v{i,2});
end
tmp=[]; tmq=[]; tmr=[]; oft=0;
for i=1:vn,
  [tma,tmb,tmc]=find(v{i,4}); tma=(oft/stp)*ones(size(tma))+tma;
  tmp=[tmp;tma]; tmq=[tmq;tmb]; tmr=[tmr;tmc]; oft=oft+v{i,5}(1);
end
tmn=size(tmp,1); clf;
for i=1:tmn,
  tma=tmp(i)-.5; tmb=tmp(i)+.5; tmc=tmq(i)-.5; tmd=tmq(i)+.5;
  fill([tma,tmb,tmb,tma],[tmc,tmc,tmd,tmd],k); hold on;
 end
axis equal; axis off;
% tiao3.m
clear all; v{1,2}=[1,1;3,4;3,8;3.5,9;4,8;4,4;6,1;4,3;4,2;3.5,1;3,2;3,3];
v{2,2}=[1,2;1,8;1.5,9;2,8;2,2;1.5,1]; v{3,2}=[1,7;3.5,9;6,7;4,8;4,2;3.5,1;3,2;3,8];
vn=size(v,1); tma=[];
for i=1:vn,
  v{i,1}=size(v{i,2},1); tmp=[v{i,2};v{i,2}(1,:)];
  for j=1:v{i,1},
    tma=diff(tmp);
  end
end
tmb=max(min(abs(tma))); mag=10; mag=mag/tmb;
for i=1:vn,
  tmc=min(v{i,2});
  v{i,4}=ceil(mag*v{i,2}-[tmc(1)*ones(v{i,1},1),tmc(2)*ones(v{i,1},1)]);
  v{i,3}=max(v{i,4});
end
for i=1:vn,
  tmz=[v{i,4};v{i,4}(1,:)]; tma=diff(tmz); tmm=tma(:,2)./tma(:,1); tmp=[]; tmq=[];
  for j=1:v{i,1},
    tms=sign(tmz((j+1),1)-tmz(j,1)); tmx=tmz(j,1):tms:tmz((j+1),1);
    tmu=tmz(j,1)*ones(size(tmx)); tmv=tmz(j,2)*ones(size(tmx));
    tmy=ceil(tmm(j)*(tmx-tmu)+tmv); tmn=size(tmy,2); tma=diff(tmy); tmb=[]; tmd=[];
    for k=1:(tmn-1),
      tmb=[tmb,tmx(k)]; tmc=min(tmy(k),v{i,3}(2)); tmd=[tmd,tmc];
      for l=1:abs(tma(k)),
        tmd=[tmd,min(v{i,3}(2),(tmc+l*tms))]; tmb=[tmb,tmx(k)];
      end
    end
    tmx=tmb; tmy=tmd; tmp=[tmp,tmx];tmq=[tmq,tmy];
  end
  tma=ones(size(tmp)); v{i,5}=sparse(tmp,tmq,tma,v{i,3}(1),v{i,3}(2));
end
for i=1:vn,
  tmz=[v{i,4};v{i,4}(1,:)]; tma=diff(tmz); tmm=tma(:,1)./tma(:,2); tmp=[]; tmq=[];
  for j=1:v{i,1},
    tms=sign(tmz((j+1),2)-tmz(j,2)); tmy=tmz(j,2):tms:tmz((j+1),2);
    tmv=tmz(j,2)*ones(size(tmy)); tmu=tmz(j,1)*ones(size(tmy));
    tmx=ceil(tmm(j)*(tmy-tmv)+tmu); tmn=size(tmx,2); tma=diff(tmx); tmb=[]; tmd=[];
    for k=1:(tmn-1),
      tmb=[tmb,tmy(k)]; tmc=min(tmx(k),v{i,3}(1)); tmd=[tmd,tmc];
      for l=1:abs(tma(k)),
        tmd=[tmd,min(v{i,3}(1),(tmc+l*tms))]; tmb=[tmb,tmy(k)];
      end
    end
    tmy=tmb; tmx=tmd; tmp=[tmp,tmx]; tmq=[tmq,tmy];
  end
  tma=ones(size(tmp)); v{i,5}=v{i,5}|sparse(tmp,tmq,tma,v{i,3}(1),v{i,3}(2));
end
tmp=[]; tmq=[]; osx=0; osy=0; tma=0; tmb=0;
for i=2:vn,
  tma=tma+v{(i-1),3}(1); tmb=0; osx=[osx,tma]; osy=[osy,tmb];
end
for i=1:vn,
  [tma,tmb]=find(v{i,5}); tmc=ones(size(tma));
  tmp=[tmp;(tma+tmc*osx(i))]; tmq=[tmq;(tmb+tmc*osy(i))];
end
tmn=size(tmp,1); clf;
for i=1:tmn,
  tma=tmp(i)-.5; tmb=tmp(i)+.5; tmc=tmq(i)-.5; tmd=tmq(i)+.5;
  fill([tma,tmb,tmb,tma],[tmc,tmc,tmd,tmd],k); hold on;
end
axis equal; axis off;
```

§ A.25 Investigating the distribution observed from within the network

```
% dstnvst.m; Kit Tiyapan, 17th November 2002
% (x,y) in square domain
clear all; Dim=2; St=sum(100*clock); rand('state',St); N=1000;
X=rand(N,Dim); Ta=delaunay(X(:,1),X(:,2)); TaN=size(Ta,1); EXX=sparse(N,N);
E=[]; DL=[]; xc=1/2; yc=1/2; LB=0.05; UB=1-LB; Xin=ones(N,1);
for i=1:N,
  if((min(X(i,1),X(i,2))<LB) | (max(X(i,1),X(i,2))>UB))
    Xin(i,1)=0;
  end
end
% or (x,y) in circular domain
clear all; Dim=2; St=sum(100*clock); rand('state',St); N=1000;
TwoPi=2*pi; X=[]; R=[]; xc=1/2; yc=1/2;
for i=1:N,
  Tmp=rand(1,2); TmpB=(Tmp(1)-xc); TmpC=(Tmp(2)-yc); TmpA=(TmpB*TmpB+TmpC*TmpC);
  while(TmpA>(1/4))
    Tmp=rand(1,2); TmpB=(Tmp(1)-xc); TmpC=(Tmp(2)-yc); TmpA=(TmpB*TmpB+TmpC*TmpC);
  end
  X=[X;Tmp]; R=[R;TmpA];
end
Ta=delaunay(X(:,1),X(:,2));
% or (r,theta) in a circle
clear all; Dim=2; St=sum(100*clock); rand('state',St); N=1000; TwoPi=2*pi;
R=rand(N,1); Th=TwoPi*rand(N,1); X=[R.*cos(Th), R.*sin(Th)]; Ta=delaunayn(X);
% and then
TaN=size(Ta,1); EXX=sparse(N,N); E=[]; DL=[]; xc=0; yc=0; B=0.95; Xin=ones(N,1);
for i=1:N,
  if(R(i,1)>B)
    Xin(i,1)=0;
  end
end
% then
T=[];
for i=1:TaN,
  include=1;
  for j=1:(Dim+1),
    if(~Xin(Ta(i,j)))
      include=0; break;
    end
  end
  if(include)
    T=[T;Ta(i,:)];
  end
end
TN=size(T,1);
for i=1:TN,
  Tmp=[T(i,:),T(i,1)];
  for j=1:(Dim+1),
    if(~EXX(Tmp(j),Tmp(j+1)))
      E=[E;[Tmp(j),Tmp(j+1)]]; dx=X(Tmp(j+1),1)-X(Tmp(j),1);
      dy=X(Tmp(j+1),2)-X(Tmp(j),2); length=sqrt(dx*dx + dy*dy);
      xm=X(Tmp(j),1)+(dx/2); ym=X(Tmp(j),2)+(dy/2); dx=xm-xc; dy=ym-yc;
      dist=sqrt(dx*dx + dy*dy); DL=[DL;[dist,length]];
      EXX(Tmp(j),Tmp(j+1))=1; EXX(Tmp(j+1),Tmp(j))=1;
    end
  end
end
EN=size(E,1);
% plot
figure(1); clf; hold on;
for i=1:EN,
  plot([X(E(i,1),1),X(E(i,2),1)],[X(E(i,1),2),X(E(i,2),2)]);
end
axis equal; axis off;
n=20; int=1/n; Tmp=zeros(n,2);
for i=1:EN,
  TmpA=ceil(DL(i,1)/int); Tmp(TmpA,1)=Tmp(TmpA,1)+DL(i,2); Tmp(TmpA,2)=Tmp(TmpA,2)+1;
end
TmpA=[];
for i=1:n,
  if(Tmp(i,2))
    TmpA=[TmpA;Tmp(i,:)];
  end
```

```
end
Ld=TmpA(:,1)./TmpA(:,2); LdN=size(Ld,1); Tmp=(int/2):int:(LdN*int); Ld=[Tmp',Ld];
figure(2); clf; bar(Ld(:,1),Ld(:,2),'m');
xlabel('Distance from centre','FontSize',13); ylabel('Average length','FontSize',13);
```

§ A.26 Miscellaneous functions

```
% findfarea.m, finds face area, (c) Kit Tiyapan, February, 2001.
function [fca, ppr] = findfarea(odvc)
nvthf=size(odvc,1); stp=floor(nvthf/3); ndpt =1+stp; rdpt =1+2*stp;
nmvc= cross((odvc(ndpt,:)-odvc(1,:)),(odvc(rdpt,:)-odvc(1,:)));
clov =[odvc; odvc(1,:)]; xsq =nmvc(1,1)*nmvc(1,1); ysq =nmvc(1,2)*nmvc(1,2);
zsq =nmvc(1,3)*nmvc(1,3); znmvc =sqrt(xsq+ysq+zsq); nmnmv =nmvc/znmvc; socp =0;
for i=1:nvthf,
  socp =socp+cross(clov(i,:),clov((i+1),:));
end
fca =(abs(dot(nmnmv, socp)))/2;
nvct =ones(3,1); crdm =[odvc(1,:); odvc(ndpt,:); odvc(rdpt,:)];
aprm =det([nvct, crdm(:,2), crdm(:,3)]); bprm =det([crdm(:,1),  nvct, crdm(:,3)]);
cprm =det([crdm(:,1), crdm(:,2), nvct]);
dprm =det([crdm(:,1), crdm(:,2), crdm(:,3)]); ppr =[aprm, bprm, cprm, dprm];
% ordervertices.m, orders the vertices in a list, (c) Kit Tiyapan, February 2001.
function [odvc] = ordervertices(vnsp)
zvnsp =size(vnsp,1);
if (zvnsp==3)
  odvc =vnsp; return;
end
vmny =1; vmxy =1;
for i=2:zvnsp,
  if (vnsp(i,2) < vnsp(vmny,2))
    vmny =i;
  elseif (vnsp(i,2) > vnsp(vmxy,2))
    vmxy =i;
  end
end
ndps =[vnsp(vmny,1:2); vnsp(vmxy,1:2)]; aprm =det([[1;1],ndps(:,2)]);
bprm =det([ndps(:,1),[1;1]]); cprm =det(ndps); vrts =vnsp(vmny,:); vlfs =vnsp(vmxy,:);
for i=1:zvnsp,
  if ((i~=vmny) & (i~=vmxy))
    lfpq =aprm*vnsp(i,1)+bprm*vnsp(i,2)+cprm;
    if (lfpq > 0)
      vrts =[vrts; vnsp(i,:)];
    elseif (lfpq < 0)
      vlfs =[vlfs; vnsp(i,:)];
    end
  end
end
zvrts =size(vrts,1); zvlfs =size(vlfs,1);
if(zvrts >= 3)
  if (vrts(1,2)==vrts(2,2))
    if (vrts(1,1) > vrts(2,1))
      vrts =[[vrts(2,:); vrts(1,:)]; vrts(1,3:zvrts)];
    end
  end
  Angle =[];
  for i=2:zvrts,
    xcd =vrts(i,1)-vrts(1,1); ycd =vrts(i,2)-vrts(1,2); ang =[ang; atan(ycd/xcd)];
  end
  vrts =sort([ang,vrts(2:zvrts,:)],1); svrts =[vrts(1,:); vrts(:,2:5)];
else
  svrts =vrts;
end
if(zvlfs >= 3)
  if (vlfs(1,2)==vlfs(2,2))
    if (vlfs(1,1) < vlfs(2,1))
      vlfs =[vlfs(2,:); vlfs(1,:)];
    end
  end
  ang =[];
  for i=2:zvlfs,
    xcd =vlfs(i,1)-vlfs(1,1); ycd =vlfs(i,2)-vlfs(1,2); ang =[ang; atan(ycd/xcd)];
  end
  vlfs =sort([ang,vlfs(2:zvlfs,:)],1); svlfs =[vlfs(1,:); vlfs(:,2:5)];
else
  svlfs =vlfs;
end
odvc =[svrts; svlfs];
% perc.m, function to find percolation, Kit Tiyapan, (c) 21st November, 2002
function [Pc,Cord,TSeries] = perc(N,LMat,UMat,NeMat)
Blocked=randperm(N); NClusA=0; Perco=0;
```

```
for i=1:N,
  Joined=0;
  for j=1:NClusA,
    if(ClusA{j,3}(1,Blocked(1,i))~=0)
      ClusA{j,1}=ClusA{j,1}+1; ClusA{j,2}(1,Blocked(1,i))=1;
      ClusA{j,3}=ClusA{j,3} | NeMat(Blocked(1,i),:); Joined=1;
    end
    if(Joined==1)
      for k=1:4,
        ClusB{1,k}=ClusA{j,k};
      end
      NClusB=1;
      if(j==1)
        Tmp=ClusA; clear ClusA;
        for k=1:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{(k+1),l};
          end
        end
      elseif(j==NClusA)
        Tmp=ClusA; clear ClusA;
        for k=1:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{k,l};
          end
        end
      else
        Tmp=ClusA; clear ClusA;
        for k=1:(j-1),
          for l=1:4,
            ClusA{k,l}=Tmp{k,l};
          end
        end
        for k=j:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{(k+1),l};
          end
        end
      end
      for k=1:(NClusA-1),
        if(sum(ClusA{k,2} & ClusB{1,3}) ~= 0)
          ClusB{1,1}=ClusB{1,1}+ClusA{k,1}; ClusB{1,2}=ClusB{1,2} | ClusA{k,2};
          ClusB{1,3}=ClusB{1,3} | ClusA{k,3}; ClusB{1,4}=ClusB{1,4} | ClusA{k,4};
        else
          NClusB=NClusB+1;
          for l=1:4,
            ClusB{NClusB,l}=ClusA{k,l};
          end
        end
      end
      if((sum(full(LMat & ClusB{1,2}))~=0) & (sum(full(UMat & ClusB{1,2}))~=0))
        ClusB{1,4}=1; Perco=1;
      end
      NClusA=NClusB; ClusA=ClusB; clear ClusB; break;
    end
  end
  if(Joined==0)
    NClusA=NClusA+1; ClusA{NClusA,1}=1; ClusA{NClusA,2}=sparse(1,Blocked(1,i),1,1,N);
    ClusA{NClusA,3}=NeMat(Blocked(1,i),:); ClusA{NClusA,4}=0;
  end
  TSeries{1}{i,1}=ClusA; TSeries{1}{i,2}=Perco;
end
Tmp=Blocked; Blocked=[];
for i=1:N,
  Blocked=[Blocked,Tmp(1,(N-i+1))];
end
Nc=0; TSnap=[];
for i=1:N,
  if(TSeries{1}{i,2})
    Nc=i; break;
  end
end
Pc=Nc/N; Cord=mean(sum(NeMat,2)); clear ClusA ClusB; NClusA=0; Perco=0;
for i=1:N,
  Joined=0;
  for j=1:NClusA,
    if(ClusA{j,3}(1,Blocked(1,i))~=0)
      ClusA{j,1}=ClusA{j,1}+1; ClusA{j,2}(1,Blocked(1,i))=1;
      ClusA{j,3}=ClusA{j,3} | NeMat(Blocked(1,i),:); Joined=1;
    end
    if(Joined==1)
      for k=1:4,
        ClusB{1,k}=ClusA{j,k};
      end
      NClusB=1;
      if(j==1)
        Tmp=ClusA; clear ClusA;
        for k=1:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{(k+1),l};
          end
        end
      elseif(j==NClusA)
        Tmp=ClusA; clear ClusA;
        for k=1:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{k,l};
          end
        end
      else
        Tmp=ClusA; clear ClusA;
        for k=1:(j-1),
          for l=1:4,
            ClusA{k,l}=Tmp{k,l};
          end
        end
        for k=j:(NClusA-1),
          for l=1:4,
            ClusA{k,l}=Tmp{(k+1),l};
          end
        end
      end
      for k=1:(NClusA-1),
        if(sum(ClusA{k,2} & ClusB{1,3}) ~= 0)
          ClusB{1,1}=ClusB{1,1}+ClusA{k,1}; ClusB{1,2}=ClusB{1,2} | ClusA{k,2};
          ClusB{1,3}=ClusB{1,3} | ClusA{k,3}; ClusB{1,4}=ClusB{1,4} | ClusA{k,4};
        else
          NClusB=NClusB+1;
          for l=1:4,
            ClusB{NClusB,l}=ClusA{k,l};
          end
        end
      end
      if((sum(full(LMat & ClusB{1,2}))~=0) & (sum(full(UMat & ClusB{1,2}))~=0))
        ClusB{1,4}=1; Perco=1;
      end
      NClusA=NClusB; ClusA=ClusB; clear ClusB; break;
    end
  end
  if(Joined==0)
    NClusA=NClusA+1; ClusA{NClusA,1}=1; ClusA{NClusA,2}=sparse(1,Blocked(1,i),1,1,N);
    ClusA{NClusA,3}=NeMat(Blocked(1,i),:); ClusA{NClusA,4}=0;
  end
  TSeries{2}{i,1}=ClusA; TSeries{2}{i,2}=Perco;
end
Nc=0; TSnap=[];
for i=1:N,
  if(TSeries{2}{i,2})
    Nc=i; break;
  end
end
Pc=[Pc,Nc/N]; Cord=[Cord,mean(sum(NeMat,2))];
```

§ A.27 Percolated traffic networks

```
% Amsterdam
clear all; twn='Amsterdam'; T=7; B=0; L=0; R=7;
V=[2.5,3.8; 2.6,3.9; 2.6,4.3; 2.5,4.2; 2.5,4.5; 2.7,4.6; 2.9,4.7; 2.8,4.3; 2.8,3.8; 2.9,3.2;
2.8,3.1; 2.5,2.9; 2.4,2.8; 2.1,3.2; 2,3.3; 2,3.8; 2.1,4.5; 2.1,5.4; 2.2,5.9; 2.3,6.5;
2.4,6.6; 2.5,6.5; 2.5,T; 2.7,6.9; 2.9,6.8; 2.6,6.5; 2.5,6.2; 2.3,6.2; 2.3,6; 2.5,5.9;
2.4,5.4; 2.2,5.4; 2.2,5.1; 2.4,5.1; 2.4,4.9; 2.2,4.9; 2.2,4.5; 2.4,4.5; 2.3,3.8; 2.2,3.8;
2.2,3.6; 2.2,3.4; 2.4,3.5; 2.6,3.6; 2.7,4.8; 2.5,4.7; 2.8,5.1; 3.1,5.1; 3.1,5.4; 2.7,5.3;
2.6,5.4; 2.8,5.5; 3.1,5.6; 2.8,5.8; 2.7,5.8; 2.8,6.5; 3,6.5; 3.1,5.9; 3.3,6.3; 3.4,6.6;
3.2,6.6; 3.1,6.7; 3.7,T; 3.5,T; 3.3,T; 3,T; 2.7,T; 2,T; 1.8,T; 1.5,T;
1.4,T; 1.1,6.8; 1,6.7; .9,6.9; .9,T; .4,6.9; .3,T; .4,6.7; .5,6.5; .5,6.4;
.6,6.3; .7,6; .3,5.8; L,5.6; L,6; L,6.3; L,6.4; L,6.6; L,6.7; L,6.9;
.9,6.1; .9,5.8; .4,5.5; .4,5.3; .9,5.5; .9,5.4; .9,5.3; .9,5.1; .6,4.9; .5,5.1;
.9,4.8; .7,4.7; .3,4.5; L,4.3; L,4.7; L,5.1; L,5.3; .9,4.5; .9,4.2; .5,3.9;
.3,3.7; .3,3.5; .6,3.7; .8,3.8; .8,3.4; .6,3.6; .5,3.5; .5,3.3; .8,3.4; .8,3.2;
.6,3.1; .4,2.9; .1,3.1; .3,2.6; .8,2.7; .9,2.6; .5,2.4; .3,2.2; L,2.1; .7,1.3;
.4,1; L,1.4; L,.9; .5,.7; L,.6; .6,.6; L,.3; .9,.5; 1.5,.4; 1.6,.3;
1.1,.3; 1.3,B; .6,B; 1.9,.1; 1.5,B; 2.3,B; 2.2,.3; 2.8,B; 2.5,.1; 2.6,.3;
```

```
2.3,.5; 2.2,.4; 1.4,1.2; 1.6,1.4; 1.8,1.5; 1.5,2; 1.6,2.2; 1.2,2.6; 1.1,2.7; 1,2.9;
1.5,2.8; 1.5,2.9; 1.6,3; 1.7,2.9; 1.4,3.1; 1.1,3.8; 1.5,3.8; 1.6,3.8; 1.6,4.5; 1.4,4.5;
1.1,4.5; 1.2,5.4; 1.5,5.4; 1.2,6.2; 1.5,6.2; 1.6,6.2; 1.3,6.6; 1.6,6.5; 1.4,6.9; 1.7,6.7;
1.8,6.9; 1.9,6.8; 1.9,6.7; 1.9,2.5; 2.1,2.5; 2.8,1.8; 2.7,1.6; 2.6,1.4; 2.4,1; 2.7,.8;
2.7,.7; 2.6,.5; 2.6,.4; 2.9,.2; 2.9,.3; 3.1,B; 3.3,.1; 3.3,.3; 3.4,.4; 3.4,.6;
3.5,.7; 3.5,.9; 3.5,1.3; 3.6,1.5; 3.6,1.7; 3.7,2.1; 3.8,2.3; 3.8,2.6; 2.7,3; 3.9,3.1;
4.1,3.3; 3.6,4.1; 3.9,4.1; 4,4.1; 4.1,4.1; 4.3,4.1; 3.7,4.4; 3.8,4.4; 4,4.3; 4.1,4.3;
4.2,4.5; 4.4,4.4; 3.7,4.6; 3.9,4.6; 3.7,4.7; 3.9,4.7; 4.2,4.7; 4.4,4.7; 4.4,4.6; 3.8,5;
3.9,5.1; 3.9,5; 4.3,5; 4.3,5.1; 3.8,5.3; 4,5.3; 4.2,5.3; 4.4,5.2; 3.9,5.6; 4.5,5.3;
4.6,5.3; 4.7,5.2; 4.5,5; 4.6,5; 4.5,4.5; 4.6,4.4; 4.7,4.6; 4.7,4.7; 4.8,4.5; 5.1,4.5;
5,4.3; 4.9,4.3; 4.8,4.1; 4.4,4.1; 4.2,3.6; 4.3,3.5; 4.5,3.7; 4.7,3.6; 4.6,3.3; 4.5,3;
5.1,3.6; 5.4,4.3; 5.2,4.4; 5.2,4.5; 5.4,4.6; 5.4,4.4; 5.5,4.5; 5.7,4.4; 5.7,4.3; 5.8,4.2;
5.7,4.1; 5.6,4.2; 5.2,3.5; 5.3,3.5; 5.6,3.4; 6.1,3; 5.9,4; 6,4.2; 6.3,4; 6.2,3.7;
6.5,3.4; 6.7,3.7; 6.6,3.5; 6.4,4.2; 6.5,4.3; 6.9,4.6; 6.7,4.8; R,5; 6.3,4.6; 6.5,4.9;
6.7,5.1; 6.9,5.2; 6,4.3; 5.9,4.7; 6,4.6; 6.1,4.7; 6.2,4.8; 5.8,5.3; 6,5.3; 6,5.2;
6.2,5.3; 5.9,5.5; 6.1,5.4; 5.8,5.3; 5.6,5.4; 5.5,5.4; 5.4,5.4; 5.4,5.2; 5.1,5.3; 5.2,5.5;
5.3,5.6; 5.4,5.8; 5.4,5.9; 5.6,5.7; 5.7,5.8; 5.9,5.8; 5.7,6; 5.5,6.1; 5.4,6.2; 5.3,6.2;
5.2,6.3; 5.1,6.1; 5,6; 4.8,5.8; 4.5,5.8; 4.7,5.7; 4.9,5.6; 5,5.5; 4.5,5.4; 4.1,5.6;
3.9,5.9; 3.2,6.1; 4.3,6.3; 4.4,6.3; 4.3,5.9; 4.6,6.3; 4.6,6.5; 4.8,6.4; 5,6.6; 5.4,6.5;
5.6,6.9; 5.1,6.8; 4.7,T; 5.3,6.8; 5.5,6.7; 5.6,6.9; 5.7,6.8; 5.8,6.7; 5.9,6.7; 6.1,6.7;
5.5,6.6; 5.7,6.5; 5.8,6.5; 5.9,6.4; 6.2,6.6; 6.4,6.5; 6.2,6.4; 6.4,6.3; 6.2,6.3; 6.5,6.1;
6.7,6.3; 6.7,6.7; 6.5,T; 6.9,6.5; 6.1,6.1; 6.1,6; 6.4,5.8; 6.6,5.6; 6.7,5.7; 6.8,5.8;
6.1,5.6; 6.2,5.7; 6.3,5.5; 6.4,5.4; 6.4,5.6; 6.5,5.5; 6.3,5.2; 6.4,5.1; 6.5,5; 6.6,5.3;
6.5,5.4; 6.7,5.4; 5.3,3; 5.5,2.9; 5.2,2.6; 5.5,2.6; 5.8,2.6; 5.8,2.5; 5.9,2.4; 5.2,2.2;
5.1,2.3; 4.5,2.2; 4.6,2; 4.4,2; 4.5,1.5; 4.5,1.3; 4.5,1; 4.5,.7; 4.1,.6; 4.1,.4;
4,.2; 4.6,.6; 4.6,.3; 4.6,.1; 4.7,.1; 4.7,.4; 4.7,.5; 4.7,.6; 4.6,1; 4.6,1.4;
4.6,1.5; 5.3,1.6; 5.3,1.6; 5.4,1.2; 5.2,1.2; 5.3,.8; 5.5,.9; 5.5,.7; 5.5,.5; 5.5,.4;
5.2,.2; 5.2,.1; 5.6,.1; 6,.5; 6.1,.4; 6.2,.2; 6.5,.3; 6.5,.5; 6.4,.6; 6.4,.6;
6.3,1; 6.3,1.2; 6.2,1.5; 6.1,1.7; 6,2; 6.3,2.6; 6.3,2.5; 6.4,2.1; 6.5,1.9; 6.5,1.6;
6.6,1.4; 6.7,1.3; R,2.9; R,2.7; T,2.4; R,2.3; R,1.8; R,1.6; R,1.4; R,.5;
2.5,1.1; 1.1,6.6; 1,6.4; L,4.8; L,3.6; L,3.3; L,3.1; L,.1; 1.6,5.4; 4.6,B;
4.3,5.5; 4.5,4.3; 5.9,4.4; 6.9,4; R,4.3; 6.1,5; 5,6.3; 4.9,6.2; 4.8,6.1; 4.9,6.8;
5,T; 5.3,T; 5.8,T; 6.3,T; 6.9,T; 6.3,6; 6.5,5.6; R,5.5; 6,5.9; 6.8,5.2;
4.8,B; 5.4,B; 6.1,B; 6.6,B; 6.6,.1; R,5.9; R,6.4];
VN=size(V,1);
E=[1,2; 1,4; 1,44; 1,39; 2,3; 2,9; 3,4; 3,6; 3,8; 3,9; 4,5; 5,6; 5,38; 5,46; 6,7; 6,45;
7,8; 7,48; 8,9; 9,44; 10,11; 10,44; 11,43; 11,209; 12,42; 12,13; 12,209; 12,207;
13,14; 13,206; 14,15; 14,164; 15,16; 15,163; 16,17; 16,40; 16,168; 17,18; 17,37; 17,169;
18,19; 18,32; 18,173; 19,20; 19,29; 19,176; 20,21; 20,183; 21,22; 21,23; 21,182;
22,24; 22,26; 22,28; 25,26; 25,66; 26,27; 26,56; 27,28; 27,30; 28,29; 29,30; 29,32;
30,31; 30,55; 31,32; 31,34; 31,51; 32,33; 33,34; 33,36; 34,35; 35,36; 35,38;
36,37; 37,38; 37,40; 38,39; 39,40; 39,41; 39,43; 40,41; 41,42; 42,43; 45,46; 45,47; 46,51;
47,48; 47,50; 48,49; 49,50; 49,53; 50,52; 51,52; 51,55; 52,53; 52,54; 53,58; 53,239;
54,55; 54,57; 54,58; 55,56; 56,57; 57,59; 57,61; 58,332; 59,60; 59,332; 60,61; 60,62;
61,62; 61,64; 62,65; 63,64; 68,182; 69,181; 70,177; 70,181; 71,72; 71,74; 72,73; 72,452;
73,74; 73,79; 74,75; 74,78; 75,76; 76,77; 76,78; 76,89; 77,90; 78,79; 78,88; 79,80; 79,87;
80,81; 80,452; 80,86; 81,82; 81,85; 81,453; 82,83; 82,92; 83,93; 83,84; 91,92; 91,453;
92,93; 92,95; 93,94; 93,107; 94,95; 94,106; 94,100; 95,96; 96,97; 96,172; 97,98; 97,100;
98,99; 98,101; 99,100; 99,102; 99,105; 100,454; 101,102; 101,108; 102,103; 103,105; 103,104;
108,171; 108,109; 109,110; 109,114; 110,111; 110,113; 111,112; 111,455; 112,113; 112,456;
113,114; 113,116; 114,115; 114,166; 115,116; 115,119; 116,117; 117,118; 117,457; 118,119;
118,121; 118,123; 119,120; 120,121; 121,122; 122,123; 124,123; 124,125; 124,127; 124,128;
125,126; 126,127; 126,156; 126,159; 127,128; 128,129; 128,130; 130,131; 131,132; 131,133;
131,134; 134,135; 134,136; 136,137; 136,138; 138,139; 138,141; 138,458; 139,140; 139,141;
140,142; 140,144; 141,142; 141,143; 144,145; 144,146; 144,147; 147,149; 147,152; 147,153;
148,149; 148,196; 149,150; 150,151; 150,193; 150,194; 151,152; 151,154; 151,189; 151,192;
153,154; 154,155; 155,156; 155,189; 156,157; 157,158; 157,451; 158,159; 158,161; 159,160;
159,162; 160,165; 160,166; 161,162; 161,184; 162,163; 162,165; 163,164; 163,168; 164,185;
165,167; 166,167; 166,171; 167,168; 167,170; 168,169; 169,170; 169,459; 170,171; 170,173;
171,172; 172,173; 172,174; 173,175; 173,459; 174,175; 174,177; 175,176; 175,178; 176,183;
176,459; 177,178; 177,179; 178,180; 179,180; 180,181; 181,182; 182,183; 184,185; 184,187;
185,186; 186,187; 186,205; 187,188; 187,204; 188,203; 188,451; 189,451; 189,190; 190,191;
190,201; 191,192; 191,204; 192,193; 192,199; 193,195; 194,195; 194,197; 195,198; 196,197;
197,460; 198,199; 198,401; 199,200; 200,201; 200,400; 201,202; 201,399; 202,203; 202,398;
202,451; 203,204; 203,397; 204,205; 204,396; 205,206; 205,395; 206,207; 206,394; 207,208;
207,392; 208,209; 208,210; 210,260; 210,211; 211,212; 211,256; 212,213; 212,217; 213,218;
214,215; 214,219; 215,216; 215,220; 216,222; 216,245; 216,254; 216,255; 217,218; 217,218;
217,223; 218,219; 218,224; 219,220; 220,221; 221,222; 221,227; 222,229; 223,224; 223,225;
224,226; 225,226; 225,230; 226,232; 226,227; 227,228; 227,233; 228,229; 228,243; 229,245;
230,232; 230,235; 231,232; 231,234; 231,236; 232,233; 233,234; 234,238; 235,236; 235,239;
236,237; 236,239; 237,238; 237,461; 238,240; 239,330; 239,461; 240,241; 240,329; 241,242;
241,243; 241,327; 242,244; 242,328; 243,244; 244,245; 245,246; 246,247; 246,462; 246,251;
247,248; 247,249; 248,249; 249,250; 250,251; 251,252; 252,253; 253,254; 253,257; 254,462;
255,256; 255,257; 256,257; 257,258; 258,259; 258,261; 258,263; 259,260; 260,383; 261,262;
261,273; 262,263; 262,272; 263,264; 264,265; 264,266; 265,267; 266,267; 266,269; 267,268;
268,269; 268,463; 269,270; 270,271; 270,293; 271,272; 271,274; 271,277; 272,273; 273,274;
274,275; 275,276; 275,277; 276,281; 276,387; 276,436; 277,278; 277,280; 278,279; 278,293;
278,284; 279,280; 279,284; 279,287; 281,283; 282,283; 282,464; 283,443; 284,285; 284,464;
285,286; 285,287; 286,287; 286,465; 287,288; 287,291; 289,290; 289,296; 289,297; 290,291;
290,379; 291,480; 291,380; 292,368; 292,382; 292,480; 293,294; 293,295; 294,295; 294,298;
295,296; 295,298; 296,297; 297,466; 297,378; 298,299; 298,304; 299,300; 299,302; 300,301;
300,466; 301,303; 301,377; 302,304; 302,316; 302,303; 303,373; 304,305; 305,306; 306,307;
```

```
306,311; 306,314; 307,308; 307,310; 308,309; 309,310; 310,311; 311,312; 312,313; 312,314;
313,315; 313,318; 314,315; 315,316; 315,317; 316,479; 316,371; 317,318; 317,353; 318,319;
318,353; 319,320; 319,352; 319,328; 320,321; 320,327; 320,351; 321,322; 321,467; 321,340;
322,468; 322,323; 323,324; 323,469; 324,325; 324,326; 325,326; 325,335; 326,327; 326,329;
327,328; 329,330; 330,331; 330,335; 331,332; 331,333; 333,334; 333,337; 333,337; 333,343;
334,337; 334,336; 335,336; 336,338; 336,469; 337,338; 337,470; 338,339; 338,467; 339,340;
339,342; 340,341; 341,342; 341,351; 341,345; 342,470; 342,344; 343,470; 344,345; 344,471;
344,472; 345,346; 346,347; 346,472; 346,473; 347,348; 347,351; 348,349; 348,352; 349,350;
349,353; 349,473; 350,351; 350,474; 351,352; 352,353; 353,354; 354,359; 355,356; 355,357;
355,474; 356,358; 356,363; 357,358; 357,359; 358,360; 359,360; 359,365; 360,361; 360,476;
360,370; 361,362; 361,364; 361,476; 361,486; 362,363; 362,364; 362,475; 364,487; 365,367;
365,366; 365,476; 366,367; 366,479; 367,368; 367,476; 367,477; 368,369; 368,477; 369,370;
369,476; 370,478; 371,372; 371,373; 372,375; 372,479; 373,374; 374,375; 374,377; 375,376;
376,381; 376,382; 376,477; 377,378; 377,466; 378,379; 378,381; 379,380; 380,381; 380,382;
382,480; 383,384; 383,385; 384,386; 384,387; 385,386; 385,391; 386,387; 387,388; 388,389;
388,391; 389,390; 389,435; 390,391; 390,393; 390,412; 391,392; 392,393; 393,394; 393,411;
394,395; 395,396; 396,397; 397,398; 398,408; 399,400; 399,402; 400,403; 400,401; 401,404;
402,403; 402,407; 403,404; 404,460; 405,406; 405,481; 405,421; 406,407; 406,419; 407,408;
407,418; 408,409; 408,416; 409,410; 409,415; 410,411; 410,413; 411,412; 412,413; 412,435;
413,414; 413,434; 414,415; 414,433; 415,416; 416,417; 417,418; 417,432; 418,419; 418,431;
419,420; 419,430; 420,421; 420,423; 421,422; 422,481; 422,482; 423,482; 423,483; 424,425;
424,429; 425,426; 425,428; 426,427; 426,483; 427,428; 427,485; 428,429; 429,430; 430,431;
431,432; 432,433; 433,434; 434,435; 436,437; 436,443; 437,438; 437,444; 438,439; 438,445;
439,440; 439,446; 440,441; 440,447; 441,442; 441,448; 442,450; 442,449; 450,485; 467,468;
468,469; 485,483; 485,484; 452,453];
EN=size(E,1);
Tmp=[128,130,.2,1.9; 128,130,.2,1.8; 128,130,.3,1.6; 128,130,.5,1.5; 208,209,3.1,2.8;
112,207,3.2,2.4; 113,206,3.1,2.3; 147,153,1.4,.8; 163,168,1.6,3.2; 452,453,1,6.5;
174,177,1.2,6.5; 175,178,1.4,6.4; 176,183,1.6,6.4; 19,20,2.2,6.3; 344,470,5,7;
248,249,4.8,4.7; 324,325,4.6,5.9; 293,295,6.1,4.5; 278,284,6.3,4.4; 285,287,6.4,4.5;
259,261,4.9,3.4; 261,272,5,3.4; 261,272,5.3,3.4; 211,212,3.7,3.7; 324,325,4.5,5.9;];
lmp=32/3/2.54; lrl=1000; vx=[L,B;R,B;R,T;L,T];
c{1,2}=[(VN+1),148,153,129]; c{2,2}=[148,460,402,189,126,129,153];
c{3,2}=[460,484,389,394,402]; c{4,2}=[402,394,13,15,126,189];
c{5,2}=[484,(VN+2),288,278,275,389]; c{6,2}=[394,389,275,210,10,13];
c{7,2}=[126,15,19,21,71,452,98,125]; c{8,2}=[129,126,125,98,105];
c{9,2}=[105,98,452,71,(VN+4)]; c{10,2}=[13,10,1,55,19,15];
c{11,2}=[71,21,23]; c{12,2}=[19,55,56,65,23,21];
c{13,2}=[55,331,343,65,56]; c{14,2}=[210,211,212,239,331,55,1,10];
c{15,2}=[211,257,308,331,239,212]; c{16,2}=[211,210,275,278,294,308,257];
c{17,2}=[294,278,288,(VN+3),474,315]; c{18,2}=[308,294,315,474,343,331];
cn=size(c,1);
% Brussels
clear all; twn='Brussels'; T=6; B=0; L=0; R=7;
V=[.7,2.2; .6,2.3; .8,2.4; 1,2.3; .9,2.1; 1.2,2.1; 1.1,1.9; 1.5,2.4; 1.2,2.6; 1.1,2.7;
 .8,3; 1.3,2.7; 1.2,2.8; .7,2.8; .5,2.6; .4,2.5; .4,2.2; .2,2.5; .3,2.7; .4,2.9;
 .2,3; .5,3.5; .7,3.3; .9,3.2; 1.4,3; 1.5,3.2; 1.2,3.5; 1.6,3.4; 1.4,3.8; 1.8,3.5;
 2.2,3.2; 1.7,2.8; 2.3,3; 2.7,2.8; 1.8,2.5; 2.1,2.5; 2.3,2.3; 2,2.3; 1.7,2.1; 2.1,2;
 1.9,1.6; 1.6,1.6; 1.7,1.4; 1.9,1; 2.6,1.5; 2.9,1.7; 2.8,1.8; 3.1,2;2.9,2; 2.4,2.1;
 2.8,2.2; 2.9,2.4; 3.2,2.5; 3.6,2.5; 4,2.3; 4.1,2.5; 3.8,2.7; 4.3,2.9; 4.5,3.1; 4.9,2.9;
 5,3.3; 4.6,3.2; 4.8,3.2; 4.4,3.6; 4.8,3.8; 4.4,4.1; 3.9,3.4; 4.1,3.2; 3.6,3; 3.4,3.3;
 3.3,2.8; 3,3.1; 3.6,3.6; 3.4,3.7; 3.2,3.8; 3,3.7; 2.8,3.8; 2.5,3.4; 2.3,3.6; 2.4,3.9;
 2,3.8; 2.1,4.1; 1.8,4.4; 1.6,4.1; 1.3,4.3; 1.2,4; 1,4; .7,3.8; .1,3.4; .1,3.7;
 .1,4.2; .3,4.3; .5,4.5; .7,4.5; 1.2,4.7; 1,4.8; .8,4.8; .3,4.9; .6,5.1; .5,5.5;
 .3,5.3; .1,5.5; .3,5.7; .6,5.6; .8,5.1; 1.3,5.4; 1.4,5.1; 1.5,5; 1.6,5.2; 1.4,4.7;
 1.3,4.4; 1.6,4.6; 2,4.7; 2.2,5.1; 2.2,5.4; 2.3,5.6; 2.1,5.7; 2,5.5; 1.9,5.4; 1.7,5.4;
 1.7,5.6; 1.8,5.9; 2.7,5.9; 2.6,5.7; 2.5,5.4; 2.7,5.6; 2.8,5.8;3.6,5.9; 3.8,5.8; 4.2,5.6;
 3.9,5.2; 3.7,5.3; 3.3,5.3; 3.2,5.4; 3,5.1; 3.2,5; 3.1,4.9; 2.9,5; 3,4.6; 2.8,4.3;
 3.3,4.1; 3.4,4.5; 3.6,4.7; 3.9,4.3; 4.1,4.1; 4.4,4.5; 4.3,4.6; 3.8,4.9; 4.4,5; 4.7,4.9;
 5.1,5.9; 5.8,5.7; 5.7,5.4; 5.4,4.7; 5.3,4.4; 5.2,4.5; 5.1,4.2; 5.3,4.2; 5.5,4.2; 5.6,4.6;
 6,4.5; 6.2,4.8; 6.2,5; 6.6,5; 6.4,4.7; 6.6,4.6; 6.4,4.4; 6.1,4.2; 6,4.2; 5.9,4.1;
 6.1,4; 5.6,3.8; 5.2,3.9; 5.6,3.5; 5.5,3.2; 5.7,3.2; 6.2,3.6; 6.2,3.3; 6.3,3.1; 6.5,3.4;
 6.8,3.5; 6.8,3.8; 6.5,3.7; 6.4,3.9; 6.4,4.1; 6.7,4.2; 6.9,3.1; 6.7,2.8; 6.6,2.4;6.1,2.5;
 5.9,1.9; 6.4,1.8; 6.2,1.2; 6.8,1.6; 6.6,1; 5.6,1.3; 4.4,1.3; 4.4,1.7;4.6,2.3; 4,1.9;
 4,1; 1.6,1.2; 2.1,1.1; 1,1.6; .9,1.3; .7,1.3; .2,1.1; .4,1.4; .3,1.5; .5,2;
 .3,2.1; .1,2.2; .7,1.9; .4,1.3; L,3; 1.6,.8; 1.8,.5; 1.5,.4; 1.4,.6; 1.9,.8;
 2.4,.6; 2.1,.6; 2.2,.5; 2.4,.4; 2.7,.5; 3.2,.6; 3.3,.1; 6,.7; 6.5,.6; 6.8,.5;
 .7,.5; .3,.6; .1,.3; 1.1,.4; 4.4,3.3; 3.2,3.5; L,3.5; L,4.5; L,4.7; L,5.6;
 .3,T; 1.5,5.7; 1.3,5.8; 1.4,T; .7,T; 2.5,T; 2.3,T; 1.8,T; 3.2,T; 3.6,T;
 4.4,T; 6.3,T; R,4.8; R,5.7; 6.3,2.8; R,3.9; R,4.3; R,3.1; R,3.1; R,2.6;
 R,2.3; R,1.6; 6.9,.9; R,.8; R,1.3; 5.5,.9; 4.1,B; L,.8; L,1.7; L,2.3;
 L,1.8; 1.3,B; 1.8,B; 2.1,B; 2.4,B; 2.8,B; 6.3,B; 6.6,B; R,.4; L,.7;
 L,.1; .9,B; L,.1; L,1.4; 3.4,B; 5.7,B; 6,T];
VN=size(V,1);
E=[1,2; 1,3; 1,5; 1,210; 2,3; 2,16; 2,17; 3,4; 3,10; 3,15; 4,5; 4,6; 4,9;5,7; 5,213; 6,7;
6,8; 6,42; 7,204; 8,9; 8,35; 8,39; 9,10; 9,12; 10,11; 10,13; 10,14; 11,14; 11,14; 11,24;
12,13; 12,25; 12,32; 14,15; 14,19; 14,20; 15,16; 16,17; 17,18; 17,211; 18,19; 18,215;
19,20; 20,21; 20,22; 20,23; 21,89; 21,215; 22,23; 22,27; 22,88; 22,89; 23,24; 24,25; 24,27;
25,26; 25,32; 26,27; 26,28; 27,28; 27,29; 27,87; 27,88; 28,30; 29,30; 29,84; 29,86;
30,31; 30,79; 30,81; 31,33; 31,78; 32,35; 33,34; 34,37; 34,51; 34,52; 34,53; 34,72;
35,36; 35,38; 36,37; 36,38; 37,40; 37,50; 38,39; 38,40; 39,40; 40,41; 40,45; 40,47; 40,50;
41,42; 41,43; 41,203; 42,43; 43,202; 43,44; 44,202; 44,203; 44,216; 44, 220; 45,46; 45,221;
```

```
46,47; 46,48;47,49; 48,49; 48,52; 48,53; 48,54; 49,51; 50,51; 51,52; 52,53; 53,54; 53,71;
54,57; 55,56; 55,57; 55,200; 56,57; 56,58; 56,199; 57,58; 57,68; 57,69; 57,71; 58,59; 58,68;
59,60; 59,62; 60,61; 60,190; 60,199; 61,63; 61,172; 62,173; 61,174; 61,175; 62,63; 62,235;
63,65; 64,65; 64,66; 64,67; 64,235; 65,66; 65,157; 66,146; 67,68; 67,69; 67,73; 69,70;
70,71; 70,72; 70,73; 70,236; 72,78; 73,74; 74,75; 74,236; 75,76; 75,141; 76,77; 76,236;
77,78; 77,80; 78,79; 79,80; 79,81;80,82; 80,140; 81,82; 81,84; 82,83; 83,84; 83,112; 83,113;
84,85; 85,86; 85,112; 86,87; 87,88; 87,93; 87,111; 88,89; 88,90; 88,92; 89,90; 89,215; 89,237;
90,91; 90,237; 91,92; 91,238; 92,93; 92,239; 93,94; 93,97; 93,98; 94,95; 94,97; 94,111;
95,96; 95,110; 95,111; 95,107; 96,97; 96,105; 97,99; 98,99; 98,101; 99,100;100,101; 100,103;
101,102; 102,103; 102,240; 103,104; 103,241; 104,105; 105,106; 106,107; 106,109; 106,242;
106,243; 106,245; 107,108; 108,109; 108,110; 108,113; 109,120; 110,111; 110,112; 113,114;
114,115; 114,125; 114,119; 115,116; 115,118; 116,117; 116,246; 117,118; 117,122; 117,247;
118,119; 118,121; 119,120; 120,121; 121,122; 122,248; 122,242; 123,124; 123,127; 124,125;
124,126; 125,126; 126,127; 127,249; 128,129; 128,133; 128,250; 129,130; 129,132; 130,131;
130,251; 130,149; 131,132; 131,148; 132,133; 133,134; 133,136; 134,135; 135,136; 135,138;
136,137; 136,143; 137,138; 137,139; 138,139; 139,140; 139,142; 140,141; 141,142; 142,143;
142,144; 143,148; 144,145; 144,147; 145,146; 146,147; 146,150; 147,148; 147,149; 149,150;
150,151; 151,152; 152,153; 152,164; 152,287; 153,154; 153,161; 153,163; 154,155; 154,156;
154,160; 155,156; 155,158; 156,157; 157,158; 158,159; 158,173; 159,160; 159,170; 160,161;
161,162; 161,169; 162,163; 162,165; 163,164; 163,252; 164,165; 164,166; 165,167; 166,167;
166,253; 166,254; 167,168; 167,185; 168,169; 168,171; 169,170; 170,171; 170,172; 171,177;
172,173; 172,174; 174,175; 175,176; 176,178; 177,178; 177,183; 178,179; 178,180; 179,180;
179,255; 180,183; 180,181; 181,182; 181,187; 182,183; 182,186; 182,256; 183,184; 184,185;
185,186; 186,257; 187,188; 187,258; 187,251; 188,189; 188,260; 189,190; 189,261; 190,191;
190,255; 191,192; 191,196; 192,193; 192,194; 193,195; 193,196; 193,228; 194,195; 194,261;
194,262; 195,229; 195,263; 196,266; 197,266; 197,201; 197,198; 198,199; 198,200; 200,201;
201,227; 201,267; 202,204; 204,205; 204,213; 205,206; 205,216; 205,231; 206,208; 206,232;
207,208; 207,209; 207,268; 207,284; 208,209; 208,213; 209,210; 209,269; 210,211; 211,212;
211,269; 212,270; 212,271; 218,219; 218,272; 218,217; 216,231; 216,219; 216,217; 217,220;
217,222; 217,273; 286,228; 286,266; 219,234; 220,221; 221,222; 221,224; 221,225; 221,226;
222,223; 223,224; 223,274; 224,225; 224,275; 225,226; 225,276; 225,227; 226,227; 227,285;
228,229; 229,230; 229,277; 230,278; 230,263; 230,279; 231,232; 231,233; 232,233; 232,268;
233,280; 233,281; 234,282; 234,283; 242,243; 243,244; 263,264; 263,265];
EN=size(E,1);
Tmp=[11,13,1,3; 11,13,1.2,2.9; 20,23,.6,3.1; 34,51,2.6,2.4; 34,53,3,2.6; 42,43,1.5,1.5;
45,221,2.7,1.3; 45,221,2.6,1.2; 45,221,2.5,1; 45,221,2.4,.7; 56,58,4.3,2.7; 63,65,4.8,3.6;
65,66,4.6,4; 72,78,2.7,3.3; 129,132,3.7,5.6; 130,131,4.1,5.3; 130,149,4.6,5.4;
136,143,3.3,4.9; 136,143,3.3,4.8; 138,139,2.7,4.7; 152,164,6.5,5.4; 154,156,5.3,4.7;
176,255,5.7,3; 179,180,6.4,3.2; 200,201,3.6,1.4; 201,267,4.1,.5; 221,226,2.9,.9;
226,227,3.3,.5];
lmp=20/(3*2.54); lrl=1000; vx=[L,B;R,B;R,T;L,T];
c{1,2}=[281,233,216,205,208,269,284,(VN+1)]; c{2,2}=[285,227,221,45,216,233];
c{3,2}=[227,197,199,57,54,46,45,221,227]; c{4,2}=[286,266,197,227,285];
c{5,2}=[264,193,286,(VN+2)]; c{6,2}=[261,189,193,264];
c{7,2}=[286,193,189,60,199,197,266]; c{8,2}=[216,45,46,47,40,9,11,14,269,208,205];
c{9,2}=[269,14,21,215]; c{10,2}=[14,11,83,110,96,92,90,21];
c{11,2}=[46,54,53,70,76,83,11,9,40,47]; c{12,2}=[199,60,61,63,65,66,146,70,53,54,57];
c{13,2}=[189,187,166,152,61,60]; c{14,2}=[253,166,187,189,261];
c{15,2}=[253,166,152,287,(VN+3)]; c{16,2}=[61,152,287,151,146,66,65,63];
c{17,2}=[70,146,151,249,83,76]; c{18,2}=[83,249,248,110]; c{19,2}=[110,248,241,96];
c{20,2}=[96,241,(VN+4),239,92]; c{21,2}=[92,239,215,21,90]; cn=size(c,1);
% Freiburg
clear all; twn='Freiburg'; T=5; B=0; L=0; R=6;
V=[4.2,1.9; 3.5,2.1; 2.3,2.9; 2.4,3.3; 2.8,4.5; 3.3,T; 2.4,4.7; 2.4,T; 2.1,4.8; 2.2,T;
1.6,4.9; 1.2,T; 1.9,1.7; .9,4.3; .1,4.7; .3,T; L,4.8; L,4.4; .7,3.7; .3,3.2;
.7,3.1; 1.2,3; 1.4,2.5; 1.5,2.9; 1.6,3.6; 2,3.4; 1.9,2.9; 1.5,2.3; 1.7,2; 1.9,2;
2.3,2; 2.5,1.3; 3.4,1.2; 4.1,1.1; 4.8,1.2; 5.4,1.4; 5.7,1.5; R,1.5; R,1.8; 5.7,1.7;
5.4,1.9; 5.6,2.2; 4.9,1.9; R,3.4; 5.8,1.1; 4.7,.7; 4.1,.7; 3.3,.7; 4.7,.1; 5.3,B;
5.7,B; R,1.1; 4,.2; 3.3,.2; 3.2,B; 3.9,B; 2.8,.7; 2.6,B; 2,B; 2.1,.6;
2.7,.4; 2.1,.8; 1.5,.6; 1.3,.4; 1.1,.7; 1.2,1.3; .8,B; .4,1.5; L,1.6; L,1.4;
L,1.8; .2,2.2; .6,1.8; 2.2,1.3; L,2.4];
VN=size(V,1);
E=[1,2; 1,34; 1,43; 2,3; 2,33; 3,4; 3,27; 3,31; 4,5; 4,26; 5,6; 5,7; 7,8; 7,9; 9,10; 9,11;
11,12; 11,14; 12,14; 13,29; 13,74; 13,75; 14,15; 14,19; 15,16; 15,17; 15,18; 19,20; 19,22;
19,25; 20,21; 20,75; 21,22; 22,23; 22,24; 23,24; 23,28; 24,25; 24,27; 25,26; 26,27; 28,29;
28,30; 29,30; 30,31; 31,32; 32,33; 32,74; 33,34; 33,48; 34,35; 34,47; 35,36; 35,46; 35,43;
36,37; 36,41; 36,45; 37,40; 37,38; 37,45; 40,41; 40,42; 41,42; 41,43; 42,44; 45,46; 45,51;
45,52; 46,47; 46,49; 47,53; 47,48; 48,54; 48,57; 49,50; 49,53; 53,54; 53,56; 54,55; 57,61;
57,62; 58,61; 59,60; 60,61; 60,62; 62,66; 62,63; 62,74; 63,64; 63,65; 64,65; 64,67; 65,66;
66,68; 68,69; 68,70; 71,72; 71,73; 72,73; 72,75;
73,74];
EN=size(E,1);
Tmp=[11,14,1.3,4.2; 12,14,1,4.6; 13,75,1.9,1.6; 13,76,1.8,1.5; 19,20,.5,3.6;  28,30,1.8,2.4;
31,32,3.2,1.9; 34,35,4.5,1.1; 36,41,5.3,1.7; 64,65,1,.5; 73,74,2,1.2];
lmp=2.9; lrl=500; vx=[L,B;R,B;R,T;L,T];
c{1,2}=[67,74,71,(VN+1)]; c{2,2}=[55,33,74,67]; c{3,2}=[52,36,33,55,(VN+2)];
c{4,2}=[44,42,41,36,52]; c{5,2}=[36,41,42,4,74,33]; c{6,2}=[74,4,14,75,71];
c{7,2}=[75,14,17]; c{8,2}=[17,14,12,(VN+4)]; c{9,2}=[12,14,4,6]; c{10,2}=[6,4,42,44,(VN+3)];
cn=size(c,1);
% Manchester
clear all; twn='Mancheter'; T=9; B=0; L=0; R=9;
V=[.7,.8; 1.9,.8; 1.9,.1; 2.6,.8; 2.6,.3;3.5,.3; 3.8,.3; 3.8,1; 3.9,.3; 4.1,.3;
```

```
 4.3,.3; 4,1; 5,.3; 5.3,.6; 5.1,1; 5.6,.1; 3.8,1.3; 3.5,1.5; 3.5,1.7; 3.8,1.8;
 4.8,1.5; 5.3,1.5; 5.4,1.3; 5.9,1; 6.3,.4; 6.8,.8; 6.6,1.4; 6.4,1.8; 6.2,1.9; 6.1,2;
 6.1,2.3; 6,2.4; 6.3,2.3; 6.4,2.2; 6.6,2.1; 6.3,2.6; 7.2,.2; 8.4,1; 8.6,.9; 8.8,1.1;
 8.7,1.2; 8.5,1.3; 8.1,2.4; 7.9,2.7; 7.5,2.6; 7.4,2.8; 7.7,3.1; 7.4,3.6; 8.1,3.6; 8.4,3.7;
 7.7,4.1; 7.3,3.8; 7.8,4.2; 8.8,4.3; 8.6,4.6; 8.3,4.9; 8,5.2; 7.7,4.8; 7.5,4.7; 7.3,4.6;
 7,4.4; 6.5,4; 6.5,3.8; 6.5,3.6; 5.9,3.2; 5.6,3.1; 5,2.7; 5.4,2; 4.4,2.3; 3.7,2.4;
 2.9,2.7; 2.1,2.2; 1.7,2.2; 1.8,2; 1.9,1.8; 2.1,1.9; 1.9,2.1; .7,1.3; .2,2.6; .5,3.6;
 .4,3.7; .6,3.9; .7,3.7; 1.2,3.6; 1.2,3.5; 2.1,3.3; 2.6,3.1; 3,3; 3.3,2.9; 3.8,2.8;
 4.7,3.1; 5.2,3.6; 5.5,3.3; 5.7,3.5; 6.1,3.8; 6.3,4; 6.1,4.5; 6,4.4; 5.8,4.2; 5.6,4;
 5.4,3.7; 5,3.7; 4.6,3.4; 4.3,3.4; 4.5,3.6; 4.8,3.9; 5.3,4.4; 5.4,4.2; 5.7,4.6; 5.5,4.7;
 5.3,4.8; 5,4.6; 4.6,4.2; 4.1,4.6; 4,4.3; 4,4; .7,B; 4.1,3.6; 4.1,3.5; 3.4,3.6;
 3.5,3.8; 3.2,3.5; 3.3,3.9; 2.8,4.1; 3.3,4.2; 3.4,4.5; 3.5,4.8; 3.6,5.2; 2.8,5.2; 2.8,5;
 2.5,5.1; 2.3,4.2; 1.9,4.3; 1.4,4.4; 1.5,4.7; 1.4,5.6; 2,5.6; 3.3,6.3; 2.5,7.1; 1.4,8.3;
 .3,5.6; 1.2,5.6; 3.5,7.7; 4.4,8.2; 4.8,7.4; 5.4,7.3; 6.3,7; 6.9,6.3; 7.1,6.2; 7.3,6;
 7.4,5.8; 7.1,5.5; 6.9,5.6; 6.7,5.7; 6.5,5.8; 6,6.2; 5.8,6.4; 5.3,6.8; 4.7,7.1; 4.6,6.7;
 4.4,7; 4,6.9; 3.9,6.7; 3.9,6.6; 3.6,6.6; 3.8,6.3; 3.7,5.9; 4.2,5; 4.7,4.9; 4.3,5.7;
 4.5,6.1; 4.6,6.4; 4.2,6.6; 4.2,6.2; 5.4,5.8; 5.1,6.3; 5.7,4.9; 6,5.3; 6.2,5.6; 6.5,5.5;
 6.3,5.2; 6.4,5.1; 6.6,5.1; 6.9,5.3; 6.7,5.4; L,.4; L,1; L,2.1; L,2.6; L,3.7;
 L,4.1; L,5.5; 1.3,T; 2.8,T; 4.6,T; 8.2,T; R,7.3; R,4.9; R,4.2; R,3.6;
 R,3.1; R,2.9; R,1.7; R,1.1; R,.1; 7.3,B; 6.8,B; 6.4,B; 5.6,B; 5.4,B;
 5.2,B; 4.4,B; 4.3,B; 4.1,B; 3.9,B; 3.4,B; 2.6,B; 1.9,B; 1.7,B; 5.5,6.2];
VN=size(V,1);
E=[1,2; 1,78; 1,117; 1,186; 2,3; 2,4; 2,75; 3,5; 3,218; 3,219; 4,5; 4,6; 5,6; 5,217; 6,7;
6,216; 7,8; 7,9; 7,215; 8,9; 8,17; 9,214; 10,11; 10,12; 10,213; 11,12; 11,13; 11,212; 12,17;
13,14; 13,211; 14,15; 14,16; 15,17; 15,21; 15,23; 16,25; 16,209; 16,210; 17,18; 17,20; 18,23;
18,76; 19,20; 19,22; 19,72; 20,21; 20,70; 21,22; 21,69; 22,30; 23,29; 24,25; 24,27; 24,67;
25,26; 25,208; 26,27; 26,37; 27,28; 27,43; 28,29; 28,31; 28,35; 29,35; 30,31; 30,34; 31,32;
31,33; 32,36; 32,66; 32,68; 33,34; 33,36; 34,46; 35,45; 36,65; 37,38; 37,206;37,207; 38,39;
38,42; 39,40; 39,205; 40,41; 40,204; 41,42; 41,203; 42,43; 43,44; 44,45; 44,47; 45,202;
46,47; 46,201; 47,48; 48,49; 48,64; 48,61; 49,50; 49,200; 50,51; 50,54; 51,52; 51,53; 53,55;
53,59; 54,55; 54,199; 55,56; 55,198; 56,57; 56,58; 57,58; 57,151; 58,59; 59,60; 60,61; 60,184;
61,62; 61,183; 62,63; 62,97; 63,64; 63,96; 64,65; 64,95; 65,66; 65,94; 66,67; 66,93; 67,68;
67,69; 67,91; 69,70; 70,90; 70,71; 71,72; 71,87; 71,88; 72,73; 72,74; 72,77; 73,74; 73,76;
73,77; 73,85; 74,75; 74,78; 75,76; 78,187; 79,80; 79,85; 79,188; 79,189; 80,81; 80,83; 81,82;
81,190; 82,83; 82,142; 83,84; 84,85; 84,86; 84,134; 86,87; 86,132; 87,88; 88,89; 88,122;
89,90; 89,120; 90,91; 90,104; 91,92; 91,103; 92,93; 92,101; 92,102; 93,94; 94,95; 94,101;
95,96; 95,99; 96,98; 97,98; 97,177; 98,99; 98,109; 99,100; 100,101; 100,108; 102,103; 102,106;
102,108; 103,105; 104,105; 104,119; 105,106; 105,118; 106,107; 106,113; 107,108; 107,110;
107,112; 108,109; 109,110; 110,111; 110,177; 111,112; 111,169; 111,175; 112,113; 112,169;
113,114; 114,115; 114,127; 114,168; 115,116; 115,126; 116,118; 116,125; 118,119; 118,121;
119,120; 120,121; 121,123; 122,123; 122,124; 123,124; 123,125; 124,125; 124,132; 125,126;
126,127; 127,128; 127,130; 128,129; 128,167; 129,130; 129,167; 130,131; 131,132; 131,137;
132,133; 133,134; 133,135; 134,135; 135,136; 136,137; 136,139; 136,142; 137,138; 138,139;
138,165; 138,167; 139,140; 139,143; 140,141; 140,193; 141,142; 141,192; 143,144; 143,162;
143,194; 144,145; 144,195; 145,146; 145,159; 146,147; 146,158; 147,148; 147,157; 147,196;
148,149; 148,155; 149,150; 149,154; 149,197; 150,151; 150,153; 151,152; 152,153; 152,184;
153,154; 153,185; 154,155; 154,180; 155,156; 155,179; 156,157; 156,175; 157,220; 158,159;
158,176; 158,220; 159,160; 160,161; 160,172; 161,162; 161,173; 162,163; 163,164; 163,165;
164,166; 164,173; 165,166; 166,167; 166,174; 167,170; 168,169; 168,170; 170,171; 171,172;
171,174; 171,175; 172,173; 172,176; 175,179; 176,220; 177,178; 178,179; 178,181; 179,180;
180,181; 180,185; 181,182; 182,183; 183,184; 184,185];
EN=size(E,1);
Tmp=[4,6,3.4,.7; 7,8,3.8,.7; 11,12,4.2,.8; 13,14,5,.5; 15,23,5.4,1.2; 17,18,3.5,1.4;
18,23,4.9,1.1; 19,20,3.5,1.8; 19,22,4.9,1.4; 21,22,5.2,1.6; 28,35,6.5,1.7; 28,35,6.6,1.8;
30,31,5.9,2.1; 38,42,8.4,1.1; 44,45,7.6,2.6; 45,202,7.8,2.8; 46,47,7.4,2.9; 46,201,7.8,3;
50,51,7.8,3.8; 56,58,7.8,4.6; 57,58,7.6,5; 71,87,2.2,2.8; 71,87,2.4,3; 71,87,2.5,3;
72,73,1.9,2.3; 82,142,.9,5.4; 82,142,1.2,5.4; 90,91,4.3,2.6; 116,118,4.2,3.9; 122,124,2.7,3.6;
124,125,2.9,4.3; 128,129,3.2,5.3; 129,167,2.9,5.5; 131,137,2.4,5.2; 135,136,1.4,4.9;
136,139,1.4,6.4; 140,141,.3,7.1; 158,220,5.4,6.7; 161,162,4.3,7.1; 161,173,4.3,6.9];
lmp=3; lrl=1000; vx=[L,B;R,B;R,T;L,T];
c{1,2}=[218,72,85,189,(VN+1)]; c{2,2}=[218,72,17,213]; c{3,2}=[213,17,15,209];
c{4,2}=[209,15,29,206]; c{5,2}=[206,29,44,205]; c{6,2}=[205,44,202,(VN+2)];
c{7,2}=[202,44,48,49,54,199]; c{8,2}=[29,44,48,64,66]; c{9,2}=[15,29,66,69];
c{10,2}=[72,17,15,69,71]; c{11,2}=[85,72,71,123,133,135]; c{12,2}=[189,85,135,136,192];
c{13,2}=[71,69,66,106,105,123]; c{14,2}=[66,64,62,97,177,106]; c{15,2}=[64,48,49,54,150,97,62];
c{16,2}=[199,54,150,147,196,(VN+3)]; c{17,2}=[177,97,150,147,220,175,111];
c{18,2}=[123,105,106,177,111,175,166,167]; c{19,2}=[136,135,133,123,167,139];
c{20,2}=[192,136,139,140,193,(VN+4)]; c{21,2}=[193,140,139,143,194];
c{22,2}=[139,167,166,162,143]; c{23,2}=[143,162,166,175,220,147,145,144];
c{24,2}=[194,143,144,195]; c{25,2}=[195,144,145,147,196];
cn=size(c,1);
% road percolation
TmpN=size(Tmp,1); Tn=sparse(VN,VN); Tnp=[];
for i=1:TmpN,
  Tn(Tmp(i,1),Tmp(i,2))=0; Tnp{Tmp(i,1),Tmp(i,2)}=[];
end
for i=1:TmpN,
  Tn(Tmp(i,1),Tmp(i,2))=Tn(Tmp(i,1),Tmp(i,2))+1;
  Tnp{Tmp(i,1),Tmp(i,2)}=[Tnp{Tmp(i,1),Tmp(i,2)};Tmp(i,3:4)];
end
clf; hold on;
for i=1:EN,
  Tmp=[V(E(i,1),:)];
  if(Tn(E(i,1),E(i,2)))
    for j=1:Tn(E(i,1),E(i,2)),
      Tmp=[Tmp;Tnp{E(i,1),E(i,2)}(j,:)];
    end
  end
  Tmp=[Tmp;V(E(i,2),:)]; TmpN=size(Tmp,1);
  for j=1:(TmpN-1),
    plot([Tmp(j,1),Tmp((j+1),1)],[Tmp(j,2),Tmp((j+1),2)]);
  end
end
plot([L,R,R,L,L],[B,B,T,T,B]); tma=B-(T-B)/20;
plot([L,(L+lmp)],[tma,tma],'LineWidth',1.5);
tmb=strcat(num2str(lrl),' metres'); text((L+lmp+abs(tma/2)),tma,tmb);
axis equal; axis off; title(twn,'FontSize',15);
% vertices
NeVMat=sparse(VN,VN); LVMat=sparse(1,VN); UVMat=sparse(1,VN); bry=[]; EV=[];
for i=1:VN,
  EV{i,2}=[];
end
for i=1:EN,
  NeVMat(E(i,1),E(i,2))=1; NeVMat(E(i,2),E(i,1))=1;
  EV{E(i,1),2}=[EV{E(i,1),2},i]; EV{E(i,2),2}=[EV{E(i,2),2},i];
  if(V(E(i,1),1)<=L)
    LVMat(1,E(i,1))=1; bry=[bry,E(i,1)];
  end
  if(V(E(i,2),1)<=L)
    LVMat(1,E(i,2))=1; bry=[bry,E(i,2)];
  end
  if(V(E(i,1),1)>=R)
    UVMat(1,E(i,1))=1; bry=[bry,E(i,1)];
  end
  if(V(E(i,2),1)>=R)
    UVMat(1,E(i,2))=1; bry=[bry,E(i,2)];
  end
end
for i=1:VN,
  EV{i,1}=size(EV{i,2},2);
end
A=V; N=size(A,1); LMat=LVMat; UMat=UVMat; NeMat=NeVMat;
[pc,cord,tsries]=perc(N,LMat,UMat,NeMat);
% edges
NeEMat=sparse(EN,EN);
for i=1:VN,
  for j=1:(EV{i,1}-1),
    for k=(j+1):EV{i,1},
      NeEMat(EV{i,2}(j),EV{i,2}(k))=1; NeEMat(EV{i,2}(k),EV{i,2}(j))=1;
    end
  end
end
LEMat=sparse(1,EN); UEMat=sparse(1,EN);
for i=1:VN,
  if(LVMat(i))
    for j=1:EV{i,1},
      LEMat(EV{i,2}(j))=1;
    end
  end
  if(UVMat(i))
    for j=1:EV{i,1},
      UEMat(EV{i,2}(j))=1;
    end
  end
end
A=E; N=size(A,1); LMat=LEMat; UMat=UEMat; NeMat=NeEMat;
[pc,cord,tsries]=perc(N,LMat,UMat,NeMat);
% cells
v=[V;vx]; vn=size(v,1); lcm=sparse(1,cn); ucm=sparse(1,cn);
for i=1:cn,
  c{i,1}=size(c{i,2},2); tmp=ones(1,c{i,1}); c{i,3}=sparse(tmp,c{i,2},tmp,1,vn);
  for j=1:c{i,1},
    tma=c{i,2}(j);
    if(v(tma,1)>=R)
      ucm(i)=1;
    end
    if(v(tma,1)<=L)
      lcm(i)=1;
    end
  end
end
b=[]; ncm=sparse(cn,cn);
for i=1:(cn-1),
  for j=(i+1):cn,
    tmn=sum(c{i,3} & c{j,3});
    if(tmn>1)
      b=[b;i,j]; ncm(i,j)=1; ncm(j,i)=1;
    end
  end
end
bn=size(b,1); A=c; N=size(A,1); LMat=lcm; UMat=ucm; NeMat=ncm;
[pc,cord,tsries]=perc(N,LMat,UMat,NeMat);
% bonds
lbm=sparse(1,bn); ubm=sparse(1,bn);
for i=1:bn,
  if(lcm(b(i,1)) | lcm(b(i,2)))
    lbm(i)=1;
  end
  if(ucm(b(i,1)) | ucm(b(i,2)))
    ubm(i)=1;
  end
end
tmp=sparse(bn,cn);
for i=1:bn,
  tmp(i,b(i,1))=1; tmp(i,b(i,2))=1;
end
nbm=sparse(bn,bn);
for i=1:(bn-1),
  for j=(i+1):bn,
    tmn=sum(tmp(i,:) & tmp(j,:));
    if(tmn)
      nbm(i,j)=1; nbm(j,i)=1;
    end
  end
end
A=b; N=size(A,1); LMat=lbm; UMat=ubm; NeMat=nbm;
[pc,cord,tsries]=perc(N,LMat,UMat,NeMat);
% plot max clusters
tma=zeros(1,N); tmb=zeros(1,N);
for i=1:N,
  for j=1:2,
    tsries{2,j}=zeros(1,N); tmn=size(tsries{1,j}{i,1},1);
    tmp=[];
    for k=1:tmn,
      tmq=[tsries{1,j}{i,1}{k,1}]; tmp=[tmp,tmq];
    end
    switch j
      case 1
        tma(i)=max(tmp);
      case 2
        tmb(N-i+1)=max(tmp);
    end
  end
end
clf; tmp=1:N; tmp=tmp/N; axes('FontSize',13);
plot(tmp,tma,'LineWidth',2); hold on; plot(tmp,tmb);
axis square; axis([0,1,0,N]); xlabel('p','FontSize',15);
ylabel('Size of the largest cluster','FontSize',15);
% plot area
clf; hold on;
for i=1:cn,
  tmp=[]; tma=c{i,2};
  for j=1:c{i,1},
    tmp=[tmp;v(tma(j),:)];
  end
  tmp=[tmp;tmp(1,:)]; plot(tmp(:,1),tmp(:,2));
end
plot([L,R,R,L,L],[B,B,T,T,B]); tma=B-(T-B)/20;
plot([L,(L+lmp)],[tma,tma],'LineWidth',1.5);
tmb=strcat(num2str(lrl),' metres'); text((L+lmp+abs(tma/2)),tma,tmb);
axis equal; axis off; twm=strcat(twn,' (fire control area)');
title(twm,'FontSize',15);
```

§ A.28 Volume, surface area, cell- and face perimeters

```
% vareac.m, vareab.m transformed, (c) Kit Tiyapan, 8th December 2002
clear all; rand('state',sum(100*clock)); cnm=100; cell=rand(cnm,3);
[vtc,tmp]=voronoin(cell); vtc(1,:)=[9,9,9]; vtc=-.5+vtc; vtcn=size(vtc,1); vca=[];
for i=1:cnm,
  vca{i,2}=tmp{i}; vca{i,1}=size(tmp{i},2);
end
vfrm=ones(1,vtcn);
for i=1:vtcn,
  if(max(abs(vtc(i,:))) > 0.5)
```

```
    vfrm(i)=0;
  end
end
frm =ones(cnm, 1);
for i=1:cnm,
  tmp=1;
  for j=1:vca{i,1},
    tma=vca{i,2}(j);
    if(~vfrm(tma))
      tmp=0;
    end
  end
  if(~tmp)
    frm(i)=0;
  end
end
vc=[]; cnt=0;
for i=1:cnm,
  if(frm(i))
    cnt=cnt+1; vc{cnt,2}=vca{i,2}; vc{cnt,1}=vca{i,1}; vca{i,3}=cnt;
  else
    vca{i,3}=0;
  end
end
cn=size(vc,1);
% cell volume
tmh=[]; tmk=[]; tmu=0;
for xpx=1:-.1:.1,
  tmu=tmu+1; tmv=0; tmk=[];
  for ypx=1:-.1:.1,
    tmz=[xpx*vtc(:,1),ypx*vtc(:,2),vtc(:,3)]; tmv=tmv+1; tms=[];
    for i=1:cn,
      tmw=0; tmp=[];
      for j=1:vc{i,1},
        tmp=[tmp;tmz(vc{i,2}(j),:)];
      end
      tmd=delaunayn(tmp); tmn=size(tmd,1);
      for j=1:tmn,
        tmq=[];
        for k=1:4,
          tma=tmp(tmd(j,k),:); tmq=[tmq;tma];
        end
        tmw=tmw+(abs(det([tmq,ones(4,1)])))/6;
      end
      vc{i,3}=tmw; tms=[tms,tmw];
    end
    tmk=[tmk,ypx]; vmn(tmu,tmv)=mean(tms); vsd(tmu,tmv)=std(tms);
  end
  tmh=[tmh,xpx];
end
vmn=vmn/vmn(1,1); vsd=vsd/vsd(1,1);
[tmc,tmd]=contour(tmh,tmk,vmn,10);
[tmc,tmd]=contour(tmh,tmk,vsd,10);
% surface area
tmh=[]; tmk=[]; tmu=0;
for xpx=1:-.1:.1,
  tmu=tmu+1; tmv=0; tmk=[];
  for ypx=1:-.1:.1,
    tmz=[xpx*vtc(:,1),ypx*vtc(:,2),vtc(:,3)]; tmv=tmv+1; tmi=[];
    for i=1:cn,
      tmp=[];
      for j=1:vc{i,1},
        tmp=[tmp;tmz(vc{i,2}(j),:)];
      end
      tmq=convhulln(tmp); tmn=size(tmq,1); tmr=0;
      for j=1:tmn,
        tmx=[];
        for k=1:3,
          tmx=[tmx;tmp(tmq(j,k),:)];
        end
        tma=sum((tmx(1,:)-tmx(2,:)).^2).^0.5; tmb=sum((tmx(1,:)-tmx(3,:)).^2).^0.5;
        tmc=sum((tmx(2,:)-tmx(3,:)).^2).^0.5; tms=(tma+tmb+tmc)/2;
        tmr=tmr+sqrt(tms*(tms-tma)*(tms-tmb)*(tms-tmc));
      end
      vc{i,5}=tmr; tmi=[tmi,tmr];
    end
    tmk=[tmk,ypx]; mnm(tmu,tmv)=mean(tmi); sdm(tmu,tmv)=std(tmi);
  end
  tmh=[tmh,xpx];
end
mnm=mnm/mnm(1,1); sdm=sdm/sdm(1,1);
[tmc,tmd]=contour(tmh,tmk,mnm,10);
```

```
[tmc,tmd]=contour(tmh,tmk,sdm,10);
% perimeter
tmx=[]; tmu=0;
for xpx=1:-.1:.1,
  tmu=tmu+1; tmv=0; tmy=[];
  for ypx=1:-.1:.1,
    tmV=[xpx*vtc(:,1),ypx*vtc(:,2),vtc(:,3)]; tmv=tmv+1;
    tmj=[]; tmw=[]; tmo=ones(3,1);
    for i=1:cn,
      tmp=[];
      for j=1:vc{i,1},
        tma=tmV(vc{i,2}(j),:); tmb=vc{i,2}(j); tmp=[tmp;tma,tmb];
      end
      tmh=convhulln(tmp(:,1:3)); tmn=size(tmh,1); tmk=[];
      for j=1:tmn,
        tmq=[];
        for k=1:3,
          tma=tmp(tmh(j,k),1:3); tmq=[tmq;tma];
        end
        tma=det([tmo,tmq(:,2),tmq(:,3)]); tmb=det([tmq(:,1),tmo,tmq(:,3)]);
        tmc=det([tmq(:,1),tmq(:,2),tmo]); tmd=det(tmq);
        tmk=[tmk;tma/tmd,tmb/tmd,tmc/tmd];
      end
      tml=ones(tmn); tmg=[]; cnt=0;
      for j=1:(tmn-1),
        if(tml(j))
          cnt=cnt+1; tmg{cnt,2}=j;
          for k=(j+1):tmn,
            if(tml(k))
              tmd=abs(tmk(j,:)-tmk(k,:)); tme=1e-6;
              if((tmd(1)<tme) & (tmd(2)) & (tmd(3)))
                tmg{cnt,2}=[tmg{cnt,2},k]; tml(k)=0;
              end
            end
          end
        end
      end
      tmn=size(tmg,1);
      for j=1:tmn,
        tmg{j,1}=size(tmg{j,2});
      end
      tmf=[];
      for j=1:tmn,
        tma=sparse(vtcn,vtcn);
        tmz=[];
        for k=1:tmg{j,1},
          tmb=[];
          for l=1:3,
            tmi=tmh(tmg{j,2}(k),l); tmc=tmp(tmi,4); tmb=[tmb,tmc];
          end
          tmb=sort(tmb); tma(tmb(1),tmb(2))=tma(tmb(1),tmb(2))+1;
          tma(tmb(1),tmb(3))=tma(tmb(1),tmb(3))+1;
          tma(tmb(2),tmb(3))=tma(tmb(2),tmb(3))+1;
        end
        [tmb,tmc,tmd]=find(tma); tmm=max(size(tmb));
        for k=1:tmm,
          if(~(1-tmd(k)))
            tmz=[tmz;tmb(k),tmc(k)];
          end
        end
        tmm=size(tmz);
        tmt=0;
        for k=1:tmm,
          tmq=tmV(tmz(k,1),:); tmr=tmV(tmz(k,2),:);
          tms=sum((tmq-tmr).^2).^0.5; tmt=tmt+tms;
        end
        tmf=[tmf,tmt];
      end
      tma=sum(tmf)/2; tmb=mean(tmf); vc{i,4}=tma; vc{i,6}=tmb; tmj=[tmj,tma]; tmw=[tmw,tmb];
    end
    tmy=[tmy,ypx]; prm(tmu,tmv)=mean(tmj); prs(tmu,tmv)=std(tmj);
    pfm(tmu,tmv)=mean(tmw); pfs(tmu,tmv)=std(tmw);
  end
  tmx=[tmx,xpx];
end
prm=prm/prm(1,1); prs=prs/prs(1,1);
[tmc,tmd]=contour(tmx,tmy,prm,10);
[tmc,tmd]=contour(tmx,tmy,prs,5);
[tmc,tmd]=contour(tmx,tmy,pfm,10);
[tmc,tmd]=contour(tmx,tmy,pfs,6);
% area of face
tmx=[]; tmu=0;
```

```
for XCompression=1:-.1:.1,
  tmu=tmu+1; tmv=0; tmy=[];
  for YCompression=1:-.1:.1,
    tmV=[XCompression*Vertices(:,1),YCompression*Vertices(:,2),Vertices(:,3)];
    tmv=tmv+1; tmo=ones(3,1); afm=[]; afs=[];
    for i=1:cn,
      tmp=[];
      for j=1:vc{i,1},
        tma=tmV(vc{i,2}(j),:); tmb=vc{i,2}(j); tmp=[tmp;tma,tmb];
      end
      tmh=convhulln(tmp(:,1:3)); tmn=size(tmh,1); tmk=[];
      for j=1:tmn,
        tmq=[];
        for k=1:3,
          tma=tmp(tmh(j,k),1:3); tmq=[tmq;tma];
        end
        tma=det([tmo,tmq(:,2),tmq(:,3)]); tmb=det([tmq(:,1),tmo,tmq(:,3)]);
        tmc=det([tmq(:,1),tmq(:,2),tmo]); tmd=det(tmq); tmk=[tmk;tma/tmd,tmb/tmd,tmc/tmd];
      end
      tml=ones(tmn); tmg=[]; cnt=0;
      for j=1:(tmn-1),
        if(tml(j))
          cnt=cnt+1; tmg{cnt,2}=j;
          for k=(j+1):tmn,
            if(tml(k))
              tmd=abs(tmk(j,:)-tmk(k,:)); tme=1e-6;
              if((tmd(1)<tme) & (tmd(2)) & (tmd(3)))
                tmg{cnt,2}=[tmg{cnt,2},k]; tml(k)=0;
              end
            end
          end
        end
      end
      tmn=size(tmg,1);
      for j=1:tmn,
        tmg{j,1}=size(tmg{j,2});
      end
      tmb=[];
      for j=1:tmn,
        tma=0;
        for k=1:tmg{j,1},
          tmc=[];
          for l=1:3,
            tmi=tmh(tmg{j,2}(k),l); tmd=tmp(tmi,1:3); tmc=[tmc;tmd];
          end
          tmd=sum((tmc(1,:)-tmc(2,:)).^2).^0.5; tme=sum((tmc(1,:)-tmc(3,:)).^2).^0.5;
          tmf=sum((tmc(2,:)-tmc(3,:)).^2).^0.5; tms=(tmd+tme+tmf)/2;
          tmj=sqrt(tms*(tms-tmd)*(tms-tme)*(tms-tmf)); tma=tma+tmj;
        end
        tmb=[tmb,tma];
      end
      tma=mean(tmb); tmc=std(tmb); afm=[afm,tma]; afs=[afs,tmc];
    end
    afmm(tmu,tmv)=mean(afm); afms(tmu,tmv)=std(afm); afsm(tmu,tmv)=mean(afs);
    afss(tmu,tmv)=std(afs); tmy=[tmy,YCompression];
  end
  tmx=[tmx,XCompression];
end
afmmn=afmm/afmm(1,1); afmsn=afms/afms(1,1); afsmn=afsm/afsm(1,1); afssn=afss/afss(1,1);
[tmc,tmd]=contour(tmx,tmy,afmmn,10);
[tmc,tmd]=contour(tmx,tmy,afmsn,10);
[tmc,tmd]=contour(tmx,tmy,afsmn,10);
[tmc,tmd]=contour(tmx,tmy,afssn,10);
```

```
% varea.m      by Kittisak N. Tiyapan, 26th April, 2001
echo off; clear all; format short g; more off;
path(path,'/home/mjkvjkt/vn');
for xcp=1:-.1:.1,
  for ycp=1:-.1:.1,
    pt1 =fopen('/home/mjkvjkt/vn/vertices.dat','r');
    sc1 =fscanf(pt1, '%d', 4); dim =sc1(1,1); vnum =sc1(2,1);
    cnum =sc1(3,1); sc2 =fscanf(pt1, '%f', [dim, vnum]);
    vtc =sc2'; voc =sparse(cnum, vnum); frm =ones(cnum, 1); vfrm=ones(vnum,1);
    for i=1:cnum,
      sc1 =fscanf(pt1, '%d', 1);
      for j=1:sc1,
        sc2 =fscanf(pt1, '%d', 1); Num =sc2+1; voc(i,Num) =1;
        if ( max(abs(vtc(Num, :))) > 0.5 )
          frm(i,1) =0; vfrm(Num,1)=0;
        end
      end
    end
```

```
vtc =([xcp*ones(vnum,1),ycp*ones(vnum,1),ones(vnum,1)]).*vtc;
fcnm =sum(frm); vpc =full(sum(voc, 2)); [a,b,vpcn] =find(vpc.*frm);
vnm =sparse(cnum, cnum); fnm =sparse(cnum, cnum);
for i=1:(cnum-1),
  for j=(i+1):cnum,
    nmsh =sum(and(voc(i,:), voc(j,:)), 2);
    if (nmsh >= 3)
      fnm(i,j) =1; fnm(j,i) =1;
    end
    if (nmsh >= 1)
      vnm(i,j) =1;  vnm(j,i) =1;
    end
  end
end
vnpc =full(sum(vnm,2)); [a,b,vncn] =find(vnpc.*frm); fnpc =full(sum(fnm,2));
[a,b,fncn] =find(fnpc.*frm); [c1of, c1of] =find(fnm); cofn =size(c1of,1);
fn =cofn/2; vof =sparse(fn, vnum); ffrm =zeros(fn,1); xcof =zeros(fn,1); fcnt =0;
for i=1:cofn,
  if (c1of(i,1) < c1of(i,1))
    fcnt =fcnt+1; vof(fcnt,:) =and( voc(c1of(i,1),:), voc(c1of(i,1),:) );
    xcof(fcnt,1) =i;
    if (frm(c1of(i,1),1)==1 | frm(c1of(i,1),1)==1)
      ffrm(fcnt,1)=1;
    end
  end
end
vpf=sum(vof,2); [a,b,vfin]=find(vpf.*ffrm); fcmx =max(fnpc);
odvf =sparse(fcnt,fcmx); fdfm =find(frm); vofn=[];
aoc =sparse(cnum,cnum); aof =sparse(fcnt,1); fppr =[];
for i=1:fcnt,
  vthf =find(vof(i,:)); vthfn =size(vthf,2); vofn=[vofn;vthfn]; vdthf =[];
  for j=1:vthfn,
    vdthf =[vdthf; [vtc(vthf(1,j),:),vthf(1,j)]];
  end
  vspn =size(vdthf,1);
  if (vdthf(1,1)<-10 | vspn==3)
    odvc =vdthf;
  else
    X1=vdthf(1,1); X2=vdthf(2,1); X3=vdthf(3,1); Y1=vdthf(1,2);
    Y2=vdthf(2,2); Y3=vdthf(3,2); Z1=vdthf(1,3); Z2=vdthf(2,3); Z3=vdthf(3,3);
    A=det([1,Y1,Z1;1,Y2,Z2;1,Y3,Z3]); B=det([X1,1,Z1;X2,1,Z2;X3,1,Z3]);
    C=det([X1,Y1,1;X2,Y2,1;X3,Y3,1]); D= -det([X1,Y1,Z1;X2,Y2,Z2;X3,Y3,Z3]);
    FarX=10; FarY=10; FarZ= (-A*FarX-B*FarY-D)/C;
    dX1=X1-FarX; dY1=Y1-FarY; dZ1=Z1-FarZ; V1=[dX1;dY1;dZ1];
    D1=sqrt(dX1*dX1+dY1*dY1+dZ1*dZ1); dist=[D1]; gMax=0;
    JgMax=0; VJs=[dX1;dY1;dZ1];
    for j=2:vspn,
      dXj=vdthf(j,1)-FarX; dYj=vdthf(j,2)-FarY; dZj=vdthf(j,3)-FarZ;
      Vj=[dXj;dYj;dZj]; VJs=[VJs,Vj]; dist=[dist;sqrt(dXj*dXj+dYj*dYj+dZj*dZj)];
      Dj=dist(j,1); gJ=acos((dot(V1,Vj))/(D1*Dj));
      if(gJ>gMax)
        JgMax=j; gMax=gJ;
      end
    end
    g=[]; gOppPt=0; jgop=0; Vr=VJs(:,JgMax);
    for j=1:vspn,
      Vj=VJs(:,j); Dj=dist(j,1); Dr=dist(JgMax,1);
      gJ=acos((dot(Vj,Vr))/(Dr*Dj)); g=[g;gJ];
      if(gJ>gOppPt)
        jgop=j; gOppPt=gJ;
      end
    end
    ddagl=zeros(vspn,1);
    for j=1:vspn,
      if(j==JgMax | j==jgop)
        ddagl(j,1)=dist(j,1);
      else
        gRatio=g(j,1)/gOppPt; Xr=vdthf(JgMax,1); Yr=vdthf(JgMax,2);
        Zr=vdthf(JgMax,3); XOpp=vdthf(jgop,1); YOpp=vdthf(jgop,2);
        ZOpp=vdthf(jgop,3); Xdl=Xr+gRatio*(XOpp-Xr); Ydl=Yr+gRatio*(YOpp-Yr);
        Zdl=Zr+gRatio*(ZOpp-Zr); dXdl=Xdl-FarX; dYdl=Ydl-FarY; dZdl=Zdl-FarZ;
        ddagl(j,1)=sqrt(dXdl*dXdl+dYdl*dYdl+dZdl*dZdl);
      end
    end
    bck1=[]; bck2=[];
    for j=1:vspn,
      if (dist(j,1)>ddagl(j,1))
        bck1=[bck1;[vdthf(j,:),g(j,1)]];
      elseif (dist(j,1)<ddagl(j,1))
        bck2=[bck2;[vdthf(j,:),g(j,1)]];
      end
    end
```

```
      if(~isempty(bck1))
        sdbck=sortrows(bck1,5); odvc=[vdthf(JgMax,:);sdbck(:,1:4);vdthf(jgop,:)];
      else
        odvc=[vdthf(JgMax,:);vdthf(jgop,:)];
      end
      if(~isempty(bck2))
        sdbck=sortrows(bck2,5);
        for j=size(sdbck,1):-1:1,
          odvc=[odvc;sdbck(j,1:4)];
        end
      end
    end
    odvdthf=odvc; odVnl =odvdthf(:,4); odvf(i,1:vthfn) =odVnl';
    odVcd=odvdthf(:,1:3); vthfn =size(odVcd,1); stp =floor(vthfn/3);
    ndpt =1+stp; rdpt =1+2*stp;
    nvec = cross((odVcd(ndpt,:)-odVcd(1,:)),(odVcd(rdpt,:)-odVcd(1,:)));
    clstv =[odVcd; odVcd(1,:)]; Xsq =nvec(1,1)*nvec(1,1); Ysq =nvec(1,2)*nvec(1,2);
    Zsq =nvec(1,3)*nvec(1,3); nvecn =sqrt(Xsq+Ysq+Zsq); nizednV =nvec/nvecn; xps =0;
    for j=1:vthfn,
      xps =xps+cross(clstv(j,:),clstv((j+1),:));
    end
    fra =(abs(dot(nizednV, xps)))/2; nvc =ones(3,1); cdmtx =[odVcd(1,:); odVcd(ndpt,:);
    odVcd(rdpt,:)]; Aprm =det([nvc, cdmtx(:,2), cdmtx(:,3)]);
    Bprm =det([cdmtx(:,1),  nvc, cdmtx(:,3)]); Cprm =det([cdmtx(:,1), cdmtx(:,2), nvc]);
    Dprm =det([cdmtx(:,1), cdmtx(:,2), cdmtx(:,3)]); pprm=[Aprm, Bprm, Cprm, Dprm];
    athf=fra; aof(i,1) =athf; fppr =[fppr; pprm];
    aoc(c1of(xcof(i),1),c1of(xcof(i),1))=athf; aoc(c1of(xcof(i),1), c1of(xcof(i),1))=athf;
  end
  facn=aof.*ffrm; [fn,b,afn]=fnd(facn); mafn=mean(afn); mnafn=afn/mafn;
  [a,b,sacin]=fnd(frm.*sum(aoc,2)); msacin=mean(sacin); mnsacin=sacin/msacin;
  cdsAndN =zeros(cnum,4); [cVect,VerticeVect] =fnd(voc); CVPairsAmount =size(cVect);
  mdc =[];
  for i=1:CVPairsAmount,
    cdsAndN(cVect(i),:)=[cdsAndN(cVect(i),1:3)+vtc(cVect(i),:),cdsAndN(cVect(i),4)+1];
  end
  for i=1:cnum,
    mdc=[mdoc;[cdsAndN(i,1),cdsAndN(i,2),cdsAndN(i,3)]/cdsAndN(i,4)];
  end
  Vc =zeros(cnum,1);
  for i=1:fcnt,
    if (ffrm(i,1)==1)
      A =fppr(i,1); B =fppr(i,2); C =fppr(i,3); D =fppr(i,4);
      Denom =sqrt(A*A+B*B+C*C); c1 =c1of(xcof(i),1);
      if (frm(c1,1)==1)
        Xp1 =mdc(c1,1); Yp1 =mdc(c1,2); Zp1 =mdc(c1,3);
        H1 =(A*Xp1+B*Yp1+C*Zp1+D)/Denom; V1 =abs((aof(i,1)*H1)/3); Vc(c1) =Vc(c1)+V1;
      end
      c2 =c1of(xcof(i),1);
      if (frm(c2,1)==1)
        Xp2 =mdc(c2,1); Yp2 =mdc(c2,2); Zp2 =mdc(c2,3);
        H2 =(A*Xp2+B*Yp2+C*Zp2+D)/Denom; V2 =abs((aof(i,1)*H2)/3); Vc(c2) =Vc(c2)+V2;
      end
    end
  end
  sVc =sum(Vc,2); Vct=sum(sVc); Vccb=Vct/fcnm; sdccb=Vccb.^(1/3);
  sdaccb=sdccb*sdccb; saccb=sdaccb*6; Facepccb=sdccb*4; cpccb=sdccb*12;
  [a,b,sVcin] =fnd(sVc.*frm); msVcin=mean(sVcin); mnVcin=sVcin/msVcin;
  cbnVcin=sVcin/Vccb; cbnafn=afn/sdaccb; cbnsacin=sacin/saccb;
  xthc=[]; Emtx=sparse(vnum,vnum); Elmtx=sparse(vnum,vnum);
  pFace=sparse(fcnt,1); poc=sparse(cnum,1);
  %figure(1); clf; hold on;
  for i=1:fcnm,
    thc=fdfm(i,1);
    for j=1:cofn,
      if(c1of(j,1)==thc | c1of(j,1)==thc)
        xthc=[xthc;j];
      end
    end
    Sizexthc=size(xthc,1);
    Fthc=[];
    for j=1:Sizexthc,
      for k=1:fcnt,
        if(xthc(j,1)==xcof(k,1))
          Fthc=[Fthc;k];
        end
      end
    end
    fthcn=size(Fthc,1); twospthc=0;
    for j=1:fthcn,
      vlstn=vofn(Fthc(j,1),1); VList=odvf(Fthc(j,1),1:vlstn);
      X=[]; Y=[]; Z=[];
      for k=1:vlstn,
          X=[X,vtc(VList(1,k),1)]; Y=[Y,vtc(VList(1,k),2)]; Z=[Z,vtc(VList(1,k),3)];
        end
        X=[X,X(1,1)]; Y=[Y,Y(1,1)]; Z=[Z,Z(1,1)]; cvlst=[VList,VList(:,1)];
        plot3(X,Y,Z,'LineWidth',1.7); pthFace=0;
        for k=1:vlstn,
          if (Emtx(cvlst(1,k),cvlst(1,(k+1)))==0)
            Emtx(cvlst(1,k),cvlst(1,(k+1)))=1; Emtx(cvlst(1,(k+1)),cvlst(1,k))=1;
            dX=X(1,(k+1))-X(1,k); dY=Y(1,(k+1))-Y(1,k); dZ=Z(1,(k+1))-Z(1,k);
            disp=sqrt(dX*dX+dY*dY+dZ*dZ);
            Elmtx(cvlst(1,k),cvlst(1,(k+1)))=disp; Elmtx(cvlst(1,(k+1)),cvlst(1,k))=disp;
          end
          pthFace=pthFace+Elmtx(cvlst(1,k),cvlst(1,(k+1)));
        end
        pFace(Fthc(j,1),1)=pthFace; twospthc=twospthc+pthFace;
      end
      poc(thc,1)=twospthc/2;
    end
    [a,b,pfin]=find(pFace); mpfin=mean(pfin); mnpfin=pfin/mpfin; cbnpfin=pfin/Facepccb;
    [a,b,pcin]=find(poc); mpcin=mean(pcin); mnpcin=pcin/mpcin; cbnpcin=pcin/cpccb;
    xthc=[]; thc=14;
    for i=1:cofn,
      if(c1of(i,1)==thc | c1of(i,1)==thc)
        xthc=[xthc;i];
      end
    end
    Sizexthc=size(xthc,1); Fthc=[];
    for i=1:Sizexthc,
      for j=1:fcnt,
        if(xthc(i,1)==xcof(j,1))
          Fthc=[Fthc;j];
        end
      end
    end
    NumFthc=size(Fthc,1);
    for i=1:NumFthc,
      vlstn=vofn(Fthc(i,1),1); VList=odvf(Fthc(i,1),1:vlstn); X=[]; Y=[]; Z=[];
      for j=1:vlstn,
        X=[X,vtc(VList(1,j),1)]; Y=[Y,vtc(VList(1,j),2)]; Z=[Z,vtc(VList(1,j),3)];
      end
      X=[X,X(1,1)]; Y=[Y,Y(1,1)]; Z=[Z,Z(1,1)]; fill3(X,Y,Z,i);
    end
    box on; axis equal; view(-20,10); title('A Voronoi c with six others','FontSize',12);
    xlabel('x','FontSize',11); ylabel('y','FontSize',11); zlabel('z','FontSize',11);
    Numfin=sum(ffrm,1);
  end;
end;
```

§ A.29 Volume in higher dimensions

```
% volnd.m, higher-d volumes, (c) Kit Tiyapan, 9th December 2002
clear all; rand('state',sum(100*clock)); cna=500;
dim=5; c=rand(cna,dim); [v,tmp]=voronoin(c); vn=size(v,1);
for i=1:cna,
  vca{i,2}=tmp{i}; vca{i,1}=size(tmp{i},2);
end
vin=sparse(1,vn);
for i=1:vn,
  if((max(v(i,:))<1) & (min(v(i,:))>0))
    vin(i)=1;
  end
end
cin=sparse(1,cna);
for i=1:cna,
  tmn=1;
  for j=1:vca{i,1},
    tma=vin(vca{i,2}(j));
    if(~tma)
      tmn=0;
    end
  end
  if(tmn)
    cin(i)=1;
  end
end
tmv=[]; tmz=[];
tmm=factorial(dim);
for i=1:cna,
  if(cin(i))
    tmp=[];
    for j=1:vca{i,1},
      tma=v(vca{i,2}(j),:); tmp=[tmp;tma];
    end
    tmd=delaunayn(tmp); tmn=size(tmd,1); tms=0;
    for j=1:tmn,
      tmq=[];
      for k=1:(dim+1),
        tma=tmp(tmd(j,k),:); tmq=[tmq;tma];
      end
      tmq=[tmq,ones((dim+1),1)]; tma=abs(det(tmq))/tmm; tms=tms+tma;
    end
    tmv=[tmv,tms]; tma=max(tmp,[],1); tmb=min(tmp,[],1);
    tmc=prod(tma-tmb); tmz=[tmz,tmc];
  end
end
tma=tmv./tmz; tmm=sum(vin); tmn=sum(cin); vrm=mean(tma); vrs=std(tma);
% test volume formula
clear all; dim=7; d=3; tmp=[0,d]';
for i=2:dim,
  tmp=[[zeros(2^(i-1),1);d*ones(2^(i-1),1)],[tmp;tmp]];
end
v=tmp; tmd=delaunayn(v); tmn=size(tmd,1); tms=0; tmm=1;
for i=2:dim,
  tmm=tmm*i;
end
for i=1:tmn,
  tmp=[];
  for j=1:(dim+1),
    tma=v(tmd(i,j),:); tmp=[tmp;tma];
  end
  tmc=[tmp,ones((dim+1),1)]; tma=abs(det(tmc))/tmm; tms=tms+tma;
end
```

§ A.30 Regular lattices in three dimensions

```
% trr.m, regular 3-d tesselation, (c) Kit Tiyapan, 16 December 2002
qn=size(q,1); in2n=size(in2,1); in3n=size(in3,1); in5n=size(in5,1);
p=[]; p{1,1}=[o;o]'; p{1,2}=sz; r=[]; s=[]; t=[];
for i=1:sz,
  r(1,i)=dx(m(i)); s(1,i)=dy(n(i)); t(1,i)=dz(z(i));
end
p{1,3}=r'; p{1,4}=s'; p{1,5}=t'; iin=size(ii,1); tmp=ones(iin,1);
tma=sparse(tmp,ii(:,1),tmp,1,sz); tmp=[]; tmb=[]; tmc=[]; tmd=[]; cnt=0;
for i=1:sz,
  if(~tma(i))
    cnt=cnt+1; tmp=[tmp;i,cnt]; tmb=[tmb;r(i)]; tmc=[tmc;s(i)]; tmd=[tmd;t(i)];
  end
end
p{2,1}=tmp; p{2,2}=cnt; p{2,3}=tmb; p{2,4}=tmc; p{2,5}=tmd; iiin=size(iii,1);
tmp=ones(iiin,1); tma=sparse(tmp,iii(:,1),tmp,1,sz);
tmp=[]; tmb=[]; tmc=[]; tmd=[]; cnt=0;
for i=1:sz,
  if(~tma(i))
    cnt=cnt+1; tmp=[tmp;i,cnt]; tmb=[tmb;r(i)]; tmc=[tmc;s(i)]; tmd=[tmd;t(i)];
  end
end
p{3,1}=tmp; p{3,2}=cnt; p{3,3}=tmb; p{3,4}=tmc; p{3,5}=tmd; iv=[ii,2*ones(iin,1)];
for i=1:iiin,
  tmp=0;
  for j=1:iin,
    if(~(iii(i,1)-ii(j,1)))
      tmp=1;
    end
  end
  if(~tmp)
    iv=[iv;iii(i,:),3];
  end
end
ivn=size(iv,1); tmp=ones(ivn,1); tma=sparse(tmp,iv(:,1),tmp,1,sz);
tmp=[]; tmb=[]; tmc=[]; tmd=[]; cnt=0;
for i=1:sz,
  if(~tma(i))
    cnt=cnt+1; tmp=[tmp;i,cnt]; tmb=[tmb;r(i)]; tmc=[tmc;s(i)]; tmd=[tmd;t(i)];
  end
end
p{4,1}=tmp; p{4,2}=cnt; p{4,3}=tmb; p{4,4}=tmc;
p{4,5}=tmd; vn=size(v,1); tmp=ones(vn,1);
tma=sparse(tmp,v(:,1),tmp,1,sz); tmp=[]; tmb=[]; tmc=[]; tmd=[]; cnt=0;
for i=1:sz,
  if(~tma(i))
    cnt=cnt+1; tmp=[tmp;i,cnt]; tmb=[tmb;r(i)]; tmc=[tmc;s(i)]; tmd=[tmd;t(i)];
  end
end
p{5,1}=tmp; p{5,2}=cnt; p{5,3}=tmb; p{5,4}=tmc; p{5,5}=tmd; vi=[ii,2*ones(iin,1)];
for i=1:vn,
  tmp=0;
  for j=1:iin,
    if(~(v(i,1)-ii(j,1)))
      tmp=1;
    end
  end
  if(~tmp)
    vi=[vi;v(i,:),5];
  end
end
vin=size(vi,1); tmp=ones(vin,1); tma=sparse(tmp,vi(:,1),tmp,1,sz);
tmp=[]; tmb=[]; tmc=[]; tmd=[]; cnt=0;
for i=1:sz,
  if(~tma(i))
    cnt=cnt+1; tmp=[tmp;i,cnt]; tmb=[tmb;r(i)]; tmc=[tmc;s(i)]; tmd=[tmd;t(i)];
  end
end
p{6,1}=tmp; p{6,2}=cnt; p{6,3}=tmb; p{6,4}=tmc; p{6,5}=tmd; vii=[iii,3*ones(iiin,1)];
for i=1:vn,
    tmp=0;
  for j=1:iiin,
    if(~(v(i,1)-iii(j,1)))
      tmp=1;
    end
  end
  if(~tmp)
    vii=[vii;v(i,:),5];
  end
end
viin=size(vii,1); tmp=ones(viin,1); tma=sparse(tmp,vii(:,1),tmp,1,sz);
tmp=[]; tmb=[]; tmc=[]; tmd=[]; cnt=0;
for i=1:sz,
  if(~tma(i))
    cnt=cnt+1; tmp=[tmp;i,cnt]; tmb=[tmb;r(i)]; tmc=[tmc;s(i)]; tmd=[tmd;t(i)];
  end
end
p{7,1}=tmp; p{7,2}=cnt; p{7,3}=tmb; p{7,4}=tmc; p{7,5}=tmd; viii=iv;
for i=1:vn,
  tmp=0;
  for j=1:ivn,
    if(~(v(i,1)-iv(j,1)))
      tmp=1;
    end
  end
  if(~tmp)
    viii=[viii;v(i,:),5];
  end
end
viiin=size(viii,1); tmp=ones(viiin,1); tma=sparse(tmp,viii(:,1),tmp,1,sz);
tmp=[]; tmb=[]; tmc=[]; tmd=[]; cnt=0;
for i=1:sz,
  if(~tma(i))
    cnt=cnt+1; tmp=[tmp;i,cnt]; tmb=[tmb;r(i)]; tmc=[tmc;s(i)]; tmd=[tmd;t(i)];
  end
end
p{8,1}=tmp; p{8,2}=cnt; p{8,3}=tmb; p{8,4}=tmc; p{8,5}=tmd; map=[]; tmp=ones(sz,1);
map{1,1,1}=sparse(p{1,1}(:,1),tmp,p{1,1}(:,2),sz,1); cnt=p{1,2}; e=q;
V=[p{1,3},p{1,4},p{1,5}]; tmp=ones(p{2,2},1);
for i=2:nx,
  map{i,1,1}=sparse(p{2,1}(:,1),tmp,cnt*tmp+p{2,1}(:,2),sz,1);
  for k=1:iin,
    map{i,1,1}(ii(k,1),1)=map{(i-1),1,1}(ii(k,2),1);
  end
  cnt=cnt+p{2,2}; % +iin;
  for k=1:qn,
    e=[e;map{i,1,1}(q(k,1)),map{i,1,1}(q(k,2))];
  end
  for k=1:in2n,
    e=[e;map{i,1,1}(in2(k,1)),map{(i-1),1,1}(in2(k,2))];
  end
  V=[V;(i-1)*dim1*tmp+p{2,3},p{2,4},p{2,5}];
end
tmp=ones(p{3,2},1);
for j=2:ny,
  map{1,j,1}=sparse(p{3,1}(:,1),tmp,cnt*tmp+p{3,1}(:,2),sz,1);
  for k=1:iiin,
    map{1,j,1}(iii(k,1),1)=map{1,(j-1),1}(iii(k,2),1);
  end
  cnt=cnt+p{3,2}; % +iiin;
  for k=1:qn,
    e=[e;map{1,j,1}(q(k,1)),map{1,j,1}(q(k,2))];
  end
  for k=1:in3n,
    e=[e;map{1,j,1}(in3(k,1)),map{1,(j-1),1}(in3(k,2))];
  end
  V=[V;p{3,3},(j-1)*dim2*tmp+p{3,4},p{3,5}];
end
tmp=ones(p{4,2},1);
for i=2:nx,
  for j=2:ny,
    map{i,j,1}=sparse(p{4,1}(:,1),tmp,cnt*tmp+p{4,1}(:,2),sz,1);
    for k=1:ivn,
      if(iv(k,3)==2)
        map{i,j,1}(iv(k,1),1)=map{(i-1),j,1}(iv(k,2),1);
      else
        map{i,j,1}(iv(k,1),1)=map{i,(j-1),1}(iv(k,2),1);
      end
    end
    cnt=cnt+p{4,2}; % +ivn;
    for k=1:qn,
      e=[e;map{i,j,1}(q(k,1)),map{i,j,1}(q(k,2))];
    end
    for k=1:in2n,
      e=[e;map{i,j,1}(in2(k,1)),map{(i-1),j,1}(in2(k,2))];
    end
    for k=1:in3n,
      e=[e;map{i,j,1}(in3(k,1)),map{i,(j-1),1}(in3(k,2))];
    end
    V=[V;(i-1)*dim1*tmp+p{4,3},(j-1)*dim2*tmp+p{4,4},p{4,5}];
  end
end
tmp=ones(p{5,2},1);
for i=2:nz,
  map{1,1,i}=sparse(p{5,1}(:,1),tmp,cnt*tmp+p{5,1}(:,2),sz,1);
  for k=1:vn,
    map{1,1,i}(v(k,1),1)=map{1,1,(i-1)}(v(k,2),1);
  end
  cnt=cnt+p{5,2};
  for k=1:qn,
    e=[e;map{1,1,i}(q(k,1)),map{1,1,i}(q(k,2))];
  end
  for k=1:in5n,
    e=[e;map{1,1,i}(in5(k,1)),map{1,1,(i-1)}(in5(k,2))];
  end
  V=[V;p{5,3},p{5,4},(i-1)*dim3*tmp+p{5,5}];
end
tmp=ones(p{6,2},1);
for i=2:nx,
  for j=2:nz,
    map{i,1,j}=sparse(p{6,1}(:,1),tmp,cnt*tmp+p{6,1}(:,2),sz,1);
    for k=1:vin,
      if(~(vi(k,3)-2))
        map{i,1,j}(vi(k,1),1)=map{(i-1),1,j}(vi(k,2),1);
      else
        map{i,1,j}(vi(k,1),1)=map{i,1,(j-1)}(vi(k,2),1);
      end
    end
    cnt=cnt+p{6,2};
    for k=1:qn,
      e=[e;map{i,1,j}(q(k,1)),map{i,1,j}(q(k,2))];
    end
    for k=1:in2n,
      e=[e;map{i,1,j}(in2(k,1)),map{(i-1),1,j}(in2(k,2))];
    end
    for k=1:in5n,
      e=[e;map{i,1,j}(in5(k,1)),map{i,1,(j-1)}(in5(k,2))];
    end
    V=[V;(i-1)*dim1*tmp+p{6,3},p{6,4},(j-1)*dim3*tmp+p{6,5}];
  end
end
tmp=ones(p{7,2},1);
for i=2:ny,
  for j=2:nz,
    map{1,i,j}=sparse(p{7,1}(:,1),tmp,cnt*tmp+p{7,1}(:,2),sz,1);
    for k=1:viin,
      if(~(vii(k,3)-3))
        map{1,i,j}(vii(k,1),1)=map{1,(i-1),j}(vii(k,2),1);
      else
        map{1,i,j}(vii(k,1),1)=map{1,i,(j-1)}(vii(k,2),1);
      end
    end
    cnt=cnt+p{7,2};
    for k=1:qn,
      e=[e;map{1,i,j}(q(k,1)),map{1,i,j}(q(k,2))];
    end
    for k=1:in3n,
      e=[e;map{1,i,j}(in3(k,1)),map{1,(i-1),j}(in3(k,2))];
    end
    for k=1:in5n,
      e=[e;map{1,i,j}(in5(k,1)),map{1,i,(j-1)}(in5(k,2))];
    end
    V=[V;p{7,3},(i-1)*dim2*tmp+p{7,4},(j-1)*dim3*tmp+p{7,5}];
  end
end
tmp=ones(p{8,2},1);
for i=2:nx,
  for j=2:ny,
    for k=2:nz,
      map{i,j,k}=sparse(p{8,1}(:,1),tmp,cnt*tmp+p{8,1}(:,2),sz,1);
      for m=1:viiin,
        if(~(viii(m,3)-2))
          map{i,j,k}(viii(m,1),1)=map{(i-1),j,k}(viii(m,2),1);
        elseif(~(viii(m,3)-3))
          map{i,j,k}(viii(m,1),1)=map{i,(j-1),k}(viii(m,2),1);
        else
          map{i,j,k}(viii(m,1),1)=map{i,j,(k-1)}(viii(m,2),1);
        end
      end
      cnt=cnt+p{8,2};
      for m=1:qn,
        e=[e;map{i,j,k}(q(m,1)),map{i,j,k}(q(m,2))];
      end
      for m=1:in2n,
        e=[e;map{i,j,k}(in2(m,1)),map{(i-1),j,k}(in2(m,2))];
      end
      for m=1:in3n,
        e=[e;map{i,j,k}(in3(m,1)),map{i,(j-1),k}(in3(m,2))];
      end
      for m=1:in5n,
        e=[e;map{i,j,k}(in5(m,1)),map{i,j,(k-1)}(in5(m,2))];
      end
      V=[V;(i-1)*dim1*tmp+p{8,3},(j-1)*dim2*tmp+p{8,4},(k-1)*dim3*tmp+p{8,5}];
    end
  end
end
en=size(e,1); Vn=size(V,1);
figure(1); clf; hold on;
for i=1:en,
  plot3([V(e(i,1),1),V(e(i,2),1)],[V(e(i,1),2),V(e(i,2),2)],[V(e(i,1),3),V(e(i,2),3)]);
end
axis off; axis equal;
clf; hold on; tms=sum(nemat,2); tmp=[]; tma=17.2; % 5x5x5 units
for i=1:en,
  if(tms(i)<tma)
    tmp=[tmp;e(i,:)];
  end
end
tmn=size(tmp,1);
for i=1:tmn,
  plot3([V(tmp(i,1),1),V(tmp(i,2),1)],...
    [V(tmp(i,1),2),V(tmp(i,2),2)],[V(tmp(i,1),3),V(tmp(i,2),3)]);
end
axis off; axis equal;
% for vertices
nvmat=sparse(Vn,Vn);
for i=1:en,
  nvmat(e(i,1),e(i,2))=1; nvmat(e(i,2),e(i,1))=1;
end
A=V; N=Vn; lmat=sparse(1,N); umat=sparse(1,N); LB=min(V(:,1)); UB=max(V(:,1));
rng=UB-LB; LBv=.05*rng+LB; UBv=UB-LBv;
for i=1:Vn,
  if(V(i,1)<LBv)
    lmat(1,i)=1;
  end
  if(V(i,1)>UBv)
    umat(1,i)=1;
  end
end
nmat=nvmat; Blocked=randperm(Vn); [pc,cord,tsries]=perc(N,lmat,umat,nmat);
% for edges
evm=sparse(en,Vn);
for i=1:en,
  evm(i,e(i,1))=1; evm(i,e(i,2))=1;
end
nemat=sparse(en,en);
for i=1:Vn,
  tmp=find(evm(:,i)); tmn=size(tmp,1);
  for j=1:(tmn-1),
    for k=(j+1):tmn,
      nemat(tmp(j),tmp(k))=1; nemat(tmp(k),tmp(j))=1;
    end
  end
end
A=e; N=en; lmat=sparse(1,N); umat=sparse(1,N);
for i=1:N,
  if((V(A(i,1),1)<=LBv) | (V(A(i,2),1)<=LBv))
    lmat(1,i)=1;
  elseif((V(A(i,1),1)>=UBv) | (V(A(i,2),1)>=UBv))
    umat(1,i)=1;
  end
end
nmat=nemat; Blocked=randperm(N); [pc,cord,tsries]=perc(N,lmat,umat,nmat);
% ccp
clear all; sz=9; nx=5; ny=5; nz=5; r=1; tmp=sqrt(2)*r;
dx=[0,2*r,4*r]; dy=[0,2*r,4*r]; dz=[0,tmp,2*tmp];
dim1=max(dx); dim2=max(dy); dim3=max(dz);
q=[1,2;1,3;1,5;1,6;2,5;3,5;4,5;5,6;5,7;5,8;5,9]; m=[1,3,1,3,2,1,3,1,3];
n=[1,1,3,3,2,1,1,3,3]; z=[1,1,1,1,2,3,3,3,3];o=[1,2,3,4,5,6,7,8,9];
ii=[1,2;3,4;6,7;8,9]; iii=[1,3;2,4;6,8;7,9]; v=[1,6;2,7;3,8;4,9];
in2=[]; in3=[]; in5=[];
```

§ A.31 Effects of channelling

```
% chl.m, effect of channelling, (c) Kit Tiyapan 17 December 2002
clear all; rand('state',sum(100*clock)); can=1000; ca=rand(can,3);
[va,vca]=voronoin(ca); van=size(va,1); tmv=sparse(1,van);
for i=1:van,
  if((max(va(i,:))<1)&(min(va(i,:))>0))
    tmv(i)=1;
  end
end
vcan=[]; ca=[];
for i=1:can,
  tmn=size(vca{i},2); in=1;
  for j=1:tmn,
    tma=tmv(vca{i}(j));
    if(~tma)
      in=0;
    end
  end
  if(in)
    tmp=[];
    for j=1:tmn,
      tma=va(vca{i}(j),:); tmp=[tmp;tma];
    end
    tmd=delaunayn(tmp); tmn=size(tmd,1); tmk=[];
    for j=1:tmn,
      tma=[];
      for k=1:4,
        tmb=tmp(tmd(j,k),:); tma=[tma;tmb];
      end
      tma=sum(tma,1)/4; tmk=[tmk;tma];
    end
    tmk=sum(tmk,1)/tmn; ca=[ca;tmk];
  end
end
tma=min(min(ca)); tmb=max(max(ca)); tmr=tmb-tma; ca=(ca-tma*ones(size(ca)))/tmr;
can=size(ca,1); [va,vca]=voronoin(ca); van=size(va,1); vcan=[];
for i=1:can,
  vcan=[vcan,size(vca{i},2)];
end
cvm=[];
for i=1:can,
  tma=ones(1,vcan(i)); cvm=[cvm;sparse(tma,vca{i},tma,1,van)];
end
ncc=sparse(can,can);
for i=1:(can-1),
  for j=(i+1):can,
    tma=sum(cvm(i,:)&cvm(j,:));
    if(tma)
      ncc(i,j)=1; ncc(j,i)=1;
    end
  end
end
tmv=sparse(1,van);
for i=1:van,
  if((max(va(i,:))<1)&(min(va(i,:))>0))
    tmv(i)=1;
  end
end
tmc=sparse(1,can);
for i=1:can,
  in=1;
  for j=1:vcan(i),
    tma=tmv(vca{i}(j));
    if(~tma)
      in=0; break;
    end
  end
  if(~(in-1))
    tmc(i)=1;
  end
end
c=[]; cnt=0;
for i=1:can,
  if(tmc(i))
    cnt=cnt+1; c=[c;ca(i,:)]; tmc(i)=cnt;
  end
end
cn=size(c,1); necc=sparse(cn,cn); [tma,tmb]=find(triu(ncc)); tmn=size(tma,1);
for i=1:tmn,
  tmp=tmc(tma(i)); tmq=tmc(tmb(i));
  if(tmp&tmq)
    necc(tmp,tmq)=1; necc(tmq,tmp)=1;
  end
end
% cells
tma=min(c(:,1)); tmb=max(c(:,1)); tmr=tmb-tma; tmd=tmr*.1; lb=tma+tmd;
ub=tmb-tmd; A=c; N=size(A,1); lmat=sparse(1,N); umat=sparse(1,N);
for i=1:N,
  if(A(i,1)<=lb)
    lmat(1,i)=1;
  elseif(A(i,1)>=ub)
    umat(1,i)=1;
  end
end
NeMat=necc; Blocked=randperm(cn); [pc,cord,tsries]=perc(N,lmat,umat,NeMat);
% cells rivulets, steepest input
b=[]; cnt=0; [tma,tmb]=find(necc); tmn=size(tma,1);
for i=1:tmn,
  cnt=cnt+1; tmp=c(tma(i),3); tmq=c(tmb(i),3);
  if(tmp>tmq)
    b(cnt,1)=tma(i); b(cnt,2)=tmb(i);
  else
    b(cnt,1)=tmb(i);b(cnt,2)=tma(i);
  end
end
bn=size(b,1);
for i=1:bn,
  tma=c(b(i,1),1:2); tmb=c(b(i,2),1:2); tmd=sum((tma-tmb).^2).^0.5;
  tma=c(b(i,1),3)-c(b(i,2),3); tmb=tma/tmd; tma=atan(tmb); b(i,3)=tma;
end
tmz=b(:,2:3); tmw=zeros(1,cn); tmx=zeros(1,cn);
for i=1:bn,
  tma=tmz(i,1);
  if(tmz(i,2)>tmw(tma))
    tmw(tma)=tmz(i,2); tmx(tma)=tmz(i,1);
  end
end
tma=sortrows([tmw',tmx'],1); tmx=tma(:,2)';
for i=1:cn,
  if(tmx(i))
    tmn=i-1; tma=tmx(i);
    break;
  end
end
tmx(1,1:tmn)=tma*ones(1,tmn); tma=min(c(:,1)); tmb=max(c(:,1)); tmr=tmb-tma; tmd=tmr*.1;
lb=tma+tmd; ub=tmb-tmd; A=c; N=size(A,1); lmat=sparse(1,N); umat=sparse(1,N);
for i=1:N,
  if(A(i,1)<=lb)
    lmat(1,i)=1;
  elseif(A(i,1)>=ub)
    umat(1,i)=1;
  end
```

```
end
NeMat=necc; [pc,cord,tsries]=perd(N,lmat,umat,NeMat,tmx);
% cells rivulets, max. sum sign
tmw=[b(:,1),b(:,3)]; tmx=zeros(1,cn);
for i=1:bn,
  tma=tmz(i,1); tmb=tmw(i,1); tmx(tma)=tmx(tma)+tmz(i,2);
  tmx(tmb)=tmx(tmb)-tmw(i,2);
end
tma=[1:cn;tmx]'; tmx=sortrows(tma,2); tmx=tmx(:,1)'; tma=min(c(:,1));
tmb=max(c(:,1)); tmr=tmb-tma; tmd=tmr*.1; lb=tma+tmd; ub=tmb-tmd; A=c;
N=size(A,1); lmat=sparse(1,N); umat=sparse(1,N);
for i=1:N,
  if(A(i,1)<=lb)
    lmat(1,i)=1;
  elseif(A(i,1)>=ub)
    umat(1,i)=1;
  end
end
NeMat=necc; [pc,cord,tsries]=perd(N,lmat,umat,NeMat,tmx);
% cells, combined
for i=1:bn,
  tma=c(b(i,1),:); tmb=c(b(i,2),:); tmd=sum((tma-tmb).^2).^0.5; tmd=tmd/2; b(i,4)=tmd;
end
cb=[];
for i=1:cn,
  cb{i,3}=[]; cb{i,4}=[];
end
for i=1:can,
  if(inc(i))
    tmp=[];
    for j=1:vcan(i),
      tma=va(vca{i}(j),:); tmp=[tmp;tma];
    end
    tmd=delaunayn(tmp); tmn=size(tmd,1); tmb=0;
    for j=1:tmn,
      tma=[tmp(tmd(j,1),:);tmp(tmd(j,2),:);tmp(tmd(j,3),:);tmp(tmd(j,4),:)];
      tmq=abs(det([tma,ones(4,1)]))/6; tmb=tmb+tmq;
    end
    cb{inc(i),1}=tmb;
  end
end
for i=1:bn,
  tma=b(i,1); cb{tma,3}=[cb{tma,3},i];
end
for i=1:cn,
  cb{i,2}=size(cb{i,3},2);
end
for i=1:cn,
  tmt=0;
  for j=1:cb{i,2},
    tma=b(cb{i,3}(j),3); tmt=tmt+tma;
  end
  for j=1:cb{i,2},
    tma=b(cb{i,3}(j),3)/tmt; cb{i,4}=[cb{i,4},tma];
  end
end
res=1000; map=zeros(res,res); cr=c*res; vr=va*res; z=max(max(cr))*.9; cin=zeros(1,cn);
for i=1:can,
  if(inc(i))
    cin(inc(i))=i;
  end
end
mac=sparse(res,res);
for i=1:cn,
  tmp=[];
  for j=1:vcan(cin(i)),
    tma=vr(vca{cin(i)}(j),:); tmp=[tmp;tma];
  end
  tmh=convhulln(tmp); tmn=size(tmh,1); tmj=[]; tmj{i,2}=[]; tmj{i,1}=0;
  for j=1:tmn,
    tmq=[];
    for k=1:3,
      tma=tmp(tmh(j,k),:); tmq=[tmq;tma];
    end
    tma=max(tmq(:,3)); tmb=min(tmq(:,3));
    in=0;
    if((tma>z)&(tmb<z))
      in=1; tma=[tmq(1,:);tmq(2,:)]; tmb=max(tma(:,3)); tmc=min(tma(:,3)); tmt=[];
      if((tmb>z)&(tmc<z))
        tmt{1}=tma; tma=[tmq(1,:);tmq(3,:)]; tmb=max(tma(:,3)); tmc=min(tma(:,3));
        if((tmb>z)&(tmc<z))
          tmt{2}=tma;
        else
          tmt{2}=[tmq(2,:);tmq(3,:)];
        end
      else
        tmt{1}=[tmq(1,:);tmq(3,:)]; tmt{2}=[tmq(2,:);tmq(3,:)];
      end
    end
    if(in)
      tmi=[];
      for k=1:2,
        x1=tmt{k}(1,1); x2=tmt{k}(2,1); y1=tmt{k}(1,2); y2=tmt{k}(2,2); z1=tmt{k}(1,3);
        z2=tmt{k}(2,3); x12=x2-x1; y12=y2-y1; z12=z2-z1; t=(z-z1)/z12; x=x1+x12*t;
        y=y1+y12*t; tmi=[tmi;x,y]; tmj{i,2}=[tmj{i,2};x,y]; tmj{i,1}=tmj{i,1}+1;
      end
      x1=tmi(1,1); y1=tmi(1,2); x2=tmi(2,1); y2=tmi(2,2); x12=x2-x1; y12=y2-y1;
      tmi=0.1; tma=sign(y2-y1); tms=tmi*tma; tme=abs(y2)-tmi*tma;
      for k=y1:tms:tme,
        x=round(x1+(k-y1)*x12/y12); map(x,round(k))=1;
      end
      tma=sign(x2-x1); tms=tmi*tma; tme=abs(x2)-tmi*tma;
      for k=x1:tms:tme,
        y=round(y1+(k-x1)*y12/x12); map(round(k),y)=1;
      end
      map(round(x2),round(y2))=1;
    end
  end
  if(tmj{i,1})
    tma=sum(tmj{i,2},1)/tmj{i,1};
    tma=round(tma);
    mac(tma(1),tma(2))=i;
  end
end
maq=map;
for i=1:res,
  maq(i,res)=-1; maq(i,1)=-1; maq(res,i)=-1; maq(1,i)=-1;
end
cnt=1;
cnc=[];
for j=2:(res-1),
  i=1;
  while(i<res)
    tmp=[];
    i=i+1;
    while(~maq(i,j))
      tmp=[tmp,i]; i=i+1;
    end
    tmn=size(tmp,2);
    if(tmn)
      tmc=0;
      for k=1:tmn,
        tma=maq(tmp(k),(j-1));
        if(tma-1)
          tmc=tma;   break;
        end
      end
      if(~tmc)
        cnt=cnt+1; tmc=cnt;
      end
      for k=1:tmn,
        maq(tmp(k),j)=tmc;
      end
    end
  end
end
[tmi,tmj,tmk]=find(mac);
tmn=size(tmi,1);
tml=[];
for i=1:tmn,
  tma=maq(tmi(i),tmj(i)); tml=[tml;tma];
end
sc=zeros(1,cnt);
for i=1:tmn,
  sc(tml(i))=tmk(i);
end
for i=1:res,
  for j=1:res,
    if(maq(i,j)>1)
      tma=tmk(sc(maq(i,j))); maq(i,j)=tma;
    end
  end
end
clf; hold on;
```

```
for i=1:20:res,
  tma=[(i-5),(i+5),(i+5),(i-5)];
  for j=1:10:res,
    tmb=[(j-5),(j-5),(j+5),(j+5)]; tmc=maq(i,j); fill(tma,tmb,tmc);
  end
end
axis equal;
axis off;
% map section to cell
[tma,tmb]=find(triu(necc));
tmn=size(tma,1);
tmx=[];  % top cells
for i=1:tmn,
  tmp=[cr(tma(i),:);cr(tmb(i),:)]; tmq=max(tmp(:,3)); tmr=min(tmp(:,3));
  if((tmq>z)&(tmr<z))
    tmp=sortrows(tmp,3); tmx=[tmx;tmp(1,:)];
  end
end
tmm=size(tmx,1);
tmp=[];
for i=1:cnt,
  tmp{i,2}=[]; tmp{i,1}=0;
end
tmn=100;
for i=1:res,
  for j=1:res,
    tma=maq(i,j);
    if(tma+1)
      if(tma&(tma-1)&(tmp{tma,1}<tmn))
        tmp{tma,2}=[tmp{tma,2};i,j]; tmp{tma,1}=tmp{tma,1}+1;
      end
    end
  end
end
tmy=zeros(cnt,3);  % grid from sect
for i=1:cnt,
  tma=sum(tmp{i,2},1)/tmp{i,1}; tmy(tma,:)=[tma(1),tma(2),z];
end
tmn=size(tmy,1); tmp=zeros(tmm,tmn);
for i=1:tmm,
  for j=1:tmn,
    tmp(i,j)=sum((tmx(i,:)-tmy(j,:)).^2).^0.5;
  end
end
gc=zeros(1,tmn);
for j=2:tmn,
  tma=[];
  for i=1:tmm,
    tma=[tma;i,tmp(i,j)];
  end
  tma=sortrows(tma,2); gc(j)=tma(1,1);
end
tmy=zeros(cnt,3);  % grid from sect
for i=1:20:res,
  for j=1:20:res,
    tma=maq(i,j);
    if(tma+1)
      if(~tmy(tma))
        tmy(tma,:)=[i,j,z];
      end
    end
  end
end
tmn=size(tmy,1); tmp=zeros(tmm,tmn);
for i=1:tmm,
  for j=1:tmn,
    tmp(i,j)=sum((tmx(i,:)-tmy(j,:)).^2).^0.5;
  end
end
for j=2:tmn,
  tma=[];
  for i=1:tmm,
    tma=[tma;i,tmp(i,j)];
  end
  tma=sortrows(tma,2); gc(j)=tma(1,1);
end
% look at each sectional cell
tmp=sparse(res,res); tma=8;
for i=1:res,
  for j=1:res,
    if(~(tma-map1(i,j)))
      tmp(i,j)=1;
    end
  end
end
% which link which section?
map1=map;
[tma,tmb]=find(triu(necc));
tmn=size(tma,1);
for i=1:tmn,
  tmp=[cr(tma(i),:);cr(tmb(i),:)]; tmq=max(tmp(:,3)); tmr=min(tmp(:,3));
  if((tmq>z)&(tmr<z))
    x1=tmp(1,1); x2=tmp(2,1); y1=tmp(1,2); y2=tmp(2,2); z1=tmp(1,3); z2=tmp(2,3);
    x12=x2-x1; y12=y2-y1; z12=z2-z1; t=(z-z1)/z12; x=x1+x12*t; y=y1+y12*t;
    map1(round(x),round(y))=1;
  end
end
% or this?
map1=map;
[tma,tmb]=find(triu(necc)); tmn=size(tma,1);
for i=1:tmn,
  tmp=[cr(tma(i),:);cr(tmb(i),:)];
  x1=round(tmp(1,1)); x2=round(tmp(2,1)); y1=round(tmp(1,2)); y2=round(tmp(2,2));
  map1(x1,y1)=1; map1(x2,y2)=1;
end
% test, successful, ie. all its faces completely cover the hull leaving no gaps
tst=[];
for i=1:cn,
  tmp=[];
  for j=1:vcan(cin(i)),
    tma=vr(vca{cin(i)}(j),:); tmp=[tmp;tma,vca{cin(i)}(j)];
  end
  tmh=convhulln(tmp(:,1:3)); tmn=size(tmh,1); tss=sparse(van,van);
  for j=1:tmn,
    tma=[tmp(tmh(j,1),4),tmp(tmh(j,2),4)]; tma=sort(tma);
    tss(tma(1),tma(2))=tss(tma(1),tma(2))+1; tma=[tmp(tmh(j,1),4),tmp(tmh(j,3),4)];
    tma=sort(tma); tss(tma(1),tma(2))=tss(tma(1),tma(2))+1; tma=[tmp(tmh(j,2),4),tmp(tmh(j,3),4)];
    tma=sort(tma); tss(tma(1),tma(2))=tss(tma(1),tma(2))+1;
  end
  [tma,tmb,tmc]=find(tss); tma=min(tmc);tst=[tst,tma];
end
```

§ A.32 Stereographic projection

```
% stp.m, stereographic projection, (c) Kit Tiyapan, January 2002
clear all; R=2; V=[1,-1,-1;1,-1,1;1,1,1;1,1,-1;-1,-1,-1;-1,-1,1;-1,1,1;-1,1,-1];
VN=size(V,1); E=[1,2;2,3;3,4;1,4;5,6;6,7;7,8;5,8;1,5;2,6;3,7;4,8];
EN=size(E,1); Res=100; El=[];
for i=1:EN,
  El{i,1}=[];
end
for i=1:EN,
  x1=V(E(i,1),1); y1=V(E(i,1),2); z1=V(E(i,1),3); x2=V(E(i,2),1); y2=V(E(i,2),2);
  z2=V(E(i,2),3); dx=x2-x1; dy=y2-y1; dz=z2-z1; d=sqrt(dx*dx+dy*dy+dz*dz);
  for t=0:(1/Res):1,
    x=x1+dx*t; y=y1+dy*t; z=z1+dz*t; El{i,1}=[El{i,1};x,y,z];
  end
end
V1=[]; V2=[]; H=[];
for i=1:VN,
  x=V(i,1); y=V(i,2);  z=V(i,3); d=sqrt(x*x+y*y+z*z); t=R/d;
  x1=x*t; y1=y*t; z1=z*t; V1=[V1;x1,y1,z1]; x2=0; y2=0;
  if(z1>=0)
    z2=-R; H=[H;0];
  else
    z2=R; H=[H;1];
  end
  I=x2-x1; J=y2-y1; K=z2-z1; z3=0; d1=z3-z1; t=d1/K;
  x3=x1+I*t; y3=y1+J*t; V2=[V2;x3,y3,z3];
end
Es=[]; E1=[]; E2=[]; He=[];
for i=1:EN,
  Es{i,1}=[]; E1{i,1}=[]; E2{i,1}=[]; He{i,1}=[];
end
for i=1:EN,
  for j=1:(Res+1),
    x=El{i,1}(j,1);  y=El{i,1}(j,2);  z=El{i,1}(j,3); d=sqrt(x*x+y*y+z*z);
    t=R/d; x1=x*t; y1=y*t; z1=z*t;
    E1{i,1}=[E1{i,1};x1,y1,z1]; x2=0; y2=0;
    if(z1>=0)
      z2=-R; He{i,1}=[He{i,1};0];
    else
```

```
   z2=R; He{i,1}=[He{i,1};1];
  end
  I=x2-x1; J=y2-y1; K=z2-z1; z3=0; d1=z3-z1; t=d1/K; x3=x1+I*t;
  y3=y1+J*t; E2{i,1}=[E2{i,1};x3,y3,z3];
 end
end
n=12;
Co=[]; Tmp=[];
for i=0:(pi/Res):pi,
 x=R*cos(i); y=R*sin(i); Co=[Co;x,y,0];
end
CN=size(Co,1); C=[]; count=0; TmpA=[Co,ones(CN,1)];
for t=(pi/n):(pi/n):(pi-pi/n),
 rx=[1,0,0,0; 0,cos(t),-sin(t),0; 0,sin(t),cos(t),0; 0,0,0,1];
 count=count+1; C{count,1}=(rx*TmpA')';
end
Ct=count; C1=[];
for i=1:Ct,
 C1{i,1}=[];
end
for i=1:Ct,
 C1{i,1}=[C1{i,1};TmpA(1,1),TmpA(1,2),TmpA(1,3)];
 for j=2:(CN-1),
  x=C{i,1}(j,1); y=C{i,1}(j,2); z=C{i,1}(j,3); x1=0; y1=0; z1=-R;
  I=x1-x; J=y1-y; K=z1-z; z2=0; d=z2-z; t=d/K; x2=x+I*t; y2=y+J*t;
  C1{i,1}=[C1{i,1};x2,y2,z2];
 end
 C1{i,1}=[C1{i,1};TmpA(CN,1),TmpA(CN,2),TmpA(CN,3)];
end
Co=[];
for i=(-pi/2):(pi/Res):(pi/2),
 x=R*cos(i); y=R*sin(i); Co=[Co;x,y,0];
end
CN=size(Co,1); count=0;
TmpA=[Co,ones(CN,1)];
for t=(pi/n):(pi/n):(pi-pi/n),
 ry=[cos(t),0,-sin(t),0; 0,1,0,0; sin(t),0,cos(t),0; 0,0,0,1];
 count=count+1; C{(Ct+count),1}=(ry*TmpA')';
end
for i=(Ct+1):(Ct+count),
 C1{i,1}=[];
end
for i=(Ct+1):(Ct+count),
 C1{i,1}=[C1{i,1};TmpA(1,1),TmpA(1,2),TmpA(1,3)];
 for j=2:(CN-1),
  x=C{i,1}(j,1); y=C{i,1}(j,2); z=C{i,1}(j,3); x1=0; y1=0; z1=-R; I=x1-x; J=y1-y;
  K=z1-z; z2=0; d=z2-z; t=d/K; x2=x+I*t; y2=y+J*t; C1{i,1}=[C1{i,1};x2,y2,z2];
 end
 C1{i,1}=[C1{i,1};TmpA(CN,1),TmpA(CN,2),TmpA(CN,3)];
end
Ct=Ct+count; Ct=Ct+1; C1{Ct,1}=Co; Tmp=[-Co(:,1),Co(:,2:3)]; Ct=Ct+1; C1{Ct,1}=Tmp;
clf; hold on;
for i=1:Ct,
 for j=1:(CN-1),
  if(mod(j,3)>1)
   tma=[C1{i,1}(j,1),C1{i,1}((j+1),1)];
   tmb=[C1{i,1}(j,2),C1{i,1}((j+1),2)];
   tmc=[C1{i,1}(j,3),C1{i,1}((j+1),3)];
   plot3();
  end
 end
end
axis off; axis equal; clf; hold on; % for below, E2 may be replaced by E1 and El
for i=1:VN,
 if(H(i,1))
  plot3(V2(i,1),V2(i,2),V2(i,3),'o');
 else
  plot3(V2(i,1),V2(i,2),V2(i,3),'.');
 end
end
for i=1:EN,
 for j=1:Res,
  if(He{i,1}(j,1))
   tma=[E2{i,1}(j,1),E2{i,1}((j+1),1)];m tmb=[E2{i,1}(j,2),E2{i,1}((j+1),2)];
   tmc=[E2{i,1}(j,3),E2{i,1}((j+1),3)];
   if(mod(j,8)>3)
    plot3(tma,tmb,tmc,'--');
   end
  else
   plot3(tma,tmb,tmc);
  end
 end
```

```
end
for i=(Ct-1):Ct,
  for j=1:(CN-1),
    tma=[C1{i,1}(j,1),C1{i,1}((j+1),1)]; tmb=[C1{i,1}(j,2),C1{i,1}((j+1),2)];
    tmc=[C1{i,1}(j,3),C1{i,1}((j+1),3)]; plot3(tma,tmb,tmc);
  end
end
axis off; axis equal; clf; hold on;
for i=1:EN,
  for j=1:Res,
    if(mod(j,5)>2)
      tma=[El{i,1}(j,1),El{i,1}((j+1),1)];
      tmb=[El{i,1}(j,2),El{i,1}((j+1),2)];
      tmc=[El{i,1}(j,3),El{i,1}((j+1),3)];
      plot3(tma,tmb,tmc);
    end
    if(mod(j,7)>2)
      plot3(tma,tmb,tmc);
    end
    plot3(tma,tmb,tmc);
  end
end
axis off; axis equal; rotate3d;
% for future developments
tx=.2; ty=.1; tz=.3; Vo=V; Tmp=ones(VN,1); V=Vo+[tx*Tmp, ty*Tmp, tz*Tmp];
V1o=V1; V2o=V2; Ho=H; Elo=El; Eso=Es; E1o=E1; E2o=E2; Heo=He;
a=.3; b=.4; c=sqrt(1-a*a-b*b) u=[a,b,c]; t=.5; q=[u*sin(t),cos(t)];
x=q(1,1); y=q(1,2); z=q(1,3); w=q(1,4); tma=[(1-2*(y*y+z*z)),(2*(x*y-w*z)),(2*(x*z+w*y))];
tmb=[(2*(x*y+w*z)),(1-2*(x*x+z*z)),(2*(y*z-w*x))];
tmc=[(2*(x*z-w*y)),(2*(y*z+w*x)),(1-2*(x*x+y*y))]; M=[tma;tmb;tmc]; V=(M*Vo')';
x1=Vo(3,1); y1=Vo(3,2); m=sqrt(x1*x1+y1*y1); c1=x1/m; s1=y1/m; c2=1; s2=0;
t=atan((c2*s1+c1*s2)/(c1*c2+s1*s2));  TmpA=cos(t); TmpB=sin(t); rz=[TmpA,-TmpB,0,0;
TmpB,TmpA,0,0; 0,0,1,1; 0,0,0,1]; hold on;
for i=1:(Ct-2),
  TmpA=[C1{i,1},ones(CN,1)]; Tmp=(rz*TmpA')';
  for j=1:(CN-1),
    if(mod(j,3)>1)
      plot3([Tmp(j,1),Tmp((j+1),1)],[Tmp(j,2),Tmp((j+1),2)],[Tmp(j,3),Tmp((j+1),3)],'-.');
    end
  end
end
```

§ A.33 Percolation of regular polygons in two dimensions

```
% ppgk.m, threshold area ratio (2^k), Kit Tiyapan, (c) 20th November, 2002
clear all; rand('state',sum(100*clock)); n=5; sz=10; k=4; stp=2^k; wen=1; num=0;
cnt=0; x=[]; rad=[]; ang=2*pi/n; tpi=2*pi; r=sqrt(1/(n*sin(ang/2)*cos(ang/2)));
while wen
  cnt=cnt+1; num=num+stp; x=[x;sz*rand(stp,2)]; rad=[rad;tpi*rand(stp,1)];
  for i=(num-stp+1):num,
    for j=0:(n-1),
      tmp=rad(i,1)+j*ang; v{i}{1,(j+1)}=[(x(i,1)+r*cos(tmp)),(x(i,2)+r*sin(tmp))];
    end
  end
  figure(cnt);clf; hold on;
  for i=1:num,
    tmx=[];
    tmy=[];
    for j=1:n,
      tmx=[tmx;v{i}{1,j}(1,1)]; tmy=[tmy;v{i}{1,j}(1,2)];
    end
    tmx=[tmx;v{i}{1,1}(1,1)]; tmy=[tmy;v{i}{1,1}(1,2)]; plot(tmx,tmy);
  end
  plot([0,sz,sz,0,0],[0,0,sz,sz,0]); axis equal; axis off; wen=~cmpc(n,sz,x,rad,v);
end
N0=num-stp; N1=num; tmn=0;
for l=1:k,
  cnt=cnt+1; num=(N0+N1)/2; tmx=x(1:num,:); tmr=rad(1:num,:); pced=cmpc(n,sz,tmx,tmr,v);
  if(pced)
    N1=num; tmn=0;
  else
    N0=num; tmn=tmn+1;
  end
  figure(cnt);clf; hold on;
  for i=1:num,
    tmx=[]; tmy=[];
    for j=1:n,
      tmx=[tmx;v{i}{1,j}(1,1)]; tmy=[tmy;v{i}{1,j}(1,2)];
    end
    tmx=[tmx;v{i}{1,1}(1,1)]; tmy=[tmy;v{i}{1,1}(1,2)]; plot(tmx,tmy);
  end
  plot([0,sz,sz,0,0],[0,0,sz,sz,0]); axis equal; axis off;
end
% ppgt.m, threshold area ratio, Kit Tiyapan, (c) 20th November, 2002
clear all; rand('state',sum(100*clock)); n=5; sz=10; stp=16; wen=1;  % wiederholen
num=0; cnt=0; x=[]; rad=[];
while wen
  cnt=cnt+1; num=num+stp; x=[x;sz*rand(16,2)]; ang=2*pi/n; tpi=2*pi;
  rad=[rad;tpi*rand(stp,1)]; r=sqrt(1/(n*sin(ang/2)*cos(ang/2)));
  for i=(num-stp+1):num,
    for j=0:(n-1),
      tmp=rad(i,1)+j*ang;
      v{i}{1,(j+1)}=[(x(i,1)+r*cos(tmp)),(x(i,2)+r*sin(tmp))];
    end
  end
  figure(cnt);clf; hold on;
  for i=1:num,
    tmx=[]; tmy=[];
    for j=1:n,
      tmx=[tmx;v{i}{1,j}(1,1)]; tmy=[tmy;v{i}{1,j}(1,2)];
    end
    tmx=[tmx;v{i}{1,1}(1,1)]; tmy=[tmy;v{i}{1,1}(1,2)]; plot(tmx,tmy);
  end
  plot([0,sz,sz,0,0],[0,0,sz,sz,0]); axis equal; axis off; wen=~cmpc(n,sz,x,rad,v);
end
N0=num-stp; N1=num;
for k=1:4,
  cnt=cnt+1; num=(N0+N1)/2; tmx=x(1:num,:); tmr=rad(1:num,:);
  pced=cmpc(n,sz,tmx,tmr,v);
  if(pced)
    N1=num;
  else
    N0=num;
  end
  figure(cnt);clf; hold on;
  for i=1:num,
    tmx=[]; tmy=[];
    for j=1:n,
      tmx=[tmx;v{i}{1,j}(1,1)]; tmy=[tmy;v{i}{1,j}(1,2)];
    end
    tmx=[tmx;v{i}{1,1}(1,1)]; tmy=[tmy;v{i}{1,1}(1,2)]; plot(tmx,tmy);
  end
  plot([0,sz,sz,0,0],[0,0,sz,sz,0]); axis equal; axis off;
end
% cmpc.m, a function, Kit Tiyapan, (c) 20th November, 2002
function [percolated] = cmpc(n,Size,X,Rad,V);
% 2-d continuum percolation of n-gons
N=size(X,1); Angle=2*pi/n; TwoPi=2*pi; R=sqrt(1/(n*sin(Angle/2)*cos(Angle/2)));
Tmp=V{1}{1,1}+(V{1}{1,2}-V{1}{1,1})/2; dx=Tmp(1,1)-X(1,1); dy=Tmp(1,2)-X(1,2);
r=sqrt(dx*dx + dy*dy); T=delaunay(X(1:N,1),X(1:N,2)); NT=size(T,1);
D=sparse(N,N); Ov=sparse(N,N); Ov1=sparse(N,1); Pair=[]; Limbo=[];
Oclock=sparse(NT,NT);
for i=1:NT,
  Tmp=[T(i,:),T(i,1)];
  for j=1:3,
    c1=Tmp(1,j); c2=Tmp(1,(j+1)); dx=X(c2,1)-X(c1,1); dy=X(c2,2)-X(c1,2);
    TmpA=sqrt(dx*dx + dy*dy); D(c1,c2)=TmpA; D(c2,c1)=TmpA; Pair=[Pair;[c1,c2;c2,c1]];
    if(TmpA<=(2*r))
      Ov(c1,c2)=1; Ov(c2,c1)=1; Ov1(c1,1)=1; Ov1(c2,1)=1;
    elseif(TmpA<=(2*R))
      Limbo=[Limbo;[c1,c2;c2,c1]];
    end
  end
  TmpB=atan(abs(dy/dx));
  if(dx>=0)
    if(dy>=0)  % Quadrant 1
      Oclock(c1,c2)=TmpB; Oclock(c2,c1)=pi+TmpB;
    else  % Quadrant 4
      Oclock(c1,c2)=TwoPi-TmpB; Oclock(c2,c1)=pi-TmpB;
    end
  else
    if(dy>=0)  % Quadrant 2
      Oclock(c1,c2)=pi-TmpB; Oclock(c2,c1)=TwoPi-TmpB;
    else  % Quadrant 3
      Oclock(c1,c2)=pi+TmpB; Oclock(c2,c1)=TmpB;
    end
  end
end
Tmp=Angle/2; Star=[];
for i=1:N,
  TmpA=[]; TmpB=Rad(i,1);
  for j=1:n,
    TmpA=[TmpA,mod((TmpB + (j-1)*Angle + Tmp),TwoPi)];
  end
  Star=[Star;TmpA];
end
Wobble=sparse(N,N); jWobble=sparse(N,N); TmpN=size(Limbo,1);
for i=1:TmpN,
  Min=10; TmpA=Limbo(i,1); TmpB=Limbo(i,2); jMin=j;
  for j=1:n,
    Tmp=Star(TmpA,j)-Oclock(TmpA,TmpB);
    if(abs(Tmp)<abs(Min))
      Min=Tmp; jMin=j;
    end
  end
  Wobble(TmpA,TmpB)=Min; jWobble(TmpA,TmpB)=jMin;
end
Tmp=Angle/2;
for i=1:2:TmpN,
  TmpA=Limbo(i,1); TmpB=Limbo(i,2);
  if(abs(Wobble(TmpB,TmpA)) >= abs(Wobble(TmpA,TmpB)))
    TmpA=Limbo((i+1),1); TmpB=Limbo((i+1),2);
  end
  J=jWobble(TmpA,TmpB); v{1}=V{TmpA}{1,J};
  if(J==n)
    v{2}=V{TmpA}{1,1};
  else
    v{2}=V{TmpA}{1,(J+1)};
  end
  J=jWobble(TmpB,TmpA); v{3}=V{TmpB}{1,J};
  if(J==n)
    v{4}=V{TmpB}{1,1};
  else
    v{4}=V{TmpB}{1,(J+1)};
  end
  Max=0;
  if(Wobble(TmpA,TmpB)>=0)
    vMin=v{1};
  else
    vMin=v{2};
  end
  d1=R*cos(Tmp-abs(Wobble(TmpA,TmpB))); TmpD=(X(TmpA,1)-X(TmpB,1));
  a=(X(TmpA,2)-X(TmpB,2))/TmpD; b=(X(TmpA,1)*X(TmpB,2)-X(TmpB,1)*X(TmpA,2))/TmpD;
  a1=a; b1=vMin(1,2)-a1*vMin(1,1); x3=v{3}(1,1); y3=v{3}(1,2); x4=v{4}(1,1);
  y4=v{4}(1,2); TmpD=x3-x4; p=(y3-y4)/TmpD; q=(x3*y4-x4*y3)/TmpD;
  TmpD=a1-p;
  x=(q-b1)/TmpD; y=(a1*q-b1*p)/TmpD; dx=x-X(TmpB,1); dy=y-X(TmpB,2);
  r2=sqrt(dx*dx + dy*dy); d2=r2*cos(Tmp-abs(Wobble(TmpB,TmpA)));
  d=D(TmpA,TmpB);
  if((d1+d2)>=d)
    Ov(TmpA,TmpB)=1; Ov(TmpB,TmpA)=1; Ov1(TmpA,1)=1; Ov1(TmpB,1)=1;
  end
end
Clus=Ov;
for i=1:N,
  Clus(i,i)=1;
end
NClus=size(Clus,1); ClusA=Clus(1,:); NClusA=1;
for i=2:NClus,
  Joined=0;
  for j=1:NClusA,
    TmpC=Clus(i,:) | ClusA(j,:);
    if(sum(Clus(i,:) & ClusA(j,:)))
      ClusA(j,:)=TmpC; ClusB=ClusA; ClusA=TmpC; NClusB=NClusA;
      NClusA=1; Joined=1; break;
    end
  end
  if(~Joined)
    ClusA=[ClusA;Clus(i,:)]; NClusA=NClusA+1;
  else
    for j=1:NClusB,
      if(sum(ClusA(1,:) & ClusB(j,:)))
        ClusA(1,:)=ClusA(1,:) | ClusB(j,:);
      else
        ClusA=[ClusA;ClusB(j,:)]; NClusA=NClusA+1;
      end
    end
  end
end
Left=sparse(1,N); Right=sparse(1,N); Margin=0.1*Size;
for i=1:N,
  if(X(i,1)<=Margin)
    Left(1,i)=1;
  elseif(X(i,1)>=(Size-Margin))
```

```
    Right(1,i)=1;
  end
end
percolated=0;
for i=1:NClusA,
  if(sum(Left & ClusA(i,:)) & sum(Right & ClusA(i,:)))
    percolated=1; break;
  end
end
```

B

§ B. Terminology and other resources

a, A

aboulia. lack of will or initiative seen with organic disease or damage to the brain.

adjacent. have a certain thing in common. ~ **edges**, edges which which have a common vertex. ~ **tiles**, tiles which have a common edge. ~ **vertices**, vertices which have a common edge.

affine. (*Lat.* affinis; *Fr.* affin) relating to a coordinate transformation that is equivalent to a linear transformation followed by a translation. **affine combination**, $p = \sum_1^k \alpha_i p_i$, where $p_1, p_2, \ldots, p_k$ are points in E^d, $\alpha_i = \mathrm{Re}$ and $\sum_{i=1}^k \alpha_i = 1$. ~ **geometry**, studies the properties which are preserved (invariant) under transformations in the affine group where A in $x' = xA + c$ is nonsingular. ~ **hull**, the smallest affine set containing L, where L is a subset of E^d. ~**ly independent points**, $p_i \in E^d, \alpha_i \in \Re, i = 1, 2, \ldots, k$ where $p_2 - p_1, \ldots, p_k - p_1$ linearly independent. ~ **mapping**, is $x' = xA + c$. ~ **set**, a linear combination $p = \alpha_1 p_1 + \alpha_2 p_2 + \ldots + \alpha_k p_k$ where $\alpha_1 + \alpha_2 + \ldots + \alpha_k = 1$, an affine set is the translation of a linear set vector subspace or it is simply *flat.*

akinesia. total lack of movement.

alternate. ~ **interior angles**, those angles which lie on the opposite sides of a transversal.

altitude. the line from the vertex of a triangle perpendicular to its opposite side.

anorexia. violent refusal to eat.

antipodal points. those points which admit no parallel supporting lines.

Archimedean polyhedron. a polyhedron whose faces are all regular polygons and whose vertices are all congruent to one another.

arteriosclerosis. A general term for the thickening, hardening, and loss of elasticity of the walls of blood vessels.

atherosclerosis. from *Gr. athero* (gruel, paste) and *sclerosis* (hardness). An arteriosclerosis which is caused by the deposition of materials, for example calcium, cellular waste products, cholesterol, fatty substances and fibrin, on the inner lining of an artery. If occurs at a carotid artery it can cause a stroke, while if at a coronary artery a heart attack.

augmented. (of polyhedra) having one or more k-sided faces replaced by a k-gonal pyramid, cupolar, or rotunda.

automatism. forced obedience to external command.

b, B

bimedian. the line joining the mid points of two opposite sides of a quadrilateral.

block. resistance to movement or thought at any level.

bond. link between two cells which share a face.

boundary. ~ **of a ball**, $\partial N_\epsilon(c) = \{x | |x - c| = \epsilon\}$ for $c \in R^m, \epsilon > 0$, a hypersphere in R^m.

bulimia. a violent and insatiable appetite .

c, C

cataclasis. granulation.

cataclasite. rock deformed by shearing and cataclasis.

categorical system. an axiomatic sytem S where each pair of its models is isomorphic with respect to S.

central projection. the one-to-one correspondence between points of the plane $x_{d+1} = 1$ (ie. a space E^d) and points on the hemisphere of S^{d+1} corresponding to $x_{d+1} > 0$.

chain. a planar straight line graph $C = (u_1, u_2, \ldots, u_p)$ with vertex set $\{u_1, u_2, \ldots, u_p\}$ and edge set (u_i, u_{i+1}), $i = 1, 2, \ldots, p - 1$.

-cingulum. *sfx* a belt of 12 triangles.

circumcentre. the centre of a circumscribed circle. ~ **of a triangle**, the point of concurrency of the perpendicular bisectors of the three sides of it.

circumcircle. or circumscribed circle is the circle which contains the three vertices of a triangle and has the circumcentre of that triangle as its centre.

claim. a small theorem to be proved, often it is a theorem presented and proved within the proof of another theorem.

close. ~**d ball**, $\mathcal{N}_\epsilon(c) = \{x | |x - c| < \epsilon\}$ for $c \in R^m, \epsilon > 0$. **vertex of** ~ **type**,v where $|T \bigoplus T_1 \bigoplus T_2| = k + 2$ if $k = N - 1$, where vertex v is common to three polygons $V(T), V(T_1), V(T_2) \in \mathrm{Vn}_k(S)$.

combinatorial geometry. geometry which characterises the geometrical objects as properties of finite subsets.

complement. $A \backslash B = \{x | x \in A, x \notin B\}$.

completeness. (of an axiomatic system S) impossible to add an independent axiom.

component. connected subgraph.

concurrent. (*Lat.* concurrere) ~ **lines**, three or more line which intersect at the same point (point of concurrency).

congruence transformation. mapping of the Euclidean plane onto itself which preserves all distances.

congruent. (*Lat.* congruentem, *nom* congruens)~ **tilings**,tiles which coincide with each other via a rigid motion of the plane or reflection. ~ **triangle**,a triangle in which a one-to-one correspondence can be established between their vertices such that corresonding sides are congruent and corresponding angles are congruent. Two relevant theorems to the congruence of triangles are the SSS, SAS and ASA theorems.

connected. comprising of only one piece. ~ **graph**, G such that $\forall g, v_i, v_j \in G, \exists v_i g v_j \in G$.

consistency. (of an axiomatic system) making no claims of contradictory statements.

convex. (*Lat.* convexus) ~ **polygon**, the region on the true side of all the half-planes of its sides. Substitution of any point coordinates into all of its half-plane equations yields negative when that point is inside the polygon. ~ **hull**, the smallest convex set containing $P \in E^d$. It defines A and b such that $\forall x \in P, Ax+b \leq 0$. ~ **set**, is a set where a line segment formed by any pair of its points lies within the set.

coplanar points. points which lie on the same plane with each other.

corner. a vertex of a polygon, in order to distinguish itself from that of a tiling.

-corona. *sfx* a crownlike structure of eight triangles. **-mega**~, such a complex of 12 triangles.

corresponding angles. two angles, one exterior the other interior, which lie on the same side of the transversal.

covering. a family of sets which completely covers a plane. ~ **lattice**,a lattice derived from another lattice by exchanging edges with vertices, the position of each edge normally being taken to be that of its mid point.

Coxeter-Todd lattice. the lattice in 12 dimensions which has the maximum packing number.

cross ratio. (of four collinear points p_1, p_2, p_3 and p_4) the ratio $c(p_1, p_2; p_3, p_4) = (p_{13}/p_{14})/(p_{23}/p_{24})$. It is an invariant under a linear transformation.

Curie point. the transition temperature where ferromagnetism changes into paramagnetism. For example, when a piece of iron gets too hot it is no longer attracted to a magnet.

d, D

Delaunay tessellation. see Delone tessellation.

Delone tessellation. decomposes a Euclidean space of m dimensions into simplexes identical with one another through linear transformations.

deltahedra. Polyhedra which has faces all equilateral triangles. There are eight convex deltahedra, namely tetrahedron, octahedron, icosahedron, triangular dipyramid, pentagonal dipyramid, heccaidecadeltahedron, tetracaidecadeltahedron, and dodecadeltahedron.

δ-slice. a portion of E^d, $d \geq 2$ contained between two hyperplanes orthogonal to a coordinate axis and at a distance of 2δ apart.

deltohedron. solids obtained by twisting one cone of the two cones in regular dipyramids by $1/(2n)$ turn. Then the result is called an n-gonal deltohedron and the original polyhedron a regular n-gonal dipyramid.

depth. (*O.E.* deop)(of a point p in a set S) the number of convex hulls or convex layers that have to be stripped from S before p is removed; (of a set S) the depth of its deepest point.

dihedral angle. the angle created by two intersecting planes.

n^p-distribution. distributions whose expected number of extreme points in a sample of size n is $\mathcal{O}(n^p)$.

dominate. (of a point) having coordinate components in all dimensions greater than another.

e, E

ϵ_0. the permittivity of free space, $\varepsilon_0 = 8.85 \times 10^{-12}$ Fm^{-1}.

edge. (*Ger.* die Kante, -n; *Lat.* acies) the arc joining two vertices or a 1-face of a d-dimensional polytope P. ~-**to**-~, (of a tiling of polygons) having all sides and edges coincides, as well as corners and vertices.

elongated. (of polyhedra) having a largest m-sided polygon replaced by an m-prism.

endpoint. (of edges) a vertex.

equiaffine. a subgroup of affine group whose $|A| = \pm 1$, the invariant of which is the volume; Euclidean distance. distance between two points represented as vectors x_i and x_j, that is $|x_i - x_j| = \sqrt{(x_i - x_j)^{\mathrm{T}}(x_i - x_j)} = \left[\sum_{k=1}^{n}(x_{ik} - x_{jk})^2\right]^{1/2}$ for n dimensions.

Euclidean space. a Cartesian space with the Euclidean distance of any dimension.

Euler line. the line containing the circumcentre O, the median point M, the orthocentre H and the centre N of the medial circle. The length of this line is $|OH|$. Then we have $|OM| = (1/3)|OH|$ and $|ON| = |NH|$.

expected complexity. estimate of the average behaviour of an algorithm.

exterior angle. (of a transversal) each of the two angles formed by a halfline, a vertex, and the halfline of a transversal on the side away from the line segment that contains both vertices; (of a triangle) an angle adjacent and supplementary to an [interior] angle of a triangle.

extreme point. (of a convex set) a point $p \in S$ convex set where $\nexists a, b \in S$ such that p lies on the open line segment $\overline{ab}$;

f, F

face. (*Ger.* die Fläche; *Lat.* hedrae).

facet. a $(d-1)$-face of a polytope in d dimensions. see also *subfacet*.

far. (*O.E.* feorr) **vertex of a** ~ **type**, is when $|T \bigoplus T_1 \bigoplus T_2| = k - 2$, if $k = 1$.

flux. the volumetric flow per cross sectional area, $j = \mathrm{d}V/(A\mathrm{d}t)$.

full period generator. a linear congruential generator whose period is k.

g, G

galactic. *adj* Pertaining or belonging to the Milky Way Galaxy.

galaxian. *adj* Pertaining or belonging to a galaxy.

n-gon. a polygon with n sides and n corners.

gyroelongated. (of polyhedra) having one largest m-sided polygon replaced by an m-gonal antiprism.

h, H

half-plane. (of a plane P with respect to a line m relative to a point $A \in P, A \notin m$) $P_1 = \{A\} \cup \{X | X \in P, X \notin m, (A, X) \cap m = \emptyset\}$ and $P_2 = \{X | X \in P, X \notin m, , (A, X) \cap m \neq \emptyset\}$.

half-space. the portion of E^d lying on one side of a hyperplane. Or if P is a plane and A a point not on P, then the two halfspaces with respect to P relative to A are $\S_1 = \{A\} \cup \{X | X \in P, (A, X) \cap P = \emptyset\}$ and $\S_2 = \{X | X \in P, (A, X) \cap P \neq \emptyset\}$.

n-hedral tiling. a tiling with n distinct prototiles.

homeomorphism. topological equivalence.

homogeneous coordinates. coordinates obtained from projection of points from the inhomogeneous or conventional coordinates represented by the hyperplane $x_{d+1} = 1$ onto the unit hemisphere S^{d+1} of E^{d+1} represents the points at infinity by letting $\varepsilon_{d+1} = 0$.

hyperboulia. excess of will, urgency.

hyperkinesia. increased speed, violence, force, and spread of movement.

hyperplane. a vector space of codimension 1. A hyperplane in an n-dimensional hyperspace is a linear space of $(n-1)$ dimensions. It separates the hyperspace into three parts, *viz.* itself and two other parts which are homeomorphic to the original space.

i, I

incentre. the point where the medians of a triangle intersect.

incidence. the membership of a point p on a line l, is an invariant in affine geometry.

incircle. (of a triangle) the circle which has the incentre of that triangle as its centre and touches all three sides of the triangle.

independence. (of a axiom) cannot be proved by using one or more other axioms within of the same set.

inscribed circle. see *incircle.*

interior angle. (made by a transversal) an angle made by the halfline transversal on the side which contains the segment between the two vertices, a vertex, and one halfline of the other line attached to that vertex.

inversion. (in E^d) a point-to-point transformation of E^d which maps a vector v applied to the origin to the vector $v' = v1/|v|^2$.

isometry. *n* a distance-preserving mapping, a synonym for congruence transformation, comprises of rotation, translation, reflection and the latter two combined which is called a glide reflection; a transformation in which a figure and its image are equal reflections of each other.

isomorphic. one to one. ~ **models**, (in an axiomatic system S) those models in which there exists at least one relation-preserving, one-to-one correspondence between each of their elements, in other words every true statement made about elements in one set is also true about the corresponding elements in the other set.

isomorphism. a one to one and onto relation.

isosceles triangle. a triangle which has two equal sides, *i.e.* where two of the sides are congruent.

isotone. monotone nonincreasing or nondecreasing.

k, K

kagome. *Jpn.* a basket pattern.

l, L

Leech lattice. the lattice in 24 dimensions which has the maximum packing number.

linear. (*Lat.* linearis) ~ **combination**, $p = \alpha_1 p_1 + \alpha_2 p_2 + \ldots + \alpha_k p_k$ for $p_i \in E^d, \alpha_i \in \Re, i = 1, 2, \ldots, k$. ~ **congruential generator**, a generator which produces random numbers $R_i \in [0, 1)$, where $R_i = X_i/k$, $X_i \in [0, k-1]$, $X_{i+1} = (sX_i + c) \bmod k$, $i = 1, 2, \ldots$, $X_0, m, c, k \in \mathcal{I}^+$, X_0 is the *seed*, m the *multiplier*, c the *increment* and k the *modulus.* ~ **set**, an affine set which passes through the origin.

link. a branch that is not a part of a tree.

lune. a combination where two triangles are attached to opposite sides of a square.

m, M

mapping. association of each preimage point in the domain subset of the source set to exactly one image point in the range subset of the target set. **one-to-one** ~, a mapping where there is no two different images with the same preimage. **onto** ~, a mapping where the range is the union of all subsets of images

medial circle. the circle which passes through the mid points of the sides of a triangle, the feet of its altitudes and the mid of the lines from the point where the altitudes intersect to the corresponding vertices. It is also known as the nine-points circle or the pedal circle because of these nine points it passes through.

median. (of a triangle) bisector of any one of its three angles. also ~ **line**,.

mesh. a closed loop in a graph. **basic** ~, a closed loop formed from the tree by one link of the graph.

metric. a generalized distance.

Möbius strips. An n^{th}-order Möbius strip is a band obtained by joining the two ends of a rectangular strip into a loop after having twisted one of them by an angle $n\pi$, where n is an integer. When the strip is cut along the centre-line, if n is odd the result is one strip having $2n + 2$ half twists which is knotted when $n \geq 3$. If n is even the result is two strips.

monohedral tilings. tilings in which every tile congruent with one another.

monotone. (*Fr.* monotonie; *Gr.* monotonos) ~ **chain**, (with respect to a straight line l) a chain which is intersected by a line orthogonal to l exactly one point. ~ **polygon**, (with respect to a straight line l) a simple polygon whose boundary is the union of two chains monotone with respect to l.

mylonite. Laminated rock with fine grain which is the product of grinding or granulation within the tectonic fault zones.

Monte Carlo. method first done during the 1940s, involve partial differential and integral equations and multi-dimensional integrals, stereotypically maps a deterministic system onto a sampling experiment, from which are collected random samples whose results of statistical analysis give an estimate solution to the problem in real system.

n, N

neighbour. that which has a certain thing in common, for example a common vertex or a common edge.

nine-points circle. see medial circle.

o, O

obtuse. (*Lat.* obtusus) ~ **angle**, an angle which is greater than a right angle, or one which is neither a right- nor an acute angle.

open ball. $N_\epsilon(c) = \{x \mid |x - c| < \epsilon\}$ for $c \in R^m, \epsilon > 0$.

order. (*O.Fr.* ordre) ~ **of**, (implies that) multiplication by constants is involved. ~ **of connectedness**, (of a graph) the maximum number of edges which can be removed without changing the number of components. ~ **of a Voronoi diagram**, $k = |T|$, $\mathrm{V}_{n_k}(S) = \bigcup_i V(T)$, $T \subset S$.

orthocentre. (of a triangle) the point where the lines drawn from the vertices normal to the respective oppositesides intersect, the point of concurrency of its three altitude lines.

orthogonal. (*Gr.* orthos; gonia) right angle. ~ **group**, intersection between the similarity and the equiaffine groups, an invariant of which is the distance. ~ **vectors**, those vectors which are at right angle to each other; $|x_1 + x_2| = |x_1 + x_2|$, $(x_1 + x_2)^2 = (x_1 + x_2)^2$, $(x_1 + x_2)^{\mathrm{T}}(x_1 + x_2) = (x_1 - x_2)^{\mathrm{T}}(x_1 - x_2)$, $x_1^{\mathrm{T}} x_2 = 0$. ~ **projection**, a set of orthogonal projections of point.

orthographic. (*Gr.* orthos; graphein; *Lat.* orthographia) ~ **projection**, a set of orthogonal projections of point.

p, P

packing. a family of non-overlapping sets in a plane.

parametral plane. (in crystallography) the plane which cuts all the three axes of a crystal.

partition. (of S) each of the two or more nonempty and disjoint subsets of S.

path. a self-avoiding path, ie a path whose all vertices are distinct.

pencil. (*Lat.* penicillus): ~ **of lines**, a set of concurrent lines.

perseveration. an uncontrollable self-stimulating and self-maintaining which causes the indefinite continuation or repetition of nervous processes.

polyhedral. (*Gr.* polyedros) ~ **set**, the intersection of a finite set of closed half-spaces in E^d.

polytope. (also d-polytope) convex d-polytope or a bounded d-dimensional polyhedral set.

projective. (*Lat.* projectum) ~ **group**, a full linear group on $d + 1$ homogeneous coordinates where $|B| = [A, 0; c, 1] \neq 0$ in $(x', 1) = (x, 1)B$

r, R

regression. a problem of best approximation in a subspace. ~ **function**, some function f^* of $d-1$ variables which minimizes the norm $| f - f^* |$, where f is a function of $d-1$ variables which represents a set of points in E^d.

regular. (*Lat.* regula) ~ **polygon**, a polygon with equal sides and equal angles. ~ **polytope**, (or a regular polyhedron) a polytope with all faces congruent regular polygons. There are only five distinct types of these and they are called the Platonic solids; (of a vertex) v_j, when there are $i < j < k$ such that (v_i, v_j) and (v_j, v_k) are edges of a graph G whose vertices are indexed in such a way that $i < j$ means either $y_i < y_j$ or, $y_i = y_j$ and $x_i > x_j$.

remote. (*Lat.* removere, remotus) ~ **exterior angle**, (of an interior angle) an angle not adjacent to the interior angle. ~ **interior angle**, (of an exterior angle) an angle not adjacent to the exterior angle.

rheology. the study of the flow of matter.

ridge. boundary element of a facet.

rigid. (*Lat.* rigere, rigidus) ~ **motions**, affine transformations which preserve distance, which are the essence of Euclidean geometry.

route. a path with possible double points.

s, S

side. edge. A side is to a corner what an edge is to a vertex.

similarity. (*Fr.* similaire; *Lat.* similis) ~ **group**, an affine group which has $AA^{\mathrm{T}} = \lambda^2 I$, the ratio of distances between points are preserved.

simple. (*O.Fr.* simple; *Lat.* simplus) ~ ***d*-polytope**, a polytope whose vertices meet exactly d edges. A simple polytope is a dual of a simplicial one.

simlex. a convex hull. **Euclidean** ~, ***d*-simplex**, a d-polytope P which is the convex hull of $(d-1)$ affinely independent points. It contains the total number of 2^{k+1} of k-faces where $k \in \mathcal{I}\, k \geq -1$, an empty set being $k = -1$. A simplex for $d = 0$ is a vertex, for $d = 1$ an edge, for $d = 2$ a triangle and for $d = 3$ a tetrahedron.

simplicial. (*Lat.* simplex, simplicis) ~ ***d*-polytope**, a polytope all the facets of which are simplices.

simply connected. is connected and contains no holes.

site. a vertex, to be distinguished from a nucleus or generator of Voronoi networks which is also a site but of the dual network.

space. (*O.Fr.* espace; *Lat.* spatium) ~ **points**, at least four points which are not necessarily coplanar.

sparsity. (*Lat.* sparsus) measure of sparseness in point distribution, a point set $S \in E^d$ has sparsity $c \in \mathcal{I} \geq 1$ for a given $\delta \in \mathcal{R}^+$ iff there are at most c points of S within any box or hypercube of side 2δ. In other words, sparsity is the scarcity of points within a given box. It is preserved through orthogonal projection.

spheno-. *prf* a wedgelike combination formed by two adjacent lunes. **di~--**, two such combinations. **hebe~--**, two lunes separated by a third one.

snub. (of polyhedra) *adj* resulting from a chiral process of rotating all faces of a polyhedron in the same direction. This creates one m-sided polygon for each vertex of degree m and two triangles for each edge. A polyhedron has the same snub as its dual.

stereohedra. set of regions whose congruent copies fill three dimensional space without overlap except at their boundaries.

stochastic. (*Gr.* stokhastikos) ~ **systems**, physical systems which involve random process evolving over time.

subfacets. the $(d-2)$-faces of a polytope P in d dimensions.

t, T

temporomandibular jaw joint disease. The condition of painful jaw joint, sometimes also ears, neck, shoulders and back. Possible symptoms include uncomfortability when openning the mouth and clicking sound when moving the jaw joint.

tessellated. (*Lat.* tessera, tessella; *Gr.* tessares) ~ **polyhedra**, polyhedral cells having the same number of faces.

tessellation. (from *Lat. tessellātus.* mosaic) a space comprised of tiles or simplexes identical to one another via linear transformation; space entirely covered with a pattern.

tesseract. a four-dimensional hypercube.

tile. each piece of, or each set in, a tiling.

tiling. (*Lat.* tegere, tegula; *O.E.* tigele) **plane** ~, a countable family of closed sets which covers a plane and leaves no gap.

tinnitus. The perception of clicking, hissing, popping, ringing, rumbling, or other sound in the ears when no external sound is present. Causes are numerous and include bad positioning of the neck and atherosclerosis. Suggested treatment ranges from taking vitamin A, E, magnesium, potassium and zinc to chiropractice.

tree. a set of branches connecting all the nodes of the graph without forming any closed loops or meshes.

truncated. (of polyhedra) having an k-gonal pyramid cut off from one or more of the vertices.

v, V

valence. (*Lat.* valere, valentia) n-~, having a vertex which is an end point of n edges.

vertex. *pl.* vertices. (*Ger.* die Ecke, -n; *Lat.* angulorum solidum) a point in any dimension. When creating a Voronoi diagram the nuclei are vertices of the corresponding Delaunay diagram, while a Voronoi vertex is the circumcentre of a facet of the Delaunay triangulation of a convex hull one dimension higher. ~ **of a pencil**, the point through which all lines in a pencil pass. Vertices are many a vertex or isolated points connected to edges, they are 0-faces of a polytope P in d dimensions.

Voronoi. *n* George Fedosevich Voronoi; Voronoi tessellation, Voronoi networks, *etc.* **generalised ~ diagram**, $V(T) = \{p : \forall v \in T, \forall w \in (S-T), d(p,v) < d(p,w)\}$ or $V(T) = \bigcap_{ij} H(p_i, p_j), p_i \in T, p_j \in (S-T)$, where $H(p_i, p_j)$ is the half-plane containing p_i that which is defined by the perpendicular bisector of $\overline{p_i p_j}$. ~ **diagram**, (also called area of influence polygons (mining), area potentially available to a tree (forestry), capillary domains, Dirichlet tessellation, domain of an atom (metallurgy), plant polygons (ecology), plesiohedra (one kind of stereohedra), Thiessen polygon, Wigner-Seitz regions (physics), Wirkungsbereich (crystallography)).

worst-case complexity. measures the performance of an algorithm over all problem instances.

§ B.1 Abbreviation

a, A

AIESEC. Association Internationale des Etudiants en Sciences Economiques et Commerciales.

ATPIJ. The Association of Thai Professionals in Japan.

ATSIST. The Association of Thai Students in Science and Technology Professions.

c, C

cf. *Lat.* `confer`, compare.

c.g.. centre of gravity.

d, D

DT. Delaunay triangulation; Dirichlet tessellation.

e, E

eg. *Lat.* `exempli gratia`, for example.

i, I

IEEE. Institute of Electrical and Electronics Engineers, Inc..

IUPAC. International Union of Pure and Applied Chemistry. `www.iupac.org`.

p, P

Pc. percolation. ~-**process**, percolation process.

p.d.f.. probability density function.

PM. porous media; porous membranes.

s, S

SEM. scanning electron microscope.

s.t.. such that.

STP. standard temperature and pressure.

t, T

TC. twentieth century.

TEM. transmission electron microscope.

TIT. Tokyo Institute of Technology.

u, U

UMIST. University of Manchester Institute of Science and Technology.

v, V

VP. Voronoi Percolation; the study of percolation on Voronoi networks; any application of Voronoi tessellation in the percolation theory.

VT. Voronoi tessellation.

§ B.2 Biographies

Abu Ja'far Muhammad ibn Musa Al-Khwarizmi. b. *circa* 780, Baghdad; d. *circa* 850.

Lidwig van Beethoven. b. 17th December 1770 , Bonn; d. 26th March 1827 , Vienna. An accomplished pianist and composer who wrote most wonderful piano sonatas and helped shape the

Romantic period of classical music. He began to have problems with his hearing facility in 1796 which developed to become a total deafness which most modern otologists decide are caused by otosclerosis of the mixed type.

Brahmagupta. b. 598, Ujjain, India; d. *circa* 670, India.

Jean Le Rond d'Alembert. b. 17^{th} November 1717 , Paris, France; d. 29^{th} October 1783 , Paris, France.

Girolamo Cardano. b. 24^{th} September 1501 , Pavia, Duchy of Milan; d. 21^{st} September 1576 , Rome.

Augustin Louis Cauchy. b. 21^{st} August 1789 , Paris, France; d. 23^{rd} May 1857 , Sceaux, France.

Arthur Cayley. b. 16^{th} August 1821 , Richmond, Surrey, England; d. 26^{th} January 1895 , Cambridge, Cambridgeshire, England. Another mathematician from a Yorkshire family who lived in Cambridge. He contributed to matrices, non-Euclidean geometry and the abstract group concept.

Edward Salisbury Dana. b. 16^{th} November 1849 , New Haven, Connecticut; d. 1935. Studied at Yale, where he received his Ph.D. in 1876, as well as in Heidelberg and Vienna, he was the son of James Dwight Dana who also wrote books as well as appendices to his father's *System of Mineralogy.*

James Dwight Dana. b. 12^{th} February 1813 , Utica, New York; d. 4^{th} April 1895 . Studied at Yale and joined the navy, he taught at Yale and married Henrietta Frances, the third daughter of Professor Benjamin Silliman whom he assisted there. He wrote a most definitive Manual of Geology, received a Ph.D. from the University of Munich on its fourth centennial celebration in 1872, and continued working very hard into the last year of his life.

Henry Philibert Gaspard Darcy. b. 30^{th} June 1803 , Dijon, Département de la Côte d'Or, France; d. 2^{nd} January 1858 , Paris. Discovered the Darcy's law of flow in porous media. Invented the modern style Pitot tube. Noticed the existence of the boundary layer in fluid flow. His name is sometimes wrongly written 'D'Arcy'. This has been verified as his school photo in 1821 already wrote the name as 'Darcy'. It is interesting that his name should have always been written in such an anglicised way as 'Henry', whereas the name of his wife, on the other hand who was originally English, used a french spelling, 'Henriette Carey'.

Jean Baptiste Louis Romé Delisle. b. 1736, Gray, eastern France; d. 1790, Paris.

Boris Nikolaevich Delone. b. 15^{th} March 1890 , St Petersburg, Russia; d. 1980.

René Descartes. b. 31^{st} March 1596 , La Haye, Touraine, France; d. 11^{th} February 1650 , Stockholm, Sweden. In 1647 he met Pascal in France and argued with him that a vacuum could not exist. The curve of the equation $x^3 + y^3 = 3axy$ which he discussed in 1638 is now called the Folium of Descartes, though it is no longer associated with flowers' petals. He solved $x^2 + ax = b^2$ with ruler and compass by writing it as $(x + a/2)^2 = (a/2)^2 + b^2$ and seeing that x is nothing but the distance from the corner A to the circle, centred at B, which has a diameter of a and touches a line segment of length b, $\overline{AC}$, at C.

Abraham de Moivre. b. 26^{th} May 1667 , Vitry near Paris, France; d. 27^{th} November 1754 , London, England.

Johann Peter Gustav Lejeune Dirichlet. b. 13^{th} February 1805 , Düren, French Empire; d. 5^{th} May 1859 , Göttingen, Hanover. The young from Richelet or *Le jeune de Richelet*, for the town in Belgium where his family came from, he is not from France as many had claimed. In his youth he showed interests in history and mathematics. He treasured his copy of Gauss's Disquisitiones arithmeticae as others might a bible. When Gauss died in 1855, he was offered his chair at Göttingen. With him the golden age of mathematics in Berlin began. His proofs are characterised by surprisingly simple initial observations followed by extremely sharp analysis of the problem.

Diophantus of Alexandria. b. *circa* 200; d. *circa* 284.

Johann Peter Gustav Lejeune Dirichlet. His name originated from *Le jeune de Richelet* [The young from Richelet, a town in Belgium where his family came from.]

Euclid of Alexandria. b. *circa* 325 BC; d. *circa* 265 BC, Alexandria, Egypt.

Leonhard Euler. b. 15^{th} April 1707 , Basel, Switzerland; d. 18^{th} September 1783 , St. Petersburg, Russia.

Edward Morgan Forster. b. 1879; d. 1970.

Johann Carl Friedrich Gauss. b. 30th April 1777 , Brunswick, Duchy of Brunswick; d. 23rd February 1855 , Göttingen, Hanover. At seven he discovered that $\sum_{i=1}^{1} 00i = 50 \times 101$. His doctoral dissertation was a discussion of the fundamental theorem of algebra. He is interested in differential geometry, where he discovered that the Gaussian curvature is invariant under isometric transformations of area in E^3. He is also interested in magnetism and worked with Weber. Together they discovered the Kirchoff's theory.

William Rowan Hamilton. b. 3rd or 4th August, 1805; d. 1865. When a child, he was taught 14 languages, and at 17 taught himself mathematics and there by discovered an error in Laplace's Celestial Mechanics. He is credited for having invented the quaternions, and sometimes for having scratched the result of that discovery, that is $i^2 = j^2 = k^2 = ijk = 1$, on the stone of the Brougham bridge on the Royal Canal. He also invented the icosian game where one is asked to find a path along a polyhedron's edges such that each node is visited only and at least once.

Abraham bar Hiyya Ha-Nasi. b. 1070, Barcelona, Spain; d. 1136, Provence, France.

Henry Selby Hele-Shaw. b. 1854, Billericay, Essex; d. 1941. He taught at University College, Liverpool. He was elected to the Royal Society in 1899 because of the fundamental investigation he had carried out regarding streamline flow of liquids.

Charles Hermite. b. 24th December 1822 , Dieuze, Lorraine, France; d. 14th January 1901 , Paris, France.

Joseph-Louis Lagrange. b. 25th January 1736 , Turin, Sardinia-Piedmont; d. 10th April 1813 , Paris, France.

Pierre Laplace. b. 1749; d. 1827. It is often said that his five volume *Mécanique Céleste* (1799–1825) is in great part a summation of works by his predecessors, as a result of which he often omitted derivations by writing them off as being obvious and easy to see.

Sir Joseph Larmor. b. 11th July 1857 , Magheragall, County Antrim, Ireland; d. 19th May 1942 , Holywood, County Down, Ireland. He was a Lucasian Professor of Mathematics at Cambridge from 1903 until 1932 when he was succeeded by Dirac.

Gottfried Wilhelm von Leibniz. b. 1st July 1646 , Leipzig, Saxony; d. 14th November 1716 , Hannover, Hanover.

Hendrik Antoon Lorentz. b. 18th July 1853 , Arnhem, Netherlands; d. 4th February 1928 Haarlem, Netherlands.

William Hallowes Miller. b. 1801, Velinde, near Llandovery, South Wales; d. 1880, Cambridge.

Hermann Minkowski. b. 22nd June 1864 , Alexotas, Russian Empire, now Kaunas, Lithuania; d. 12th January 1909 Göttingen, Germany.

Franz Ernst Neumann. b. 1798, Joachimsthal; d. 1895.

Luca Pacioli. b. 1445, Sansepolcro, Italy; d. 1517, Sansepolcro, Italy. His *Summa de arithmetica geometria, proportioni et proportionalita*, published in 1494, summarises the contemporary algebra, arithmetic, geometry, and trigonometry. In 1509 published the *Divina proportione* which deals with the golden ratio and contains illustrations by Leonardo da Vinci.

Parmenides. b. *circa* 515; d. after 450 BC. He reasoned that since a void is nothingness, if two particles were separated by a void, then they would be separated by nothing. In other words, they would not be separated at all, they would be touching (*cf* Davies, 2001).

Jean Louis Poisseuille. b. 1799; d. 1869.

Simeon-Denis Poisson. b. 1781; d. 1840.

Scipione del Ferro. b. 6th February 1465 , Bologna, Italy; d. 5th November 1526 , Bologna, Italy.

Niels Stensen (aka Nicolaus Steno). b. 1638, Copenhagen; d. 1686, Schwerin.

Robert Louis Stevenson. (1850–1894)

James Stirling. b. May 1692, Garden near Stirling, Scotland; d. 5th December 1770 , Edinburgh, Scotland.

Johannes Diderik van der Waals. b. 23^{rd} November 1837 , Leyden, The Netherlands; d. 8^{th} March 1923 , Amsterdam. It was him who coined the equation for real gas, $(p + an^2/V^2)(V - nb) = nRT$.

George Fedosevich Voronoi. b. 28^{th} April 1868 , Zhuravka, Poltava guberniya, Russia (now Ukraine); d. 20^{th} November 1908 , Warsaw, Poland. Both his master's degree, 1894, on the algebraic integers associated with the roots of an irreducible cubic equation and his doctoral thesis on algorithms for continued fractions were awarded the Bunyakovsky prize by the St. Petersburg Academy of Sciences. But he decided that he wanted to teach at the Warsaw University where he extended work by Zolotarev on algebraic numbers and the geometry of numbers. He met Minkowski in 1904 at an international conference at Heidelberg.

Egor Ivanovich Zolotarev. b. 12^{th} April 1847 , St Petersburg, Russia; d. 19^{th} July 1878 , St Petersburg, Russia. Received a silver medal from the Gymnasium in St. Petersburg, attended lectures by Kummer and Weierstrass, and discussed mathematics with Hermite, he worked with Korkin and gives complete solutions to the four- and five variable cases of the problem of finding the minimal values of n-variable quadratic forms with real coefficient.

§ B.3 Computation and softwares

AVS. *AVS can be used to find cross sections in 2– and 3–D networks.*

hull. *Hull is written in ANSI C by Ken Clarkson. It computes the convex hull of a point set of any dimension.*

Synopsis: hull -d -f<format> -A -aa<alpha> -af<format> -oN -ov -s<seed> -r -m<multiplier> -X<debug file> -i<input file> -oF<output file>
Hull takes in points as its input. The outputs are vertices of the convex hull facets, Delaunay triangulations, alpha shapes, and volumes; in postcript or OFF format for geomview.

hullio.a precursor to *hull.*

Matlab. *Matlab has an algorithm for finding 2–D voronoi diagrams. However, the output data for this is not very structured, which makes it difficult to use the data obtained for analysis purpose.*

When repeatedly running a .m file online, all variables should be cleared by 'clear all' two times, both at the beginning and at the end of the file. Failing to do so sometimes results in consistency in the results.

Submitting .m files through NQS often requires writing every path in full. However, in .m files one can write 'path(path,*directory*)' or 'path(*directory*, path)' for post-appending and pre-appending a path, for example that which contains the .m files containing functions.

I guess that everything done on a matrix in Matlab is as fully vectorised as possible, since provided that this is the case that very program can still be greatly improved, and I hold the programmers who develop it in a better regard than what would have allowed me to assume this. So we can vectorise our algorithm by simply putting the various items into a matrix and work on them, in that matrix, in parallel instead of in sequence as we would normally do. There is still a limitation in that we can subject a matrix to only one operation, and therefore can only do in parallel things which require the same operation.

Another limitation is that we can only put matrices into a structure, which in Matlab contains data of the class *cell*, but not vice versa. So the great convenience we have from working with the cell structure comes at the cost to parallelisation. In effect, this means that we can only parallelise our algorithmic details but not the whole algorithm itself. In other words, we can not simultaneously find the percolation probability for two different networks using Matlab.

qhull. *A quick hull C program for finding convex hulls, Delaunay triangulations, Voronoi vertices, furthest-site Voronoi vertices, halfspace intersection about a point, hull volume and facet area. It is written by C. Bradford Barber and Hannu Huhdanpaa at The Geometry Center, University of Minnesota (Barber et al, 1996). The program combines the 2-d Quickhull algorithm with the general dimension Beneath-Beyond algorithm. The latter is an incremental algorithm which adds a point to the convex hull of the points just processed. It processes a new point in steps as described in Algorithm 9.1. The boundary of the visible facets is the set of horizontal ridges for the point. A* *facet is visible to a point if the latter is above it. Cones of new facets are constructed from the point of its horizon ridges.*

Algorithm 9.1 *Beneath-Beyond algorithm*

```
for [each new point] do
  locate facets visible to it;
  construct a cone of new facets;
  delete the visible facets;
endfor
```

□

There are various kinds of quickhull algorithms. The one mentioned here is the work of Barber *et al* (*ibid.*) which works in the space of points and convex hulls and maintains an outside set for each facet. Being in the outside set implies that a point is above the facet. It is one of the variations of the randomised incremental algorithm proposed by Clarkson and Shor (1989) which works in the space of halfspaces and polytopes, dual to that of the present one, and maintains a conflict graph, a set of all the list of polytope edges that intersect an unprocessed halfspace. Some of the options are *d* Delaunay triangulation by lifting points to a paraboloid,

d Qu computes the furthest-site Delaunay triangulation from the upper convex hull, *f* print all fields of all facets, *FA* computes total area and volume, *FN* lists the voronoi vertices for each voronoi region, *Fp* halfspace intersection coordinates (*F* for output format), *Ft* prints a triangulation (with points (the centrums) added to non-simplicial facets), *Fv* when used with *v* option prints a (furthest-site) Voronoi diagram (Output: number of ridges \\ `< the count of indices> <1st input site> <2nd input site> <1st ridge> <2nd ridge> ... \\ ...`), *Fx* convex hull vertices, *G* Geomview output (2− to 4−*d*), *H*n,n,... computes halfspace intersection about $[n, n, 0, \ldots]$ (The point $[n, n, n, \ldots]$ lies inside $Hx + b \leq 0$, default $b = 0$), *m* Mathematica output (2− and 3−*d*, in Mathematica <*variable*>, <<, <*filename*> then *Show[Graphics3D[list]]*), *o* prints the input points and facets, *Pg* prints only good facets, *QV*n a good facet includes point *n* ($n < 0$ means a good facet does not include point *n*), *Qg* builds good facets, *TO ¡file¿*, *Tv* verify structure, convexity, and point inclusion of the result, *p* vertex coordinates, *i* vertices incident to each facet, *P* printing, *Q* qhull control, *Qb*k:n scales the k^{th} coordinate of the input points (the lower bound of the input points becomes *n*) *Qbk:0Bk:0* drops dimension *k* from the input points before the Delaunay and Voronoi to allows sub-dimensional convex hulls, *QB*k:n the upper bound becomes *n*, *Qbb* scales the last coordinate to $[0, m]$ where *m* is the maximum absolute value of the other coordinates, *QbB* scales the input points to fit the unit cube after projection to the paraboloid. The lower and the upper bounds for all dimensions are -0.5 and $+0.5$, *QJ* triangular output, *Fa* prints area for each facet, *PA*n prints the *n* largest facets, *PF*n prints facets larger than *n*, *Fs* prints a summary of the structure, *s* prints a summary to stderr, *T* tracing, *v* Voronoi diagram via the Delaunay triangulation, *v Qu* finds the furthest-site Voronoi diagram,

Examples:

```
rbox <number of points> t<seed> D3 | qhull QV5 v p Pg | qhull G > <filename>;
rbox <number of points> t<seed> D3 | qhull v Fs > <filename>;
```

A *d*-d convex hull in this algorithm is represented by its vertices and $(d-1)$-d facets or faces. Extreme points are those which are the vertices of a convex hull. Each facet has a set of vertices, a set of neighbouring facets and a hyperplane equation. A $(d-2)$-d face is a ridge of the convex hull, which is the intersection of the vertices of two neighbouring facets. An oriented hyperplane through *d* points is represented by its unit normal which points outwards and its offset from the origin. The signed distance from a point to a hyperplane is the inner product of the point and the normal plus the offset. A hyperplane defines a halfspace of points having negative distance from it. The point is above the hyperplane if this distance is positive.

In R^d, Quickhull repairs in order faults where more than two facets meet at a ridge, a facet is in another facet, a facet has fewer than *d* neighbours, a facet has a flipped orientation, a point just processed is coplanar with a horizontal facet, concave facets, coplanar facets and redundant vertices.

The program *rbox* is written in C. It generates pseudo random points for *qhull*. The arguments *t*n tells it to use *n* as a random seed. Here *D2* means 2-dimension while *D3* 3-dimension.

rbox. *This program generates random points for qhull. When used without an option gives a list of possible options.*

Sweep2. *This is a program for creating 2-dimensional Voronoi diagrams and Delaunay triangulation. It uses sweepline algorithm and is written by Steve Fortune.*

triangle. *A C program by Jonathan Richard Shewchuk for 2–D triangulation and mesh generation. It uses Ruppert's Delaunay refinement algorithm.*

volume. *Written by Joseph O'Rourke, it finds the volume of a simple polyhedron from the triangulated surface input read from stdin. Inputs are vertex coordinates represented as integers and triangle faces as vertex indices.*

§ B.4 Internet resources

`www.faqs.org` *Internet FAQ Consortium.* A site for Frequently Asked Questions which covers various areas. Useful algorithms for geometrical computation can be found here together with their references.

`www.geom.umn.edu` *Geometry Center.* Being at the University of Minnesota, this is the place to find the programs Quickhull (Qhull) and Rbox used in the beginning of this study. Qhull is incorporated into MATLAB as the commands Qhull, Convhulln, Delaunayn and Delaunay3. It is written by C. Bradford Barber and Hannu Huhdanpaa.

`www.gnu.org` *The "GNU's not Unix!" Project and Free Software Foundation.* Recursively named, this project originated by Richard Stallman offers a variety of profound softwares, most of which are for Unix and Linux developed by tinkerers. Most of the softwares come with codes, therefore are ideal for developpers. The 3-d viewing program called Geomview is only one example.

`www-groups.dcs.st-and.ac.uk` *Turnbull Server.* Named after Herbert Westren Turnbull (1885–1961), this server is valuable to anyone who has an interest in Mathematics, as well as researchers in history of this field. It houses the MacTutor History of Mathematics archive, which covers history of the subject and biographies of mathematicians. The materials offered are extensive.

`www.gutenberg.org` *Gutenberg Project.* A site containing valuable books on many topics, including but not only literatures.

`www.mathworks.co.uk` *MathWorks Developers of MATLAB and Simulink.* A useful site for users of both products. One of the strong points of products of MathWorks is the extensiveness of their help facilities. The helps available here are better organised and more explanatory than those that come with the program.

`www.nectec.or.th` *National Electronics and Computer Technology Center, Thailand.* Supposedly the only research authority in Thailand, this site hosts various other sites of thai researchers all over the world. That of ATSIST, www.nectec.or.th/bureaux/atsist, is but one of them.

§ B.5 TeXnicalities

LaTeX is a macro which runs on TeX. It gives one convenience but not without a tradeoff in understanding. Also one may not have much freedom in writing macros on LaTeX. Tiyapan used TeX for his Work Notes dated 12^{th} February 2001 .

LaTeX(Lamport, 1985) is written by Leslie Lamport. Newer versions of it has come up at a regular interval. Unlike most other macros which run on TeX, her source code is free for none but herself. Moreover, having used TeX to do what it wants, LaTeX thereby castrates her progenitor in such a way that it is impossible for her users to define new macros efficiently by using the `\def` command. With `\def` disabled, the lion has lost its fangs and users become as docile and dependent as a lobotomised patient. There can be no doubt that with the TeX users having such idea as this, sooner or later LaTeX will have to change in these respects. But this is the way things are at present.

One of the first macros written is the code to change the date format. The algorithm first sets $x = date$, then it assigns the ordinate endings *st, nd, rd* or *th* depending on the value of *ordinate*, which is calculated from

```
if $x > 30$ then $ordinate = 1$ else
  if $x > 20$ then $ordinate = x - 20$ else
    $ordinate = date$
  endif
endif
```

The macros which are either newly written or adapted from elsewhere, mainly from the *manmac* macro by Knuth, are listed in § B.5. Apart from these, this thesis uses the *plain* and the *manmac* macros. Another set of macros developed here is that which deals with languages. The definition of language here is quite wide. It contains many languages among which are those which are used here, for example the languages for Chemistry, Chinese, Czech, German, French, grammatical jargons, Japanese, Lanna, Latin, latin grammar, Mathematics, Pali, Physics, Russian, Sanskrit, and Thai

(Daiy), *etc.* Only parts of this set of macros are useful for the writing of this thesis, not least so those which are used for writing the dedication page.

In the original account of his, Tiyapan (2003, KNT8(iii)) wrote,

> When I first started using TeX instead of LaTeX, I only used the macro `plain.tex`. Then in my first book typeset with TeX (Tiyapan, 2001, KNT8(ii)), I used in addition to the plain TeX `manmac.tex` and `epsf.tex`. Now to my amazement, I have discovered many other excellent macros, for instance `rotate.tex`, and found that I could understand how they work when I read them. This is one of the benefits that comes with talking in TeX instead of, for instance, LaTeX. I also know now the difference between the primitive TeX and plain TeX, and that the latter is only one of the infinitely many possible implementations of TeX. However, since all TeX gurus I know use plain TeX as a basis, there is no reason why I should be too proud to follow the practice. Having said that, my next plan is to improvise on the primitive TeX without any direct reference to the plain TeX macro.

The citation programme BibTeX was intended to be used with LaTeX. Karl Berry and Oren Patashnik have written `btxmac.tex` which makes BibTeX usable from plain TeX. But for the present purpose I merely use my own macros, which are much simpler, and do not need BibTeX. Ultimately such database programme as BibTeX would have been extremely useful. But I wish to develop something similar to it on my own.

There are still some unsolved problems in the TeX macros, for example the page references which are embedded within groups are sometimes slightly wrong, that is they may appear to be one more or one less than their actual position. Since publishing macros play but a minor part here compared with mathematics and physics, this problem has been systematically minimised and then tolerated. The solution and explanation of it will be dealt with and published elsewhere.

When lines of text appear beside a picture there are macros which make the latter always stay next to the outer rim of the page. These work satisfactorily well, and those cases in the results which appear to say otherwise are in fact the result of some other more primitive macros earlier written.

§ B.6 Voronoi statistics of earlier simulations

All the mean numbers for each face of the network of 527 Voronoi cells (§ 3.7) are 4.9091, 5.0769, 5.2500, 5.1429, 4.8000, 4.9091, 5.2500, 5.1429, 5.1429, 5.0769, 5.2000, 5.2000, 5.0000, 5.2500, 4.8000, 5.2941, 5.0769, 5.2000, 4.9091, 5.0769, 4.6667, 5.3684, 5.2000, 5.3333, 4.6667, 5.2500, 5.0000, 4.5000, 5.0000, 5.4286, 5.2000, 5.2500, 5.0769, 5.2500, 5.1429, 4.9091, 5.0000, 5.0000, 5.3333, 4.8000, 5.3333, 5.2000, 5.3684, 5.2500, 5.0000, 5.0769, 5.0769, 5.2941, 4.8000, 5.0000, 5.0769, 5.4286, 5.3333, 5.1429, 4.6667, 4.6667, 5.4783, 5.2941, 5.2500, 5.2500, 4.9091, 4.9091, 5.4000, 5.2500, 5.1429, 5.2941, 5.4545, 5.1429, 5.0000, 4.9091, 5.2500, 5.1429, 5.1429, 5.4545, 4.8000, 5.1429, 5.0000, 5.2941, 5.2941, 5.0000, 5.4286, 5.0769, 5.1429, 5.1429, 5.2941, 5.0769, 4.9091, 5.2000, 5.2000, 5.4545, 5.1429, 5.2500, 5.1429, 5.2000, 5.2500, 5.1429, 5.1429, 5.0000, 5.3684, 5.3684, 5.2941, 5.4000, 5.4783, 5.3684, 4.9091, 5.2941, 5.2500, 4.8000, 5.2500, 5.2941, 5.1429, 5.2500, 5.2500, 5.2000, 5.0769, 5.4000, 5.0000, 5.3333, 5.4286, 4.8000, 5.2500, 5.3333, 5.0769, 4.9091, 5.3333, 5.3333, 5.2941, 5.2000, 5.3333, 5.3333, 5.2500, 5.2500, 5.2500, 5.2000, 5.2000, 5.1429, 5.0000, 4.8000, 5.3684, 5.1429, 5.2000, 5.2500, 5.0769, 5.3684, 5.2941, 5.1429, 5.2500, 5.2000, 5.1429, 4.8000, 5.0000, 5.3333, 5.2000, 5.0769, 5.2000, 5.3333, 4.5000, 5.0000, 5.3333, 5.3333, 5.0000, 5.2500, 5.2000, 4.9091, 5.1429, 5.0000, 5.3684, 5.2500, 5.0000, 5.4000, 5.3333, 5.2500, 4.8000, 5.5000, 5.1429, 5.2000, 5.2500, 5.2941, 5.2500, 5.2941, 5.4000, 5.2941, 5.2941, 4.9091, 5.2000, 5.2000, 5.4286, 5.0769, 5.3684, 5.2500, 5.3333, 5.0000, 4.9091, 5.3684, 4.0000, 5.1429, 5.3333, 4.9091, 5.3333, 5.2000, 5.1429, 5.1429, 5.1429, 5.0000, 5.0769, 5.1429, 5.3684, 4.9091, 5.2941, 5.0000, 5.2941, 5.2941, 5.2500, 5.1429, 5.0769, 5.2000, 5.4000, 5.1429, 5.0769, 5.2000, 4.6667, 5.3684, 5.4286, 5.0769, 5.2500, 5.1429, 5.3684, 5.0769, 5.2500, 5.3684, 5.3684, 5.0769, 4.8000, 5.3333, 5.0769, 5.1429, 5.2000, 4.9091, 5.2000, 4.9091, 5.2500, 5.2000, 4.9091, 5.2500, 5.2941, 5.5000, 5.2000, 5.2500, 5.4783, 5.2941, 5.2000, 5.0000, 5.2000, 5.2941, 5.2000, 5.2941, 4.9091, 5.4286, 5.2500, 5.1429, 5.5000, 5.1429, 5.4286, 5.0000, 5.3684, 5.2500, 5.3684, 5.4000, 5.2500, 5.2941, 4.8000, 4.9091, 4.8000, 5.1429, 5.3333, 5.2000, 5.0000, 5.0769, 5.0000, 5.0769, 5.1429, 5.1429, 5.0769, 5.0769, 5.4545, 5.0769, 5.2000, 5.0769, 5.4545, 5.4286, 4.8000, 5.2000, 5.0769, 5.3684, 5.3333, 5.3333, 5.4000, 5.2000, 5.0000, 5.0000, 5.2941, 5.2000, 4.9091, 5.2500, 5.3333, 5.0000, 5.3684, 5.3333, 5.3333, 5.0000, 5.4545, 5.1429, 5.0769, 5.0769, 5.0000, 5.5200, 5.0769, 5.1429, 5.2000, 5.3333, 5.0000, 5.2941, 5.2500, 5.2500, 5.2500, 5.2000, 5.2000, 5.2000, 5.2000, 5.2000, 5.2000, 5.2000, 5.2500, 5.2500, 4.9091, 5.4000, 5.2941, 5.2000, 5.2500, 5.2500, 5.1429, 5.0769, 5.0769, 5.2000, 5.2941, 5.2941, 5.4783, 5.3333, 4.8000, 5.3333, 5.2941, 5.0769, 5.3333, 5.1429, 5.3684, 5.2941, 5.1429, 5.1429, 5.4545, 5.3333, 4.9091, 5.1429, 5.2941, 5.2500, 5.0769, 5.3333, 4.9091, 5.0769, 5.3333, 5.0000, 4.9091, 5.3333, 5.2941, 5.3684, 5.2941, 5.1429, 5.2500, 5.3333, 5.1429, 5.0769, 5.2000, 5.2500, 5.3333, 5.0769, 5.0000, 5.2941, 4.9091, 5.0769, 5.0769, 5.4545, 5.2000, 5.2941, 4.8000, 5.2941, 5.2500, 5.3684, 5.2000, 4.9091, 5.2000, 5.2000, 5.1429, 4.6667, 5.2000, 5.2500, 4.6667, 5.3333, 5.2941, 5.0769, 5.0769, 5.0000, 5.3333, 5.0000, 5.4000, 5.4545, 5.3333, 5.2500, 5.1429, 5.2000, 4.9091, 5.1429, 5.2000, 5.3333, 5.2941, 5.5000, 5.2500, 5.1429, 5.2000, 5.3684, 5.1429, 5.4286, 5.0769, 5.2941, 5.2941, 5.2500, 5.1429, 5.1429, 5.2500, 5.1429, 5.3333, 4.8000, 5.2000, 5.2000, 4.8000, 5.4286, 5.2000, 5.2941, 5.2500, 5.5000, 5.2941, 5.0000, 5.2941, 5.2500, 4.6667, 5.2941, 5.0000, 5.2500, 5.2500, 4.9091, 5.2000, 5.2941, 5.0769, 5.3684, 4.9091, 4.9091, 5.2000, 5.4000, 5.2000, 5.0769, 5.3684, 5.2000, 5.2500, 5.0769, 5.2500, 5.0000, 5.2941, 4.5000, 5.3333, 5.2000, 5.2941, 5.1429, 5.0000, 5.0769, 4.5000, 5.4783, 5.3684, 5.1429, 4.9091, 5.3684, 5.2000, 5.2500, 5.0000, 5.2941, 5.0000, 5.0769, 5.0000, 5.3333, 5.2000, 5.2000, 5.2500, 5.2000, 5.2000, 5.2000, 4.8000, 5.0769, 5.2000, 5.2941, 5.2941, 5.2500, 5.0000, 5.4286, 5.2000, 5.2500, 5.3333, 5.2000, 5.2000, 5.0000, 5.1429, 5.2000, 5.2941, 5.2941, 5.3333, 4.8000, 5.3333, 4.6667, 5.1429, 5.1429 and 5.2000.

All these standard deviations are 1.4460, 1.5525, 1.5706, 2.3157, 1.2293, 1.5783, 1.5275, 1.0995, 1.0271, 1.7541, 1.4736, 1.2649, 1.5954, 1.2910, 0.6325, 1.5315, 0.9541, 1.2071, 0.7006, 0.8623, 1.0000, 1.3829, 1.5675, 1.7150, 1.2247, 1.6533, 1.5374, 0.9258, 1.1282, 1.8323, 1.9346, 1.3416, 0.6405, 1.8439, 1.6575, 1.3003, 1.4142, 1.5374, 1.5339, 1.1353, 1.5718, 0.8619, 2.1137, 1.7321, 1.5954, 1.6053, 1.3205, 1.7235, 0.7888, 1.2792, 1.7059, 1.6301, 1.4142, 1.6575, 1.3229, 0.8660, 1.9038, 1.9610, 1.9149, 1.8074, 0.9439, 1.5136, 1.9841, 1.3904, 1.6575, 1.2632, 2.0172, 1.7913, 1.5954, 1.9212, 1.4376, 1.9945, 1.6104, 1.5954, 1.1353, 1.8752, 1.4142, 1.7235, 1.4902, 1.4142, 1.5991, 1.3821, 1.8337, 1.0271, 1.6494, 1.0377, 2.0226, 1.6987, 1.9712, 1.2994, 1.5119, 0.9309, 1.9555, 1.0142, 1.0646, 1.2924, 2.1070, 1.8586, 2.0873, 1.4225, 2.2573, 1.7290, 1.7547, 1.9779, 1.6404, 1.7946, 2.2361, 1.4757, 1.6125, 1.4038, 0.9493, 1.6533, 1.5706, 1.6125, 1.8467, 1.4290, 1.0445, 1.7489, 1.5675, 1.3984, 1.1832, 1.9704, 0.9541, 1.3003, 2.1420, 1.6803, 1.6111, 2.2104, 1.9704, 1.5718, 1.2383, 1.3416, 1.6125, 2.1778, 1.5675, 1.6104, 1.1282, 1.3166, 2.0605, 1.9945, 1.2649, 1.4832, 1.3821, 2.0873, 1.4902, 1.4601, 1.2910, 1.5213, 1.4601, 0.6325, 1.8586, 2.5437, 1.7809, 1.3821, 1.5675, 2.1420, 1.0690, 1.0445, 1.6803, 1.8787, 1.2792, 1.4832, 1.4736, 1.2210, 1.6104, 0.6030, 1.6059, 1.5706, 1.4771, 1.6026, 1.6803, 1.3416, 1.3166, 2.0430, 1.1673, 1.6125, 1.6125, 1.6111, 1.9833, 1.6494, 1.8180, 1.6111, 1.4476, 0.9439, 1.6987, 1.5675, 1.6301, 1.3821, 1.8016, 2.1448, 1.0290, 1.7056, 1.4460, 2.1137, 0.8944, 1.8752, 1.9704, 1.4460, 1.1376, 1.5213, 1.8337, 1.4601, 1.6575, 1.4142, 1.5525, 1.7478, 2.2413, 1.1362, 1.3585, 1.5374, 1.6111, 2.0238, 1.3904, 1.4601, 1.0377, 1.7809, 1.5009, 2.1432, 1.7541, 2.3964, 1.0000, 1.4985, 1.9640, 2.0191, 1.7701, 2.4450, 1.3829, 0.8623, 1.9833, 1.8918, 1.8622, 1.4412, 1.7512, 1.3284, 1.5525, 1.2924, 1.7809, 1.1362, 1.2649, 1.3003, 1.6931, 1.6562, 1.0445, 1.4376, 1.5315, 1.8415, 2.1112, 1.7701, 1.9038, 1.9289, 1.1464, 1.2060, 1.3202, 1.6494, 1.2071, 1.7235, 1.8141, 2.2265, 1.7701, 1.4064, 2.0000, 1.4064, 1.7768, 1.0445, 1.8321, 2.1134, 1.9779, 1.6026, 1.5706, 1.5718, 1.5492, 0.9439, 1.5492, 0.9493, 1.4552, 1.8593, 0.9535, 1.1875, 1.9069, 1.3205, 1.4601, 1.5119, 1.1875, 1.7059, 1.7107, 1.4979, 1.6987, 1.1152, 1.9451, 1.8860, 1.3984, 1.3732, 1.4979, 1.7388, 1.7150, 1.9403, 1.1877, 1.7809, 1.0445, 1.2060, 1.3585, 1.4243, 1.3003, 1.1832, 1.5339, 0.9535, 2.2659, 1.1882, 1.6450, 1.4142, 1.7655, 0.9493, 1.3205, 1.7059, 1.7581, 1.2949, 1.0377, 1.8337, 1.3732, 1.4951, 1.5374, 1.4038, 1.6931, 1.6125, 1.4832, 1.8974, 1.8205, 1.5675, 1.3202, 1.4736, 1.9712, 1.8974, 1.9149, 1.5275, 1.8141, 1.6026, 1.5315, 1.7809, 2.5690, 1.8074, 1.7033, 1.6053, 0.9541, 1.3202, 1.7235, 1.7594, 1.2746, 1.1882, 1.3984, 1.4142, 1.6111, 1.3205, 1.9097, 1.7913, 1.8016, 1.3117, 1.4601, 1.6575, 1.6250, 2.0000, 1.6404, 1.6104, 1.4476, 1.5706, 1.3205, 1.3284, 1.1362, 1.8467, 1.6088, 1.8091, 1.5783, 1.4552, 2.2013, 1.6401, 1.6494, 1.8752, 1.9494, 1.6450, 1.7033, 1.7059, 2.1112, 1.8074, 1.5718, 1.1875, 1.1282, 1.3585, 1.1362, 1.9774, 1.1875, 2.0639, 1.2649, 1.1600, 1.1353, 1.8962, 1.5706, 1.8321, 1.8974, 1.8141, 1.3202, 1.4736, 1.7033, 1.4142, 1.5675, 1.5275, 0.7071, 2.2229, 1.4038, 1.2558, 1.5525, 1.5954, 1.3284, 2.0449, 2.2572, 1.8702, 1.6088, 1.4376, 1.0995, 1.7809, 0.8312, 1.7478, 1.6125, 1.3720, 2.0544, 1.8178, 2.0494, 1.6575, 1.9712, 2.0873, 1.7033, 1.7768, 1.7541, 1.9610, 1.2632, 1.4832, 1.4601, 1.3506, 1.7321, 1.5619, 1.9097, 1.5492, 1.7809, 1.3202, 1.2293, 1.5991, 1.7403, 1.9289, 1.3904, 1.7937, 1.4476, 1.0445, 1.6494, 1.6533, 1.3229, 1.7594, 1.2792, 1.6931, 1.7321, 0.8312, 1.0142, 1.8962, 1.6564, 1.5352, 1.5136, 1.0445, 1.6987, 1.5694, 1.6987, 1.7059, 1.7065, 1.7403, 1.2383, 1.5525, 1.8074, 1.2792, 2.1437, 1.0690, 1.5718, 1.6562, 2.0544, 1.2924, 1.8091, 2.3616, 1.5119, 1.4419, 1.4985, 1.1673, 1.2210, 2.1657, 1.7403, 2.0166, 1.4142, 1.9610, 1.3484, 1.0377, 1.4771, 2.1693, 1.4243, 1.5213, 1.2383, 2.0771, 1.7809, 1.7809, 0.6325, 1.5525, 1.3202, 0.9196, 1.6111, 2.1448, 1.7581, 1.7485, 1.6125, 1.8074, 1.9704, 1.7809, 0.8619, 1.2792, 1.8337, 1.7809, 1.6494, 1.7235, 1.3720, 0.7888, 1.7489, 0.8660, 1.3506, 1.5119 and 1.5213.

From a 6-dimensional Voronoi structure in § 3.7 there are three cells with 2729, 3213, 3246, 3421, 3490, 3606, 3938, 4143, 4398, 4417, 4442 and 4970 vertices. There are two having 1854, 2549, 2557, 2634, 2712, 2720, 2722, 2751, 2804, 2843, 2869, 2878, 2882, 2920, 2931, 2957, 2973, 2996, 3013, 3090, 3260, 3322, 3329, 3343, 3366, 3393, 3418, 3424, 3446, 3513, 3520, 3560, 3577, 3583, 3585, 3610, 3631, 3677, 3714, 3732, 3753, 3767, 3770, 3819, 3835, 3861, 3907, 3919, 3924, 3937, 3939, 4045, 4053, 4091, 4117, 4122, 4123, 4163, 4178, 4219, 4251, 4261, 4269, 4272, 4298, 4317, 4327, 4355, 4461, 4470, 4473, 4474, 4542, 4563, 4572, 4576, 4586, 4618, 4624, 4629, 4640, 4646, 4648, 4700, 4729, 4792, 4810, 4851, 4942, 4975, 4984, 4985, 5058, 5064, 5097, 5161, 5195, 5235, 5287, 5347, 5387, 5455, 5467, 5492, 5529, 5606, 5650, 5838, 5913, 6112, 6193 and 6455 vertices. And there is only one cell with each of the following numbers of vertices, 1198, 1612, 1672, 1686, 1688, 1717, 1727, 1787, 1827, 1864, 1906, 1915, 1921, 1947, 2016, 2052, 2056, 2060, 2123, 2190, 2200, 2223, 2231, 2236, 2267, 2280, 2281, 2286, 2289, 2344, 2353, 2356, 2371, 2372, 2385, 2408, 2423, 2426, 2429, 2455, 2475, 2487, 2499, 2500, 2503, 2504, 2508, 2513, 2542, 2547, 2551, 2560, 2561, 2565, 2575, 2578, 2580, 2585, 2586, 2596, 2617, 2619, 2622, 2650, 2654, 2658, 2664, 2669, 2673, 2686, 2692, 2694, 2704, 2708, 2716, 2718, 2723, 2732, 2733, 2742, 2743, 2755, 2771, 2779, 2781, 2783, 2791, 2800, 2807, 2808, 2811, 2820, 2835, 2837, 2853, 2858, 2864, 2867, 2870, 2873, 2875, 2880, 2884, 2887, 2893, 2894, 2895, 2902, 2907, 2910, 2914, 2916, 2925, 2928, 2934, 2949, 2955, 2956, 2960, 2967, 2974, 2978, 2983, 2985, 2989, 2993, 3007, 3012, 3014, 3027, 3051, 3074, 3102, 3107, 3114, 3116, 3119, 3129, 3130, 3136, 3137, 3141, 3148, 3152, 3164, 3173, 3176, 3180, 3182, 3186, 3188, 3190, 3192, 3199, 3202, 3204, 3206, 3208, 3219, 3221, 3229, 3231, 3237, 3239, 3244, 3245, 3255, 3256, 3259, 3267, 3270, 3271, 3274, 3275, 3285, 3295, 3296, 3305, 3307, 3312, 3316, 3317, 3318, 3319, 3333, 3337, 3344, 3351, 3352, 3354, 3357, 3360, 3370, 3373, 3374, 3378, 3384, 3390, 3394, 3396, 3402, 3403, 3408, 3427, 3428, 3436, 3440, 3443, 3444, 3452, 3453, 3454, 3486, 3487, 3492, 3499, 3502, 3505, 3506, 3509, 3519, 3521, 3522, 3523, 3536, 3541, 3544, 3545, 3547, 3549, 3551, 3555, 3556, 3573, 3575, 3578, 3581, 3582, 3587, 3589, 3594, 3595, 3599, 3604, 3609, 3611, 3620, 3628, 3634, 3635, 3636, 3639, 3647, 3652, 3656, 3658, 3663, 3666, 3668, 3680, 3685, 3686, 3687, 3689, 3691, 3694, 3695, 3696, 3706, 3709, 3710, 3713, 3716, 3717, 3727, 3728, 3735, 3739, 3740, 3742, 3743, 3744, 3751, 3764, 3772, 3775, 3776, 3778, 3782, 3787, 3789, 3792, 3794, 3798, 3803, 3804, 3806, 3808, 3810, 3815, 3818, 3821, 3830, 3834, 3837, 3838, 3842, 3847, 3849, 3850, 3863, 3867, 3868, 3883, 3884, 3889, 3890, 3899, 3900, 3902, 3903, 3908, 3912, 3914, 3918, 3920, 3925, 3929, 3942, 3943, 3944, 3947, 3949, 3956, 3957, 3960, 3962, 3963, 3978, 3980, 3981, 3986, 3988, 3996, 4008, 4016, 4017, 4019, 4027, 4030, 4033, 4035, 4038, 4039, 4044, 4049, 4057, 4072, 4073, 4075, 4077, 4082, 4086, 4092, 4106, 4111, 4112, 4124, 4126, 4128, 4134, 4140, 4145, 4146, 4147, 4148, 4150, 4153, 4157, 4162, 4176, 4177, 4179, 4188, 4189, 4190, 4193, 4195, 4203, 4206, 4212, 4214, 4217, 4218, 4225, 4227, 4232, 4238, 4239, 4244, 4245, 4246, 4247, 4257, 4262, 4271, 4273, 4279, 4283, 4288, 4291, 4293, 4300, 4308, 4313, 4316, 4320, 4322, 4324, 4326, 4331, 4334, 4335, 4337, 4339, 4342, 4346, 4349, 4350, 4351, 4352, 4354, 4357, 4361, 4376, 4377, 4386, 4388, 4395, 4399, 4405, 4410, 4411, 4413, 4425, 4426, 4428, 4447, 4449, 4451, 4453, 4459, 4471, 4475, 4480, 4488, 4493, 4496, 4501, 4504, 4506, 4512, 4515, 4518, 4522, 4523, 4532, 4548, 4552, 4556, 4561, 4574, 4580, 4585, 4587, 4595, 4598, 4602, 4605, 4610, 4614, 4619, 4621, 4626, 4628, 4632, 4634, 4639, 4649, 4651, 4657, 4663, 4665, 4674, 4681, 4684, 4697, 4702, 4703, 4705, 4710, 4711, 4714, 4725, 4728, 4736, 4738, 4740, 4751, 4754, 4755, 4761, 4765, 4770, 4779, 4781, 4814, 4816, 4826, 4829, 4831, 4844, 4850, 4873, 4877, 4880, 4885, 4890, 4892, 4896, 4899, 4900, 4902, 4903, 4905, 4908, 4911, 4914, 4915, 4919, 4924, 4933, 4937, 4944, 4955, 4960, 4967, 4969, 4974, 4992, 4994, 4997, 5000, 5002, 5003, 5007, 5019, 5021, 5028, 5029, 5035, 5039, 5050, 5074, 5080, 5083, 5092, 5093, 5104, 5108, 5110, 5117, 5119, 5120, 5131, 5138, 5156, 5158, 5166, 5167, 5176, 5185, 5192, 5194, 5200, 5203, 5219, 5225, 5226, 5233, 5236, 5239, 5251, 5259, 5265, 5270, 5271, 5274, 5285, 5288, 5290, 5313, 5316, 5318, 5325, 5328, 5336, 5337, 5346, 5364, 5389, 5405, 5408, 5415, 5432, 5439, 5440, 5448, 5457, 5460, 5463, 5464, 5466, 5468, 5470, 5474, 5482, 5484, 5530, 5549, 5567, 5589, 5591, 5598, 5612, 5627, 5632, 5664, 5676, 5680, 5718, 5719, 5720, 5724, 5738, 5744, 5747, 5748, 5749, 5752, 5764, 5802, 5806, 5824, 5841, 5859, 5880, 5885, 5892, 5896, 5899, 5927, 5928, 5947, 5956, 5957, 5966, 5998, 6006, 6014, 6017, 6031, 6035, 6038, 6040, 6054, 6055, 6075, 6081, 6084, 6089, 6092, 6113, 6145, 6166, 6169, 6180, 6192, 6195, 6221, 6237, 6265, 6272, 6273, 6286, 6303, 6305, 6310, 6313, 6320, 6356, 6367, 6456, 6469, 6510, 6515, 6520, 6521, 6524, 6541, 6561, 6609, 6615, 6642, 6677, 6767, 6771, 6831, 6852, 6883, 6918, 6945, 6985, 7056, 7093, 7201, 7379, 7387, 7461, 7560, 7588, 7685, 7764, 7765, 7979, 8308, 8460, 8899, 9079 and 9923.

The following results represent the earlier simulation works. None of the programs used here are used again at the later stage. However there were hundreds of tables similar to the one shown on the following page. These represent the statistical values of the Voronoi structures under study. Here only one page is shown to give an idea of how the project looked at its early stages. The Voronoi structures were then generated by `qhull`, which has been replaced by `voronoin` on Matlab. The latter, though came from the former, is much easier to use.

Result on 700 cells

```
Box size: 10
No compression
Number of cells: 700
Number of vertices: 4380
Number of cells in frame: 341
Time for counting stats: 2191.39 seconds
Number of faces connected to the first vertice at infinity: 181
Time for finding cell volumes: 3.1500 seconds
```

The table which follows was drawn by TEX's `plain` macro written by Donald Knuth who is the creator of TEX. This thesis is compiled with the help of Knuth's `manmac` macro and the *thshead* macro that I have written. Both the `tdhead` that I have used for typesetting my books (Tiyapan, 2003, KNT8(iii) to KNT8(ix)). My book Voronoi Translated (Tiyapan, 2001, KNT8(ii)), however, used only `manmac`. I am now using `bkhead`, which has been developed from `tdhead`. I am now writing a book on TEX in which you may be able to find `bkhead` and `tdhead` being discussed in more details.

n_c 700	n_v 4,380	$n_{f,1^{st}v}$ 181	$n_{c_{in}}$ 341	$t_{CPU,stat.}$ 2191.39 sec.	$t_{CPU,A}$ 137.13 sec.	$t_{CPU,V}$ 3.15 sec.	
$\min n_{v_c}$ 10	$\max n_{v_c}$ 46	$n_{\bar{v}_c}$ 25.106	$\sigma^2_{n_{v_c}}$ 46.784	$\sigma_{n_{v_c}}$ 6.8399	$g, \bar{n}_{v_c}$ 24.169	$h, \bar{n}_{v_c}$ 23.208	n_{v_c}
$\widetilde{n_{v_c}}$ 24	$\delta_\mu(n_{v_c})$ 5.4023	$M^2(n_{v_c})$ 46.717	$M^3(n_{v_c})$ 1.4240×10^2	$M^4(n_{v_c})$ 6.5877×10^3	$\kappa_{n_v,c}$ 3.0184		
$\min n_{v,c_{in}}$ 10	$\max n_{v,c_{in}}$ 46	$v_{\bar{c}_{in}}$ 26.246	$\sigma^2_{v,c_{in}}$ 40.974	$\sigma_{v,c_{in}}$ 6.4011	$\overline{V}_{,gc_{in}}$ 25.464	$\overline{V}_{,hc_{in}}$ 24.668	$n_{v,c_{in}}$
$\widetilde{n}_{v,c_{in}}$ 26	$\delta_\mu(n_{v,c_{in}})$ 5.1101	$M^2(n_{v,c_{in}})$ 40.854	$M^3(n_{v,c_{in}})$ 1.0338×10^2	$M^4(n_{v,c_{in}})$ 4.9173×10^3	$\kappa_{n_{v,c_{in}}}$ 2.9461		
$\min n_{\aleph_v,c_{in}}$ 8	$\max n_{\aleph_v,c_{in}}$ 26	$\bar{n}_{\aleph_v,c_{in}}$ 16.123	$\sigma^2_{n_{\aleph_v,c_{in}}}$ 10.244	$\sigma_{n_{\aleph_v,c_{in}}}$ 3.2006	$\bar{n}_{g,\aleph_v,c_{in}}$ 15.809	$\bar{n}_{h,\aleph_v,c_{in}}$ 15.495	$n_{\aleph_v,c_{in}}$
$\widetilde{n}_{\aleph_v,c_{in}}$ 16	$\delta_\mu(n_{\aleph_v,c_{in}})$ 2.5550	$M^2(n_{\aleph_v,c_{in}})$ 10.214	$M^3(n_{\aleph_v,c_{in}})$ 12.922	$M^4(n_{\aleph_v,c_{in}})$ 3.0733×10^2	$\kappa_{n_{\aleph_v,c_{in}}}$ 2.9461		
$\min n_{\aleph_e,c}$ 8	$\max n_{\aleph_e,c}$ 27	$\bar{n}_{\aleph_e,c}$ 15.669	$\sigma^2_{n,\aleph_e,c}$ 12.439	$\sigma_{n,\aleph_e,c}$ 3.5269	$\bar{n}_{g,\aleph_e,c}$ 15.279	$\bar{n}_{h,\aleph_e,c}$ 14.894	$n_{\aleph_e,c}$
$\widetilde{n}_{\aleph_e,c}$ 15	$\delta_\mu(n_{\aleph_e,c})$ 2.7792	$M^2(n_{\aleph_e,c})$ 12.422	$M^3(n_{\aleph_e,c})$ 22.761	$M^4(n_{\aleph_e,c})$ 4.8871×10^2	$\kappa_{n,\aleph_e,c}$ 3.1673		
$\min n_{\aleph_e,c_{in}}$ 8	$\max n_{\aleph_e,c_{in}}$ 26	$\bar{n}_{\aleph_e,c_{in}}$ 16.123	$\sigma^2_{n(\aleph_e),c_{in}}$ 10.244	$\sigma_{n(\aleph_e),c_{in}}$ 3.2006	$\bar{n}_{g,\aleph_e,c_{in}}$ 15.809	$\bar{n}_{h,\aleph_e,c_{in}}$ 15.495	$n_{\aleph_e,c_{in}}$
$\widetilde{n}_{\aleph_e,c_{in}}$ 16	$\delta_\mu(n_{\aleph_e,c_{in}})$ 2.5550	$M^2(n_{\aleph_e,c_{in}})$ 10.214	$M^3(n_{\aleph_e,c_{in}})$ 12.922	$M^4(n_{\aleph_e,c_{in}})$ 3.0733×10^2	$\kappa_{n(\aleph_e),c_{in}}$ 2.9461		
$\min n_{\aleph_f,c}$ 8	$\max n_{\aleph_f,c}$ 27	$\bar{n}_{\aleph_f,c}$ 15.629	$\sigma^2_{n(\aleph_f),c}$ 12.274	$\sigma_{n(\aleph_f),c}$ 3.5034	$\bar{n}_{g,\aleph_f,c}$ 15.242	$\bar{n}_{h,\aleph_f,c}$ 14.857	$n_{\aleph_f,c}$
$\widetilde{n}_{\aleph_f,c}$ 15	$\delta_\mu(n_{\aleph_f,c})$ 2.7635	$M^2(n_{\aleph_f,c})$ 12.256	$M^3(n_{\aleph_f,c})$ 21.051	$M^4(n_{\aleph_f,c})$ 4.6791×10^2	$\kappa_{n(\aleph_f),c}$ 3.1149		
$\min n_{\aleph_f,c_{in}}$ 8	$\max n_{\aleph_f,c_{in}}$ 26	$\bar{n}_{\aleph_f,c_{in}}$ 16.123	$\sigma^2_{n(\aleph_f),c_{in}}$ 10.244	$\sigma_{n(\aleph_f),c_{in}}$ 3.2006	$\bar{n}_{g,\aleph_f,c_{in}}$ 15.809	$\bar{n}_{h,\aleph_f,c_{in}}$ 15.495	$n_{\aleph_f,c_{in}}$
$\widetilde{n}_{\aleph_f,c_{in}}$ 16	$\delta_\mu(n_{\aleph_f,c_{in}})$ 2.5550	$M^2(n_{\aleph_f,c_{in}})$ 10.214	$M^3(n_{\aleph_f,c_{in}})$ 12.922	$M^4(n_{\aleph_f,c_{in}})$ 3.0733×10^2	$\kappa_{n(\aleph_f),c_{in}}$ 2.9461		
$\min A_c$ 0.59209	$\max A_c$ 83.165	$\bar{A}_c$ 3.4086	$\sigma^2_{A_c}$ 2.0373	σ_{A_c} 2.3046×10^2	$\bar{A}g, c$ 5.0726	$\bar{A}h, c$ 1.0502	A_c
$\widetilde{A}_c$ 1.1935×10^2	$\delta_\mu(A_c)$ 4.0384	$M^2(A_c)$ 2.8300	$M^3(A_c)$ 2.0530	$M^4(A_c)$ 1.3800	κ_{A_c} 2.0572		
$\min A_{c_{in}}$ 0.59209	$\max A_{c_{in}}$ 3.4086	$\bar{A}_{c_{in}}$ 2.0373	$\sigma^2_{A_{c_{in}}}$ 1.0502	$\sigma_{A_{c_{in}}}$ 4.0384	$\bar{A}g, c_{in}$ 2.8300	$\bar{A}h, c_{in}$ 2.0530	$A_{c_{in}}$
$\widetilde{A}_{c_{in}}$ 1.3800	$\delta_\mu(A_{c_{in}})$ 2.0572	$M^2(A_{c_{in}})$ 2.9782	$M^3(A_{c_{in}})$ 3.9125	$M^4(A_{c_{in}})$ 3.1031	$\kappa_{A_{c_{in}}}$ 1.0800		
$\min A_{fr,c}$ 8.5015×10^{-8}	$\max A_{fr,c}$ 1.1941×10^{-5}	$\bar{A}_{fr,c}$ 4.8942×10^{-7}	$\sigma^2_{A_{fr,c}}$ 2.9253×10^{-7}	$\sigma_{A_{fr,c}}$ 3.3091×10^{-5}	$\bar{A}g, fr, c$ 7.2835×10^{-7}	$\bar{A}h, fr, c$ 1.5079×10^{-7}	$A_{fr,c}$
$\widetilde{A}_{fr,c}$ 1.7137×10^{-5}	$\delta_\mu(A_{fr,c})$ 5.7986×10^{-7}	$M^2(A_{fr,c})$ 4.0635×10^{-7}	$M^3(A_{fr,c})$ 2.9478×10^{-7}	$M^4(A_{fr,c})$ 1.9816×10^{-7}	$\kappa_{A_{fr,c}}$ 2.9538×10^{-7}		
$\min A_{fr,c_{in}}$ 7.5700×10^{-4}	$\max A_{fr,c_{in}}$ 4.3579×10^{-3}	$\bar{A}_{fr,c_{in}}$ 2.6048×10^{-3}	$\sigma^2_{A_{fr,c_{in}}}$ 1.3427×10^{-3}	$\sigma_{A_{fr,c_{in}}}$ 5.1632×10^{-3}	$\bar{A}g, fr, c_{in}$ 3.6182×10^{-3}	$\bar{A}h, fr, c_{in}$ 2.6248×10^{-3}	$A_{fr,c_{in}}$
$\widetilde{A}_{fr,c_{in}}$ 1.7644×10^{-3}	$\delta_\mu(A_{fr,c_{in}})$ 2.6301×10^{-3}	$M^2(A_{fr,c_{in}})$ 3.8077×10^{-3}	$M^3(A_{fr,c_{in}})$ 5.0023×10^{-3}	$M^4(A_{fr,c_{in}})$ 3.9674×10^{-3}	$\kappa_{A_{fr,c_{in}}}$ 1.3808×10^{-3}		
$\min V_c$ 0.88511	$\max V_c$ 6.2052×10^6	$\overline{V}_c$ 2.7455×10^4	$\sigma^2_{V_c}$ 8.9303×10^{10}	σ_{V_c} 2.9884×10^5	$\overline{V}g, c$ 22.548	$\overline{V}h, c$ 6.3296	V_c
$\widetilde{V}_c$ 7.6212	$\delta_\mu(V_c)$ 5.1126×10^4	$M^2(V_c)$ 8.9176×10^{10}	$M^3(V_c)$ 4.3202×10^{17}	$M^4(V_c)$ 2.3781×10^{24}	κ_{V_c} 2.9904×10^2		
$\min V_{c_{in}}$ 0.88511	$\max V_{c_{in}}$ 1.8958×10^3	$\overline{V}_{c_{in}}$ 20.542	$\sigma^2_{V_{c_{in}}}$ 1.5264×10^4	$\sigma_{V_{c_{in}}}$ 1.2355×10^2	$\overline{V}g, c_{in}$ 5.9195	$\overline{V}h, c_{in}$ 4.4392	$V_{c_{in}}$
$\widetilde{V}_{c_{in}}$ 5.4355	$\delta_\mu(V_{c_{in}})$ 27.858	$M^2(V_{c_{in}})$ 1.5219×10^4	$M^3(V_{c_{in}})$ 2.2911×10^7	$M^4(V_{c_{in}})$ 3.9178×10^{10}	$\kappa_{V_{c_{in}}}$ 1.6915×10^2		
$\min V_{fr,c}$ 4.6056×10^{-8}	$\max V_{fr,c}$ 0.32288	$\overline{V}_{fr,c}$ 1.4286×10^{-3}	$\sigma^2_{V_{fr,c}}$ 2.4179×10^{-4}	$\sigma_{V_{fr,c}}$ 1.5550×10^{-2}	$\overline{V}g, fr, c$ 1.1733×10^{-6}	$\overline{V}h, fr, c$ 3.2935×10^{-7}	$V_{fr,c}$
$\widetilde{V}_{fr,c}$ 3.9656×10^{-7}	$\delta_\mu(V_{fr,c})$ 2.6603×10^{-3}	$M^2(V_{fr,c})$ 2.4145×10^{-4}	$M^3(V_{fr,c})$ 6.0865×10^{-5}	$M^4(V_{fr,c})$ 1.7433×10^{-5}	$\kappa_{V_{fr,c}}$ 2.9904×10^2		
$\min V_{fr,c_{in}}$ 1.2636×10^{-4}	$\max V_{fr,c_{in}}$ 0.27064	$\overline{V}_{fr,c_{in}}$ 2.9326×10^{-3}	$\sigma^2_{V_{fr,c_{in}}}$ 3.1108×10^{-4}	$\sigma_{V_{fr,c_{in}}}$ 1.7638×10^{-2}	$\overline{V}g, fr, c_{in}$ 8.4507×10^{-4}	$\overline{V}h, fr, c_{in}$ 6.3374×10^{-4}	$V_{fr,c_{in}}$
$\widetilde{V}_{fr,c_{in}}$ 7.7598×10^{-4}	$\delta_\mu(V_{fr,c_{in}})$ 3.9770×10^{-3}	$M^2(V_{fr,c_{in}})$ 3.1017×10^{-4}	$M^3(V_{fr,c_{in}})$ 6.6659×10^{-5}	$M^4(V_{fr,c_{in}})$ 1.6273×10^{-5}	$\kappa_{V_{fr,c_{in}}}$ 1.6915×10^2		

§ C. Publications and submissions of papers

§ C.1 Critical probability of 2-d tessellation

Critical probability and other properties of 2-d tessellation†

Kittisak Tiyapan‡ G. A. Davies§ David J. Bell‡

2^{nd} September 1996

Abstract

The critical probabilities of Voronoi tessellation and some other uniform lattices, including triangular, square, honey-comb, and Kagomé lattices, are identified by mean of an algorithm developed. The properties of these uniform lattices agree with existing well known results. This work provides the bond critical probability for Voronoi tessellation.

Introduction

A tessellation is an aggregate of cells that cover the space without overlapping. A Voronoi polygon is also known as a Dirichlet polygon, a Wigner-Seitz polygon, a Theissen polygon, a Blum's transform, an S polygon, a cell model, a plant polygon, Wirkkungsbereich, etc.

Definition 1. *Let Φ be a distribution of a countable set of nuclei $\{x_i\}$ in R^d, and let $x_1, x_2, x_3, \ldots$ be the coordinates of the nuclei. Then, the Voronoi region is*

$$\Pi_i = \{x | d(x, x_i) < d(x, x_j) \forall j \neq i\}$$

where $d(x, y)$ is the Euclidean distance between x and y.

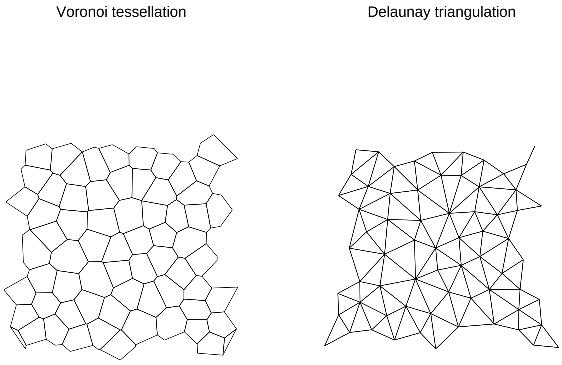


Figure 1.. Duality of Voronoi tessellation and delaunay triangles.

The number of Voronoi points (nuclei) and edges are both $\mathcal{O}(n)$ where $n \in N$, and the number of Delaunay triangles and edges are also $\mathcal{O}(n)$ (Ahuja and Schachter, 1983).

A bond-problem of any lattice L can be translated into a site-problem on L^c, its covering lattice constructed (Shante and Kirkpatrick, 1971) by the following procedure,

1 . Replacing each β_i by α_i^c.
2 . Creating β_i^c by the rule that $\forall \alpha_i^c, \alpha_j^c \in L^c$, α_i^c and α_j^c are connected by a bond of L^c if and only if their corresponding β_i and $\beta_j \in L$ have a common terminal atom of L.

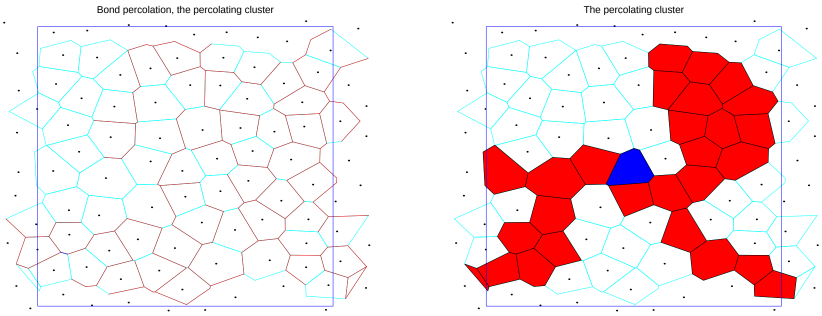
---

† This pape was presented at the MTNS96 conference held in St. Louis, U.S., June 1996, by the first author.

‡ Department of Mechanical and Environmental Informatics, Tokyo Institute of Technology, 2–12–1 O-okayama, Meguro-ku, Tokyo 152, Japan.

§ professor G. A. Davies, Department of Chemical Engineering, University of Manchester Institute of Science and Technology, Manchester M60 1QD.

‡ Professor D. J. Bell, Department of Mathematics, University of Manchester Institute of Science and Technology, Manchester M60 1QD.



(a) *Example of a bond percolation* (b) *Example of a site percolation*

Figure 2 *Example of* (a) *a bond and* (b) *a site percolation.*

A list of applications can be summarised into two groups according to the two main groups. *Bond percolation* finds application in hydrology (movement of water in dam structure, intrusion of sea water in coastal areas, filter beds, *etc*), petroleum engineering (petroleum and natural gas production, exploration, logging, *etc*), chemical engineering (heterogeneous catalysis, flow through packed beds, gel permeation chromatography, porous polymer films used in separation processes, biological membranes, inorganic membranes, *etc*), medicine and biomedical engineering (biological membranes, biological filters, flow of blood and other body fluids, electro-osmosis, *etc*), electrochemical engineering (porous electrodes, permeable and semipermeable diaphragms for electrolytic cells, *etc*), and in communication (performance of communication networks with blockage). *Site percolation* finds applications in permeation through filtration membranes (Bell *et al*, 1995), sieve blinding (Wilkinson and Davies, 1989), membrane fouling, (for example, of the Anotec (ANOPORE) microfiltration membrane) and in the form of pore clogging, effect of back-flushing and crossflow microfiltration have been studied and close agreement with experiments was obtained.

At and above the *critical probability* p_c (Shante and Kirkpatrick, 1971) a percolating cluster, a cluster that spans infinite length, occurs. A mathematical definition of p_c can be found in Bousquet-mélou (1996). The exact value of p_c for some of the uniform lattices can be found by the method of series expansion (Onody and Neves, 1992; De'Bell and Essam, 1983). But for a Voronoi lattice, at this moment, there is no deterministic method. The only possible way is by doing simulations on the lattice as has been done here.

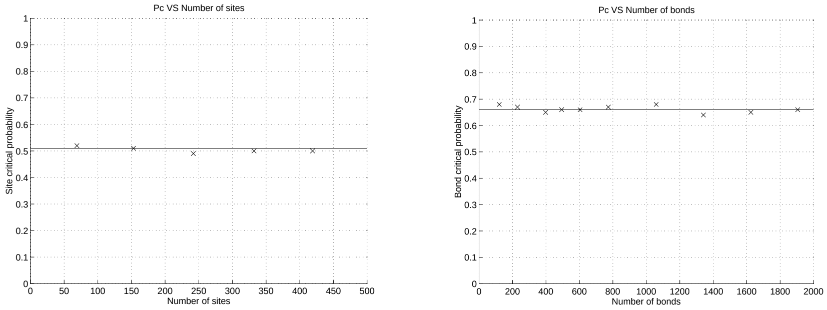
Results

Simulations are made by a developed algorithm (Tiyapan, 1995) based on Monte Carlo method. The results are best presented graphically.

Here $p^s_{c,\text{avg}}$ and $p^b_{c,\text{avg}}$ are the critical probability of site, and bond percolation respectively, averaged over a reasonably large amount of simulation; $n(i)$ is the number of clusters which have i members, where $i \in I$; n is the total number of elements, that is sites for a site problem and bonds for a bond problem; $n(i)_{\max}$ is the size of the largest cluster at a specified p; and p is the ratio between the number of blocked elements and the total number of elements. The coordination numbers for a Voronoi lattice is assumed to be the number of neighbours of each element, averaged over every element within the network. To make clearer a pictorial demonstration of the result, in most of the pictures only the first cluster which percolates is shown.

Discussion

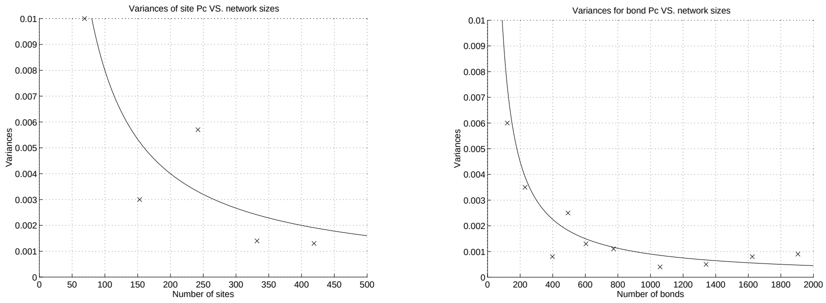
From Figure 3 (a) and (b) it is clear that the critical probability is independent of network size. Thus p_c is a property that is intrinsic for each type of network, and further studies show that it differs from one type of network to another. The idea of infinite cluster is also confirmed.



(a) (b)

Figure 3 (a) *$p^s_{c,avg}$'s plotted against number of sites, (b) $p^b_{c,avg}$'s plotted against number of bonds*

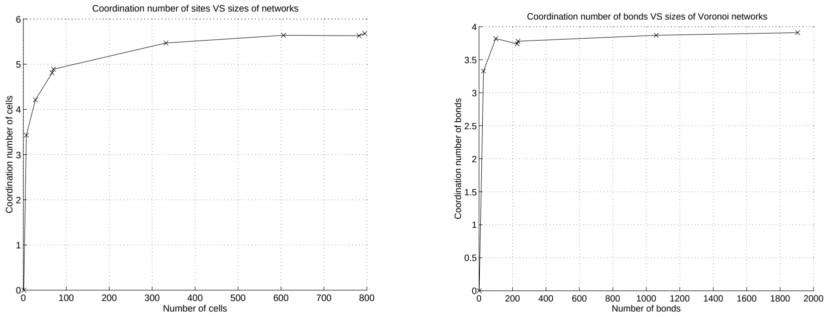
Figure 4 (a) and (b) show that the value of p_c's can be accurately obtained by either doing one simulation on a very large network, or by doing many simulations on a smaller network, since they show that the variance of the results reduces with increasing sizes.



(a) (b)

Figure 4 (a) *$\sigma^2_{p^s_c}$'s, and* (b) *$\sigma^2_{p^b_c}$'s, plotted against sizes of networks.*

Coordination numbers (mean contact numbers) of sites and bonds of Voronoi networks are plotted against network sizes in Figure 5 (a) and (b) respectively. The values seem to be approaching, but never reaching, 6 and 4 for site and bond problems respectively.



(a) (b)

Figure 5 (a) *Coordination number of sites, and* (b) *Coordination number of bonds, plotted against network sizes.*

Figure 6 shows summary of p_c's obtained from all simulations.

Figure 6 [The files for these pictures are lost.] *p_c's obtained for (a) square lattice,* 220 *bonds,* $p^b_c = 0.53$, $p^b_{c,avg} = 0.47$*; (b) triangular lattice,* 305 *bonds,* $p^b_c = 0.33$, $p^b_{c,avg} = 0.34$*; (c) honeycomb lattice,* 111 *bonds,* $p^b_c = 0.65$, $p^b_{c,avg} = 0.64$*; (d) Kagomé lattice,* 174 *bonds,* $p^b_c = 0.55$, $p^b_{c,avg} = 0.52$*; (e) [399 nuclei, 240 cells,* $p_c = 0.47$, *limit* $= 0.04$*] site problem for Voronoi lattice,* $p^s_{c,avg} = 0.51$*; and (f) [400 nuclei, 324 cells, 1044 bonds, bond percolation, limit* $= 0.04$, $p_c = 0.65$*] bond problem for Voronoi lattice,* $p^b_{c,avg} = 0.66$.

Conclusion

Critical probability is a value which is intrinsic to each type of networks. The value for each uniform or random lattices is constant and does not depend on the size of network.

[All the simulation] result shown in this paper was done by MATLAB running on UNIX workstations. The program used for the generation of Voronoi lattices was adapted from Jafferali (1995). A Kagomé lattice can be generated from either a triangular lattice or a honey-comb lattice (Tiyapan, 1995).

References

N. Ahuja and Bruce J. Schachter. *Pattern models.* John Wiley & Sons. 1983.

D. J. Bell, P. Deckmyn and G. A. Davies. Percolating clusters on Voronoi lattices and the relationship to particle fouling on filters. in *Proceedings of ICIAM 95.* ICIAM. Hamburg. 1995.

M. Bousquet-mélou. Percolation models and animals. *European Journal of Combinatorics.* **17**, 343–369. 1996.

K. De'Bell and J. W. Essam. Estimates of the site percolation probability exponents for some directed lattices. *Journal of Physics A. Mathematical General.* **16**, 3145–3147. 1983.

R. Jafferali. A stochastic model to simulate the structure and performance of asymmetric porous media and the growth of animal cell cultures. PhD thesis, UMIST. 1995.

R. N. Onody and U. P. C. Neves. Series expansion of the directed percolation probability. *Journal of Physics A. Mathematical General.* **25**, 6609–6615. 1992.

V. K. S. Shante and Scott Kirkpatrick. An introduction to percolation theory. *Advances in Physics.* **20**, 325–357. 1971.

K. Tiyapan. Computation of fluid flow. MSc thesis, UMIST. 1995.

E. T. Wilkinson and G. A. Davies. The application of percolation theory to the analysis of sieve blinding. *Chemical Engineering Science.* **44**, 459–470.

§ C.2 Abstracts of books I wrote

Interesting English. ISBN 974–346–182–5

This book is written in Thai with the title *Bhạṣạ angkṛiṣ an nàsoncaị* during the time of economic gloom for both myself and my country Thailand. It is built around three major experiences which I consider to be the turning points of my life, namely my studying the Sixth Form English in New Zealand (1983), my acquaintance with a British scholar Amnad Khitapanna since 1985, and my experience in Budapest and Europe (1991). In 104 pages I try to pass on as much as many things as possible those things which I consider valuable from all the three sources mentioned. With my teaching a class in English Literature at the Pradíṣtḥ Center in Bangkok as a catalyst the book was published at the Chulalongkorn University Press (2000) under the trademark *Kittix* which has now become my own. From my study of the Sixth Form Engish comes the sections on false friends, alliteration and assonance, onomatopoeia, jargons, Macbeth, several poetic quotations, and the section on books and movies, from discussions with Amnad the seemingly prefixed words for example *dishevelled* and *unkempt*, and songs by Andrew Lloyd Webber, and from my experience of working in Eastern Europe during 1991 the game *twenty questions*. The rest of the materials are mainly what I used in teaching the English class mentioned. They are tragedies *vs* comedies, Titus Andronicus, Julius Caesar, Romeo and Juliet, poems by Don Marquis, and homophones. Apart from that this book also mentions some word games the examples of which are riddles, aeiou-ordered words (for example, *abstemious*), spoonerisms, tongue-twisters, anagrams, scrabbles, crosswords, Targets, doublets, syzygies, word squares, and acrostics. There is a figure which shows the relative degrees of adjectives on a straight line, and also a Venn diagram showing the comparative domains of the English words *wok* and *pan*, and the Thai word *kada*. I had no laser printers and camera-ready copy of the book was printed from a laser printer at the Control System Engineering group at the Electrical Engineering Department, Chulalongkorn University with the courtesy of my former supervisor Assistant Professor Watharapong Khovidhungij [Wacàràbongṣ́ Ḳhọviḍụrkic] whom I deeply thank. But the quality of the printing is not very good, which results in some parts of the book being difficult to read. To answer the question I have been asked, the picture in the biography section was from the plastic badge which I used to wear when working at Jasmine in Bangkok.

Free translation of English. ISBN 974–346–765–3

The title of the book written above is the English translation of the actual title *Plạe kled́ Angkṛiṣ*, the word *plạe kled́* being an adaptation from the jargon of the old thai sword fighting *fan ḳled́* which means *an improvising practice*, or to do this when the word is a verb. Again the camera-ready copy being printed at the Control System Engineering Department except this time with the courtesy of Dr. Manop Wongsaisuwan who is my senior both at the Electrical Engineering Department, Chulalongkorn Unversity, as well as at the Furuta Laboratory, Tokyo Institute of Technology, whom I consider a wizard in the computational matters, and to whom I thank. This is a book about translation written from my experience in translating and inspired by my teaching in a translating class at the same Pradíṣdḥ Center already mentioned. It is a book of translation both from Thai into English as well as from English into Thai. As such, none of the various examples given is easy and straight forward, and all of them have baffled some translators in the past. It was written at the height of my artistic and literarical ability and at the time when I also enrolled in and attended classes in European (English included) literatures at the Ramkamhaeng University in Bangkok. The book starts off with the phrase *kạrplạe gụe kạrplạe gụe kạrplạe*, literally *translation is translation is translation* but which I deliberately translated as *to write is to write is to write* all languages and writings being but translations of the thought. Historically the romans seldom translated but used Greek in their writings instead, and the greeks never did for theirs were the originals. The book talks about creativity in translation and plagiarism. In general rules may be broken at the right time. Styles of writing are mentioned again without repetition of my previous book. Namely these are the registers and modes of use, alliterations, assonances, onomatopoeias, metaphor, simile, repetition, coupling, rhetoric, allusions, direct and indirect speeches, mythological materials, burlesques, comparative usages of punctuation marks of both languages. The books says that simplicity is the mother of invention. Archaism and parallellism are mentioned, and so are jargons from various disciplines. Examples from literature includes those by Shakespeare, Lewis Caroll, Yeats, and Sundaurbhụ̀. In particular the inventive usages of Cockney English is problematic in translation, no negative pronouns exists in Thai so one needs to try hard to find their equivalences. Some of the jargons I mentioned are those of thai arts, fruits, spices, illnesses, anatomy, fauna, and flora. Some of the humourous clippings from the internet obtained from my friend and computer professional Ken Labinjok in Bristol are also shared with the readers in Thai, some of which needed some explanation to become funny. As for the rest, most of the things I wish to say the book has hardly left out; I would never have thought that I had put so many of them in these 106 pages had I not looked at it again just now. By the way, does anyone happen to know that *novemdecillion* in American is 10^{60} but in English is 10^{114} I wonder. Both my first book and my second one have one problem alike, that is that the computer that I use is of japanese make and can fluently write Japanese but not Thai. One problem which persisted and baffled me throughout is the mysterious disappearance of all the *ỵ*'s. Ironically enough the alphabet *ỵ* is in Thai called *ỵạu phụ̀ying*, literally *w for women*! The page on biography is a little more comprehensive than that of the first book. Having mentioned a picture in the previous one, the picture in this book was taken in front of my house in

Bangkok.

Voronoi Translated. (ISBN 974–13–1503–1)

That succinct title was followed by a more descriptive one, *Introduction to Voronoi tessellation and essays by G. L. Dirichlet and G. F. Voronoi*, which summarises this book. This is the first book, and hopefully the last one, I have written which has a grey cover, *grey* being the symbol of *Gandalf the Great* which in turn rhymes with *Graham Davies* the name of my supervisor and sponsor from end 2000. It is also the first book which I typeset with TeX which I consider complicated but works, while both of the previous two books were typeset using LaTeX which I consider simple and works, the reason for the change being the upgrade of the latter to LaTeX2ε which I consider complicated and (presumeably) works together with the fact that it has TeX working in the background. The switching over thus helps reduce one shell around the equivalences of the only desirable attribute left which is that of *complicated but works.* Here the two venerable alphabets *v* and *n* rules throughout, starting from the title *Voronoi*, the name of a russian mathematician of the nineteenth century, then the picture taken in *Vienna* on the front, and that in *Venice* on the back covers; if you look carefully enough, somewhere in the book you will find another picture of a stone masonry with a caption saying *Verona* which Professor Davies decidedly says is not a Voronoi tessellation. The introduction gives some historical backgrounds and pictures of various orders of covering lattices of the Voronoi tessellation in two dimensions, which can be used to represent for example a realistic picture of the conglomeration of grains within some kind of matrix. The triangular and the hexagonal lattices are examples of dual lattices. The picture of the hexagonal and kagome lattices superimposed on top of each other is not a picture of the Eden Project in Cornwall but an example of covering lattices. This is not a free translation of the seminal works by G. F. Voronoï and G. L. Dirichlet. I tried to retain the original as much as possible, namely where I thought the grammar in the original was incorrect or the structure of the sentence is convoluted I coined up the equivalence, though this is by no mean how I always translate. As a result the translation is necessarily more difficult to read than it would have been had it been freely summarised. My reason for doing it in this style is firstly in order to transfer as much nuances across as possible, and secondly simply because I am not a mathematician and therefore in no position to summarise the work of one. Last but not least must be my own shortcoming and inexperience in translating from French and German. The preface of the book gives a brief history of the *Journal für die reine und angewandte Mathematik* while the introduction that of the Voronoi tessellation. Also one knows from the latter that the name *kagome* is a Japanese word which means *basket interstice or pattern*, the fact that I realised while reading the Japanese language in Tokyo. *Me* is the word for *eye* as well as *pattern*, and in Japnanese *kago* means *an intersticed basket* as much as *taklạ̀* does in Thai. There is hardly discipline in which Voronoï tessellation has not found applications and the list of relevant areas is endless, ranging from management to computation to physics. This book has 282 pages of contents.

Percolation within percolation and Voronoi Tessellation. ISBN 974–91036–1–0

This book contains both my original thesis, as submitted to UMIST in March 2003, as well as the translation of Schumann's Liederkreis, Op. 24, from German into English and Thai that I did for Aṃnạc Gịtàbarrṇạ who was once the president of the Bangkok Music Society. The preface says that this is not the final version of the thesis. In fact only God knows what the final thesis will look like. There are 585 pages in all, and approximately half of these are appendices. As usual I write this book and thesis in TeX. As such it is my most complex project on TeX to date. A TeX macro I wrote is given which in its packed form is over 600 lines long. It predominantly deals with how to display pictures, as well as index and cross references. Another TeX macro is also given which contains my works on transcription systems of various languages from Lanna to Chinese to ASL, that is the American Sign Language. Other programs given are what I had used on Matlab to produce the results mentioned in the contents. The introduction and literature survey are eclectic verging on desultory, which is partly due to the multidisciplinary nature of the work. Parts of the introduction contain the contents of the emails I wrote to various people concerned when there were serious problems affecting the progress of the work. Future researchers and academicians may be able to learn from this and avoid facing or incurring them. The literature survey is distributed throughout the book instead of centralisedly staying together in one place. I give some flavour of the different disciplines concerned. These include mathematics and geometry, physics and percolation, statistics and statistical physics, tessellation and Voronoi Tessellation, quadratic forms and quadratic equations. Observing the clusters and their development is important in the study of percolation. The sudden accelerated growth of the cluster size heralds the onset of percolation. Applications of percolation theory to traffic congestion and economics transition are also mentioned. The value of percolative thresholds are obtained from the simulation for the Voronoi Tessellation, 2 homohedral tilings, cities traffic, and n-gons and spheres in continuum. Also in the appendices are the collection of my writings both technical and nontechnical since 1994. The range of the topics of these is rather wide, for example antimony trioxide extraction, cyberspace, computer worm, economic and traffic modellings, variable structure control and singular perturbation. There are also articles on languages. And I have included translations I have made of the seminal works by Dirichlet and Voronoi on the Voronoi Tessellation.

Thai grammar, poetry and dictionary. ISBN 974–17–1861–6

In this book I present my a new transcription system for Thai using the roman alphabet. The tonal symbols used in this system are adopted from the *pīnyīn* system of Chinese. The consonants either follow the conventions in Pali and Sanskrit or are invented by means used in these two languages. These latter ones are namely /[ḳh, z, ỵ, ḍ́, d́, ḅ, fh, f, ʻ.] and *ḥ*. Not only all the consonants but also all the romanised vowels are standardised such that they represent an isomorphism, that is one-to-one and onto, of the Thai script. The usefulness of this is immense because one can then write in English and yet refer to any Thai words no matter how difficult with no ambiguities and with ease. In order to prove the usability of this system, I have used it to write no less than 100 examples of all the kinds of poem that I know. Also the dictionary at the end of the book in a way provides another proof of its user-friendliness. For many reasons some of which are beyond my control while others could and should have been avoided had I been more experienced, there are *many* spelling errors in the book. Some of these are the following, each correct word being given following the page number where it appears, and in brackets its meaning and misspelt form: 13, Rạmgaṃhạeng ([name of a king], Rạmgamhạeng), 15, khoad́ (bottle; khoadt); 18, pàṃṃ̈pọ́eë (lapsesome; pàṃṃ̈pọ́ee), dharrm (dhamma; damr), dharrmȧd́ạ (normal; damrȧd́ạ); 19, ạu ([a long vowel]; ạ̇u [a hidden vowel]) (this error, since it is in the macro, occurs everywhere, therefore there are two possibilities if you see an *ạ̇u* in the book and most of the times it means this *ạu*); 22, ywạk (to lust; yạk), d́uan (moon; d́oen); 24, lāo (a pen for animals; lạ̄); 25, tàung (have to; tạ̀ng), dondạn (enduring; tontạn), saṃbhao (a Chinese junk; saṃghao); 26, aukmạ (coming out from; aumạ); 27, graị (who; gaị); 29, d́ị (good; dị), tọayặng (example; tụayặng); 43, c_{1l} ([defined to be one of the 24 low leading consonants]; $c = 1l$); 45, $\circ_1$ ([the first syllable to be rhymed with in this stanza]; $\circ, 1$), ḅăụ (not; băụ); 46, ḅauk (tell; bauk); 47, ngạn (work, hgạn), glạ̄ew (miss by a hair breadth; gạ̄ew); 48, graị (anybody; grai); 51, Ịk (and also; Ḷk), Ḅanggom ([homage done while sitting on one's own knees]; Ḅanggkom); 55, brāum (together with; grāum); 57, jọ̀ay (to help; jọ̀y); 58, prayọjṇ̇a (usefulness; prayọjṇa), lavẹ̄n (avoid; lavẹ̀n); 62, tặe (but; tặen); 64, d́ek (child; dek); 65, Sạmon ([a literarical character]; Saq.mon); 69, kaỵjạ (marijuana; kayỵjạ); 71, mȧhọʻ.ạr (enormous; mȧhọlạr), sȧnạmkịʻ.ạ (sport stadium; sȧnạmkịlạ), sȧmạgom (society, club; lȧmạgom); 72, ỵhăị (big; ỵhāi); 74, jāo (morning; jạ̄); 75, lòang (breach; lọ̀ng); 77, thoed́ (Please!; toedz.); 80, śọkạ (sadness; śọkjạ); 81, krạḅvhài (pay homage to; klạḅvhài); 84, Bhȧgȧvadȧgịtạ ([the Sanskrit epic *Bhagavad Gītā*]; Bhȧgȧvatȧgịtạ); 87, muddạ (print; mudadạ); 88, durẹś (feel disgusted; dhurẹś); 90, bloen (enjoying, absorbed; boen); 97, ywạk (want; yhạk); 100, ḅăụ (not; ḅặu). The aleph symbols are for example, in pages 41, 5.19; in page 42 respectively 5.1 and 5.2. As quoted at the end of the book, 'Es gibt Dinge, die findet man nur, wenn man etwas ganz anderes gesucht hat', may be you will find something good from the book after you have searched for all the errors it contains! I shall be happy if you let me know this is the case.

A Lanna in town. ISBN 974–17–1860–8

This is the first one of the pentalogy I wrote during the first half of 2003. The others are *A Kiwi Lanna, A British Lanna, Edokko no Lanna* and *The Siamese Lanna.* All these five books have exactly the same length, that is 112 pages two of which contains no contents. A simple process of addition tells us that they would have formed a 550-page volume if put together into one book. I want to do this in the future when I shall correct all the errors in them, spelling and all. These books contain only two themes one of which is the things I relate from my experiences while the other what they teaches me and what research I have done in order to understand them. The part which provides or tries to give understandings is definitely a more interesting one. But it is difficult to put it into the abstract without going into such a great length as including the whole book. The setting of this book begins in Thailand where I was born. Then it goes on to New Zealand, Bangkok, Surrey, Budapest, Manchester, and Tokyo. These are only a nonredundant listing of those places where I have lived. It is already too complicated without my trying to put all the other places to where I have visited.

A Kiwi Lanna. ISBN 974–91237–3–5

I was lucky to have become a part of the *saṃnakd́ạḅ* (sword school) Śrị Ȧyudhyạ and I am lucky to be here in New Zealand studying the sixth form. The former experience has taught me compassion, the latter how to do research. This last one is basically how to read and write. To read and write is not difficult, but to know *how* to do so that is another thing. It amounts to knowing how to think for yourself. Now I know, for instance, that good books always have two themes, and the more profound one is always the one in the background, a more abstract one and never a story. This is why it is always difficult to turn these books into a film where you can only show the less important theme and it is nearly impossible to express the remaining one. I learn all this in my Sixth Form English class here. Night and day I recite lines from Macbeth. Before I came here I had read Julius Caesar, but this is by far better. Not only this, I also study Art History, Mathematics and Music. Now I am a writer, thanks to Mr Lonsdale my English teacher then. I am also a mathematician, thanks to Mr Thompson who taught me the subject. I can not wait to become an artist and a composer which are what the other two of my teachers, the latter of whom is Vicky, had taught me to be. In this book, however, I only talk predominantly about literature.

A British Lanna. ISBN 974–91237–4–3

It is England which turns me into a Christian. This is by no means a compliment to the country, but it certainly is one for the people. At the Moberly Hall where I live I read the translation of Hugo's Les

Misérables, the book which greatly moves me and even before having been through the whole of it had made me profess Christianity. And here is my interpretation now, that God is the Superset. Therefore, if we believe in the one and only one Creator by definition we have Christ in our heart but mathematically speaking we are the Me in Christ in the Universe in God. Here I say Christ as a definition as distinguishable from Jesus who is also a person. A similar definition to the one above, and which is what I prefer now, is this, that it is instead a Me in Universe in Christ in God. It is easy to see from this, that we should explore the universe because they are the same family as ours. In other words they are one of us, though they may not know it, so we need to find and preach the Gospel to them. England and my teachers here have taught me many thing. It is also here that I volunteer in the Community Action and through it got to know Clair. But I had better leave you to read about her yourself.

Edokko no Lanna. ISBN 974–91341–9–2

The word *Edokko* is Japanese. It means 'Edokkite' or 'Tokyoite' in English. *No* is Japanese for 'of'. It also renders a noun preceding it into an adjective, the function which is obviously the case here. The story is set for most parts in Japan, but also contains the excursions I made from Tokyo to New Zealand, Thailand and the US. After having been in Japan for one year, and only after then, I have come to love the country so much. The first instance this happened was when I was travelling on a train across the countryside through the inner part of Kyūshū. Thereafter this has been strengthened by my working as an interpreter in Tokyo and Yokohama, my learning the Japanese archery or Kyūdou, and my having fallen in love with the fermented beans *nattou*.

The Siamese Lanna. ISBN 974–91341–8–4

I really thought when I was writing this book that I was going to die before I could see another book published. This is why I have put into this single volume all the knowledge I have about the Siamese martial arts as well as everything I have learnt from my *saṃnakd́ạḅ* (sword school) Śrị Ȧyudhyạ. I feel that I have already told you everything in this book, so there are nothing more I may add now. You should somehow find the book and read it if you are interested in the Siamese heritage. Because my great-grandmother was an Ayudhyaite, and because my Thai is way better than my Lanna, I identify myself as more a Lanna Siamese (or perhaps a Lannaese Siamese) than a Siamese Lanna despite what the title of the book says. The poem in pages 21 and 22 is quite old and is very dear to me. I have known it since I was a child, even though I do not know who wrote it. I would have liked to think it is very old, dating back to the Ayudhya Empire of Siam, but it can not be that old. Because it calls the name of the country by its new name *Daiy* (Thailand) instead of the name used until the beginning of the 20th century, *Syạm* (Siam), it means that it must be less than a century old. What I plan to do next is to do more research into, and write about this and other arts of fighting in more detail.

D

§ D. Translation

§ D.1 G. L. Dirichlet, 1848

On the reduction of positive quadratic form with three indeterminate integers.

([Lecture in physical- mathematical class meeting of the academy, on $31^{st} July, 1848$ †]), [by G. L. Dirichlet]
[translated by K N Tiyapan]

It is well known that Lagrange had pointed out for the first time that every binary quadratic form reduces, ie. can transform into another equivalent one the coefficients of which satisfy certain inequality conditions, and at the same time had proven that in every class of positive forms there always exists only one such form, so that in this case the various values of a given determinant corresponding to reduced forms can serve as the representatives of the different classes. Later on after in the "Disquisitiones arithmaticae" the ternary form were looked at from a general point of view did it become necessary for the further development of this theory to extend the study for the positive binary forms by Lagrange to the ternary ones, ie. to find out such inequality conditions between the coefficients that would satisfy one and only form in all classes.

This expansion linked with great difficulties is achieved by *Seeber* in a work specifically devoted to the positive ternary forms, the principal contents of which settles it and which *Gauss* characterises in a most interesting announcement † as follows:

> *We must do full justice to the spirit of the thoroughness by which these facing ‡ us have gone through, and when we for all that have to feel sorry that a great and perhaps much discouragingly complicated nature is attached to it, here the solution of the problem takes 41 pages and the proof 91 pages, thus we will see this by no means as a respected criticism. If a difficult problem or theorem to solve or to prove exists, then the first seeming idea is always to be recognised as a step that a solution or a proof has been found after all, and the question whether this were not of an easier and simpler way would be possible as long as in so doing such a futile question is not considered as of practicability. Therefore we look upon it as untimely to dwell on this question.*

The great complication of Seeberian method has for a longtime stimulated me to set up the theory of reduced ternary forms by a simpler method. As I now allow myself to communicate to the class the result of my effort directed towards this, I think in the interest of briefness and, if I could say so, of the lucidity of the presentation, to have to abide by the geometrical form, in which I have conducted the investigation to which I have laid down as basis the noteworthy relations which occur among the quadratics with two or three elements and with known spatial forms. I begin with the explanation of the outline already given by Gauss in the mentioned announcement on these relations.

§1

The ternary form:

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'xz + 2c'xy = \varphi, \tag{1}$$

in which we regard x, y, z as first, second, third element, called positive when φ does not become negative for real values of these elements; in one such form the coefficients:

$$a, b, c$$

are always positive, while the coefficient combinations:

$$a'^2 - bc, b'^2 - ac, c'^2 - ab, aa'^2 + bb'^2 + cc'^2 - abc - 2a'b'c' = -D, \tag{2}$$

the last $-D$ of which is called the determinant of the form, are negative. † Owing to these conditions there are always three through the equations:

$$\cos\lambda = \frac{a'}{\sqrt{bc}}, \cos\mu = \frac{b'}{\sqrt{ac}}, \cos\nu = \frac{c'}{\sqrt{ab}}$$

fully determined acute or obtuse angles λ, μ, ν, from which a three-edged corner can be built, here the condition necessary to this:

$$\cos^2\lambda + \cos^2\mu + \cos^2\nu - 2\cos\lambda\cos\mu\cos\nu < 1$$

with $D > 0$ coincided. Here nevertheless with the same angles λ, μ, ν two corners symmetrical to each other could be built, therefore we will agree to always choose the corner from these two, with which the edges, as they lie opposite to these angles in sequence, follow one another from left to right with regard to a straight line directed from the vertex 0 to the inside of the corner which can be thought of as going upward. If we now look at the three edges as the positive axes of a coordinate system we could connect the entire infinite space with our form in which we view the product $x\sqrt{a}, y\sqrt{b}, z\sqrt{c}$ as the coordinates of an arbitrary point of the space, and then φ expresses the square of the distance of this point from the vertex, or more general still the square of the distance of two points, the correspondent coordinates of which have those products to differences.

If one establishes now with three new indeterminate elements x', y', z' the linear expressions:

$$x = \alpha x' + \beta y' + \gamma z', y = \alpha' x' + \beta' y' + \gamma' z', z = \alpha'' x' + \beta'' y' + \gamma'' z', \tag{3}$$

---

† Crelle's Journal, V. 20, p. 312

† Disquisitiones arithmeticae, art. 271

of which only one restriction shall take place, that the determinant set up from the 9 coefficients α, β, γ, α', β', γ', α'', β'', γ'':

$$\alpha\beta'\gamma'' + \beta\gamma'\alpha'' + \gamma\alpha'\beta'' - \gamma\beta'\alpha'' - \alpha\gamma'\beta'' - \beta\alpha'\gamma'' = E \tag{4}$$

is not zero, then φ changes into a new form φ', with regard to which all correspondings shall be indicated with the accented alphabets. If one lets the new form again correspond to an infinite space, then through it two infinite spaces connect point for point with each other, while every two points correspond to each other when in the expressions of their coordinates:

$$x\sqrt{a}, y\sqrt{b}, z\sqrt{c};\ \ x'\sqrt{a'}, y'\sqrt{b'}, z'\sqrt{c'}$$

the elements x, y, z and x', y', z' are linked with one another through the equation (3). If the expressions just written are the coordinate differences for two pairs of corresponding points, then apparently the same relation among $x, y, \ldots$ still holds, out of which from the above and as a result of $\varphi = \varphi'$ it follows immediately that the distance of every two points of a space is equal to the distance of corresponding ones of another. The two spaces connected with each other are therefore either congruent or symmetrical, ie. they can, while the beginning points 0 and $0'$ are laid on each other, come to such a position that either every point falls on its correspooonding one or on the opposite point of the latter, when we call for short opposite points two points of the same space which lie from the beginning point at the same distance and in the opposite direction. In order to decide which of these two cases takes place, one has lines to draw in the one space from the vertex to three arbitrary points, and then to investigate whether the straight lines drawn in the other from its vertex to the corresponding points present a corresponding series or the opposite one. If one takes for example in the second space the lines from the points with the coordinates:

$$\sqrt{a'}, 0, 0;\ 0, \sqrt{b'}, 0;\ 0, 0, \sqrt{c'}$$

drawn, lines falling on the positive axes of the second space, then these follow one another from the agreement dealt with above from right to left. For the corresponding points in the first space one has the coordinates:

$$\alpha\sqrt{a'}, \alpha'\sqrt{a'}, \alpha''\sqrt{a'};\ \beta\sqrt{b'}, \beta'\sqrt{b'}, \beta''\sqrt{b'};\ \gamma\sqrt{c'}, \gamma'\sqrt{c'}, \gamma''\sqrt{c'}.$$

In order to determine whether the lines directed to these points follow one another from left to right, ie. as the axes of the first space, or follow in the reverse order, one can make use of the theorem which is known or easily derivable from known properties ‡, from which the straight lines drawn to the three points $(\xi, \eta, \zeta), (\xi', \eta', \zeta'), (\xi'', \eta'', \zeta'')$ present the same series as the axes of ξ, η, ζ or the opposite one, according to the determinant built from the 9 coordinates, when one gives the term $\xi\eta'\zeta''$ in it the positive sign, is positive or negative. For our case this determinant becomes $E\sqrt{a'b'c'}$; therefore congruence symmetry holds according as E is positive or negative.

Til now the elements x, y, z had arbitrary values. If we let them now only further mean integers, then instead of the integral space we have an infinite system of points parallelly arranged, ie. a point system of which through the intersections of three lines parallel equidistant planes would be created. If we assume now further that the substitution coefficients α, β, γ are also integers and E has the values ± 1, then every integral combination x, y, z would represent an integral combination x', x', z' and vice versa. The parallelepipedal systems thus connected with one another would as a result coincides with the other or with the opposite points of the latter. Yet, here the opposite points of points of one such system again make the same system, the two cases are not different from each other, and this becomes evident also from the circumstance that φ' remains unchanged when one takes α, β, γ with opposite signs through which E changes into $-E$. The two systems are therefore always congruent, and one sees that systems which correspond to two equivalent ternary forms φ and φ' are the same spatial structure in two different patterns. Conversely equivalent forms represent any two different parallelepipedal patterns of the same system. If one takes namely any one point of the system as the common starting point, then one has between the coordinates relation to the two axis systems and therefore also between the elements x, y, z, x', y', z' proportional to them linear equations without constant term, ie. equations of the form (3), and here from our supposition, when x, y, z are integers, x', y', z' must also have the same characteristic and vice versa, therefore it follows that $\alpha, \beta, \gamma, \ldots$ are likewise integers and that $E = \pm 1$. On the other hand one has for the homogeneous entire values of the elements the equation $\varphi = \varphi'$, which accordingly also identically takes place, q.e.d.

Similar interrelations occur between a positive binary form:

$$lx^2 + 2mxy + ny^2$$

and a system of points parallelogrammatically arranged. One takes here two axes leant against each other under the angle θ determined through the equation $m = \sqrt{ln}\cos\theta$, while one always invariably proceed with the discrimination of these axes and for example chooses the second on the left-hand side of the first one, after a fixed side of the plane is denoted as the higher one and $x\sqrt{l}, y\sqrt{l}$ viewed as coordinates, one would obtain a system of points completely determined through the quadratic form, which could be considered as the intersection of two series of equidistant parallel lines. If then between two forms the so-called proper equivalence takes place, so that $\alpha\delta - \beta\gamma$ in the substitution equations $x = \alpha x' + \beta y', y = \gamma x' + \delta y'$ is equal to the positive unity, then the corresponding systems can be brought to the coincidence through movement in the plane, while in the other case where $\alpha\delta - \beta\gamma = -1$, to say in general, one of the systems must be shifted for this purpose.

§2

After we have established in the foregoing the connection between the quadratic forms and certain geometrical patterns, there are a few further properties of these patterns to develop, whereby we would for short call a system of points arranged parallelogrammatically or parallelepipedally a *system of second* or *third order*, and infinite series of equidistant points in straight line a *system of first order*.

---

‡ Disquisitiones generales circa superficies curvas auctore, C. F. Gauss §2. VII

It seems that the common character of all three types of the system consists in that when such a system is brought into another position through a movement without rotation, which we wish to know a displacement of, that a point of it changes into the position occupied by another in the beginning, the same happens for all points, therefore that the system in its new position fully coincides with the system in the original one. It can be easily proved that the movability just discussed completely characterises all three types of the systems, and that a system endowed with this characteristic, when it lies in a plane and contains three points not lying in a straight line such that finally a system contains points at least four of which are not found in a plane, will be respectively a system of first, second or third order.

If one has for example a system of points which lie all together in the same straight line, and a and a' are two adjacent points of it, then through a displacement through which a gets to a', a' would get to a'' which is as far from a' as a' is far apart from a; the point a'' therefore also belongs to the system, and the system has no point between a' and a'', here one such point would be known before the movement between a and a'. Here this inspection could be pursued for both sides in the indeterminate, therefore the assertions is proved.

Now let two adjacent points a and a' be in a planar system with the characteristic feature of movability, so that no point of the system is found in the line aa' between a and a'. Here through the displacement from a to a' the infinite straight line aa' moves along by itself, therefore it follows that the entire points of the system in this straight line makes up a system of first order $\ldots {}''a'aaa'a'' \ldots$. Here then from the assumption the system still has at least one point outside this straight line, therefore let b be one of the points closest to this straight line. If now enters a displacement through which a gets to b, then the system of first order changes into the new position $\ldots {}''b'bbb'b'' \ldots$ and belong in this position to the original system; it is immediately clear that a point of the system can neither be found among the points $\ldots, {}''b, {}'b, b, b', b'', \ldots$ nor among the lines $\ldots {}'bbb' \ldots, \ldots {}'aaa' \ldots$. If one concludes directly, one sees that the entire system can be parallelogrammatically aranged, and that one can choose $aa'b'b$ for a basic parallelogram of it. We further add that through the given construction apparently all parallelogrammatical patterns of which the system is capable could be obtained. It follows from this that the choice of a' up to the obviously necessary restriction that no point lies between a and a' is totally arbitrary, and that b can be taken arbitrarily in the nearest parallel line.

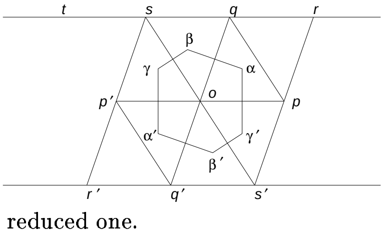
One has finally a system with the characteristic feature of the movability which contains points at least four of which are not lying in the same plane, therefore one lay a plane through any three points of this system not lying in a straight line. Here through any parallel displacement effected with this plane this is moved into itself, consequently points found in this plane build a system of second order from the previous system. As a result one has partitioned this system parallelogrammatically somehow or other, one takes one of the remaining points of the spatial system which lie closest to the plane, and administer a displacement to the system through which an arbitrary point in the plane comes to the point chosen well outside that plane. Through repeated application of this [displacement] and through the movement opposite to it one apparently obtains a parallelepipedal pattern of the given system, and it is immediately clear that the construction specified has the due generality, here the choice of the first plane, the pattern of the system of second order in this plane and finally the choice of points in the neighbouring plane can happen at will.

In the end of this paragraph we will point out further that, as one also partition the same system of second or third order, the parallelogram or parallelepiped lying at the basis of the respective partitioning always retains the same capacity, the geometrical consequence of the sentence is that equivalent forms have the same determinants. If one imagines namely in the plane of a system of second order a line returning to itself, for example a circle line, designates with z the surface area enclosed by it and with s the number of points in the inside of the line, in the course of which it makes no difference whether one wants to include the points on the periphery or not, then obviously the quotient $\frac{z}{s}$ has with growing radius the capacity of a basic parallelogram to the boundary, from which, here s and z are independent of the type of the pattern, the theorem for systems of second order becomes evident. Totally in the same fashion follows the soundness of the assertions for spatial systems.

§3

We will now point out that a system of second order would always admit partitioning by a basic parallelogram, the sides of which are not larger than its diagonals.

I. Let o be an arbitrary point of the system. The remaining points of this system always lie pairwise in the same distance and opposite direction from o. Now let p be one of the points of the pairs, for which the distance from o is smaller than for every other pair. The same smallest distance holds for more than one pair, therefore one would choose p at will in one of these. The given system consists of an infinite quantity of systems of first order congruent among one another and of the same distance, one of which is that for which o and p belong. In one of the two adjacent to this latter one, one takes the point q which is next to o, or, supposing the same shortest distance should occur for two points, arbitrarily takes one of the two. The parallelogram $poqr$ thus obtained has the desired property, here in accordance with the construction $op \leqq oq$, $oq \leqq or$, $oq \leqq os = pq$. A basic parallelogram which satisfies these conditions shall be called a reduced one.



II. We have now the relation between one such parallelogram and the planar system in which it belongs to establish. If $poqr$ is a reduced parallelogram we would be able to, without breaking the generality, assume the angle poq as not obtuse, here in the opposite case the angle at o for the parallelogram adjacent to the same structure is an acute one, and likewise we could assume $op \leqq oq$. Thereupon $or > oq$ is apparent, and we have only the condition $pq \geqq oq$ still to consider. If this is supposed and if we put for the reduction $op = \sqrt{l}$, $oq = \sqrt{n}$, so that consequently $l \leqq n$ the connection of our parallelogram to the entire point system would possibly be described to the effect that the minimum of the distance of any point of the system from

o is equal to $\sqrt{l}$, and that after one has chosen a point at this distance, in all distances still remaining, ie. outside the straight line drawn from o to the former one, the second minimum is equal to $\sqrt{n}$. The precisely stated holds true all in general; what we now add, that namely the first minimum only occurs for the point p (when we always only choose one of two opposite points), the second only for q, holds true with the following exceptions:

1. If $op < oq$, $oq = pq = os$, then the first minimum takes place only for p, the second for q and s.
2. If $op = oq$, $oq < pq = os$, then the minima are equal, and one can exchange p and q with one another.
3. If finally $op = oq = pq = os$, then one can choose one of the points p, q, s as first point and then one of the remainings as second one.

In order to demonstrate the precisely asserted, we have obviously, the opposite points are always equally far from o, only to point out that q lies closer to o, *firstly* than all remaining points in the straight line sqr, with exception of the point s, the distance from o of which according to the assumption is equal to $os = pq \geqq oq$, and *secondly* than all points of the subsequent parallel lines.

Here $pq \geqq op$, $pq \geqq oq$ and the angle poq is not obtuse, therefore the triangle opq and consequently also the oqs congruent with it has no obtuse angle; therefore the perpendicular dropped from o on qs lies between s and q (inclusive), with which
one sets:

$$\cos poq = \frac{m}{\sqrt{ln}},$$

where consequently m is not negative, therefore one has:

$$\overline{pq}^2 = l - 2m + n \geqq \overline{oq}^2 = n,$$

and hence:

$$2m \leqq l, 2m \leqq n, 4m^2 \leqq ln.$$

If one further sets the square of the height of our parallelogram ($op = \sqrt{l}$ regarded as base line) equals k, one obtains for the square Δ of its volume:

$$\Delta = lk = ln - m^2 \geqq \frac{3}{4} ln,$$

and hence:

$$\sqrt{k} \geqq \frac{1}{2}\sqrt{3n}.$$

According to this the second line is at least $\sqrt{3n} = oq\sqrt{3}$ away, and the second point is also established.

III. Here the successive minima $\sqrt{l}, \sqrt{n}$ are decided through the system as such and are independent of any fixed pattern and on the other hand, as we have just seen, correspond in quantity with the sides of the reduced parallelogram, therefore one sees that when the system permits various patterns of this fashion, the sides of the reduced parallelograms will always contain the value $\sqrt{l}$ and $\sqrt{n}$. One would essentially obtain as a result all possible basic parallelograms if one draws lines from o to all adjacent points (always with exemption of the opposite points) and then takes the nearest or the two nearest points in one of the respective nearest parallel lines; and here from the definitively demonstrated (II) this nearest or these nearest points lie closer to o than all points of the subsequent parallel lines, therefore one can see from the condition that the second points are to be taken in the first parallel line. Therefore all possible patterns would be produced if one successively connect o with all point pairs for which the successive minima take place, from which at once follows with consideration from (II) that in general and in the second one of the singular cases obtained then there is only one such pattern, in the first and third exception case however there are respectively two and three patterns of the systems.

In our present reference the precisely obtained singular cases correspond with the suppositions $2m = l < n$, $2m < l = n$, $2m = l = n$.

§4

We have so far only dealt with properties of the geometrical structures which is to be looked at from the theory of forms as the constructive representation of well known theorems and are already indicated in the article cited in the introduction. It is now to solve another problem of another kind, the problem namely when a system of second order is given and a fixed point o of it is examined, to determine the part of the plane every point of which lies nearer to o than to any other points of the system. Here the condition that a point does not lie farther from o than from any other v, therein consists that the point with o on the same side of the perpendicular drawn up in the middle of ov, so we would consequently have o to combine with all remaining points of the system and the convex polygon built from all corresponding perpendiculars to construct. But from these perpendiculars in infinite quantity only a limited number comes into question, while the remaining ones do not meet the polygon determined by it. We abide by all suppositions attended to, so that consequently $op \leqq oq$ in the reduced parallelogram ($poqr$), the angle poq is not obtuse and opq, oqp are acute. This supposes, it is easy to understand, that one has only the six vertices p, q, s, p', q', s' of the four parallelograms meeting at o to take into consideration, and that the perpendiculars corresponding to s and s' and the building diagram in the particular case, when poq is a right angle, only touch, which then the same happens for the perpendiculars corresponding to r and r'. If one draw the straight line pq, os, $p'q'$, os', one obtains the congruent triangles:

$$poq, qos, sop', p'oq', q'os', s'op.$$

If one consider only the points p, q, s, p', q', s', one has to draw a perpendicular in the middle of the straight lines going from o to these points, ie the same construction to make as when one wished t find the middle point of circumscribed circles for the designated triangles. Here no obtuse angle is found in the triangles, therefore each two successive perpendiculars not outside the corresponding triangle intersect. One obtains therefore the hexagon $\alpha\beta\gamma\alpha'\beta',\gamma'$ with the centre o and equal opposite angles and sides as the space, inside of which every point is less far apart from o than from one of the points p, q, s, p', q', s', and one is easily convinced that, with exception of r and r', the perpendiculars corresponding to the remaining points do not meet our hexagon. This requires, as a result of symmetry, only for the points in and above the line pop' to be established. For the former ones it is clear; for the latter ones it would hence appear that their distance from o is larger than the diameter of the circle traced around the hexagon. If one designate the square of its radius ρ, then:

$$4\rho\Delta = ln(l - 2m + n),$$

from which as a result of $2m \leqq l, 2m \leqq n, \Delta \geqq \frac{3}{4}ln$, it follows:

$$4\rho \leqq \frac{4}{3}(l - 2m + n) \leqq \frac{8}{3}n.$$

Here now for the points of the second and the subsequent parallel lines, as already remarked, the square of their distance from o amounts to at least $3n$, therefore there still remain simply the points in $tsqr$ apart from s, q, r to examine. From all of these none is closer to o than t, for which the square of the distance is equal to $4l - 4m + n$, and that this is larger than 4ρ, one immediately sees when one multiplies with Δ and then looks at the inequalities $2m \leqq l \leqq n$. As for the point r, one is also convinced by the same manner that the square of its distance from o is equal to $l + 2m + n > 4\rho$, the only case excluded, where $m = 0$, in which the corresponding perpendicular touches. It is thus demonstrated that every point in the inside of the hexagon $\alpha\beta\gamma\alpha'\beta'\gamma'$, and only one such hexagon, lies closer to the point o than any other of the system. On any side the distance from o would be equal to the distance from a second point, which for example for $\alpha\beta$ is the point q, and every vertex of the diagram is of the same distance from o and another two other points of the the system. The latter statement undergoes a modification only in the special cases when the angle poq is a right angle; thereupon β and γ as well as β' and γ' coincide, and the hexagon turns into a rectangle, of which the corner from o and another three other points of the system are equally far apart.

It goes without saying that one will always obtain the same hexagon whose reduced parallelogram one also lay as foundation of the construction in the singular cases, where more than one exists, just as also that the hexagon or quadrangle corresponding to all the points of the system are congruent and cover the whole plane of it.

We notice further that, as one is easily convinced, the expression:

$$\rho = \frac{ln(l - 2m + n)}{4(ln - m^2)}$$

decreases when one therein, assuming l and n constant, allows m to grow from zero up to its limit $\frac{1}{2}l$, so that consequently:

$$\rho \leqq \frac{1}{4}(l + n) \leqq \frac{1}{2}n. \tag{1}$$

Also in addition the following inequality takes place:

$$2\Delta(n - \rho) \geqq ln^2, \tag{2}$$

the soundness of which is immediately evident when one multiplies with 2, moves everything to one side and then applies $\Delta = ln - m^2$, $4\Delta\rho = ln(l - 2m + n)$, by the mean of which it changes into $ln(l - 2m) + 2mn(n - l) \geqq 0$.

§5

We come now to our true topic and have to prove that every system of third order can be arranged according to a parallelepiped whose faces are reduced parallelograms and whose edges, from which every four are equal to one another, do not exceed their diagonals.

After one has fixed an arbitrary point (0) of the systems, one would choose in pairs of opposite points for which the distance from (0) is a minimum, or when the minimum of the distance exists for several pairs, would arbitrarily choose a point (1) in one of these pairs. From all points outside the straight line (01) one would again choose one of the two nearest (2), through which again the selection under several pairs, for which the same shortest distance takes place, can be arbitrarily made. Here in the whole system, with exception of the points in (01), no point lies closer to (0) than to (2), so the same is valid also for the plane (102), and (102) is a reduced parallelogram for the system which contains this plane (§3, III). One now takes in one of the two nearest parallel planes the point which is closest to (0) or, when the minimum occurs for more than one, one of the nearest ones and connect (0) with the chosen point (3), therefore the parallelepiped would with the edges (01), (02), (03), as is easy to see, suffice the requirement. Next from the construction it follows: (01) $\leqq$ (02) $\leqq$ (03). Here for the bases of the parallelepiped (we would always indicate as such each face opposite to one another in which are found edges two of which do not exceed the third one in size, and the term side faces apply to the four remaining ones) it is already proven that they are reduced, therefore we have in virtue of the precisely noted doubled inequality only to point out further that the four diagonals of the side faces, just as the four diagonals of the body, are not smaller than (03). Now the eight diagonals mentioned above will agree, as one immediately sees, in size with the eight connecting lines which could be drawn from (0) to the eight points lying around (3) in the plane of the higher bases if we indicate this way for convenience the eight vertices of the four parallelograms meeting at (3). The fact that from the afore-mentioned connection lines none is smaller than (03) follows from the condition by which (3) is being chosen.

After we have convinced ourselves that a system of third order can always be partitioned by a reduced parallelepiped, we now have to establish the relation between such parallelepiped and the system and particularly to compare the distance of the point of systems from (0) with one another. We set $(01) = \sqrt{a}, (02) = \sqrt{b}, (03) = \sqrt{c}$ and always hold fast the assumption $a \leqq b \leqq c$.

1. In the plane of the base the conditions discused above (§3, II) occurs, so that consequently the successive minima of the distance are always $\sqrt{a}, \sqrt{b}$ in size, whereby then in the singular cases mentioned there there is an arbitrariness in the choice of the points.

2. We now look at the points outside the plane of the base underneath namely first of all the one in the plane of the base above. Here from the assumption that our parallelepiped is a reduced one, the line (03) is not longer than one of the straight lines drawn from (0) to the eight points lying around (3), so as a result the foot of the perpendicular dropped from (0) on to the plane of the base above would not be farther apart from (3) than from one of the eight points mentioned. This foot point therefore does not fall outside the hexagon or quadrangle constructed to belong to (3) in the last paragraph. Of those eight points can exceptionally, when the foot point falls on one side, one, or it could, when the foot point meets with a vertex, two (three, when the polygon becomes a rectangle,) lie equally close to the foot point as the point (3), while all remaining points of the plane are further apart from that foot point. It follows from this that the shortest distance (amounting to $\sqrt{c}$) from (0) to the a point in the base above is valid in general only for the point (3), but can exceptionally take place for one, two or even three other points.

3. For the consideration of the following parallel planes we have a boundary for the square h of the perpendicular already mentioned to determine. Here the foot point of the perpendicular does not fall outside the hexagon which belongs to (3), therefore, when ρ denotes the square of the radius of the circumscribed circle:

$$h \geqq c - \rho.$$

Now also from §4: $\rho \leqq \frac{1}{2}b \leqq \frac{1}{2}c$, consequently $h \geqq \frac{1}{2}c$. Here therefore the second parallel plane is at least $\sqrt{2c}$ away, therefore there is over the higher base only points, the distance from (0) of which is greater than $\sqrt{c}$.

If one summarises what has been said, one will see that the minimum of the distance for the entire system has the value $\sqrt{a}$, that, after a point is chosen at this distance, the minimum in the still remaining directions amounts to $\sqrt{b}$, and that finally after the second point is also fixed, for all points outside the plane, which is determined through (0) and the first two points, the smallest distance from (0) is reduced to $\sqrt{c}$. If the successive minima $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are also always completely determined in quatity, the same minimum in local relation will not be true without several exceptions which are easy to specify. If for example $a \leqq b$, $b < c$, the first two points are to be chosen in the lower base, whereby the singular cases mentioned in §3, II could occur, while the third point lies in the higher base, has a fixed position there in general, in singular cases however can occupy two, three or four different places. One ever so easily overlook that varieties in the other two cases, where $a < b = c$ or $a = b = c$, could happen.

Here from the assumption of a reduced parallelepiped with the edges $\sqrt{a} \leqq \sqrt{b} \leqq \sqrt{c}$ the length of these edges have yielded themselves as the successive minima of the system, thus it immediately follows that when several reduced parallelepiped exist from which the system can be arranged, these all become in agreement with one another with regard to the lengths of their edges, and it could also be easily pointed out that three of the lines directed from (0) to points of the systems of the lengths $\sqrt{a}, \sqrt{b}, \sqrt{c}$ when they only do not lie in the plane, are always the edges of a reduced parallelepiped. It requires therefore only the easy consideration already applied in a similar case (§3, III). Here after this the entire reduced parallelepipeds would be obtained when one construct the successive minima of every possible types, therefore it becomes evident that when this can happen in only one way (to which we also consider the case where, with the equation of two of the quantities $\sqrt{a}, \sqrt{b}, \sqrt{c}$ or with the equation of all three, the three lines are locally completely determined and only an exchange between two or all three can occur) that spatial system would allow only one pattern from a reduced parallelepiped. In all other cases there are several such patterns, the parallelepipeds of which form the basis, which could be either all different from one another or only different in part or even could be all congruent to one another. (Similarly in the two singular cases of a system of second order mentioned above the reduced parallelograms underlying the two or three various patterns were congruent to one another.)

To the determination of the question whether a system of third order permits only one or more than one pattern from a reduced parallelepiped, it would consequently only need the knowledge of a single pattern of the system, and the first case would always and exclusively take place when the reduced parallelepiped given through this pattern is of such a property that all lines which can not be exceeded by others actually exceed this parallelepiped, that is when all diagonals of the faces are larger than their sides and all diagonals of the parallelepiped are similarly larger than the edges of the bodies.

§6

As we now apply the results of the last paragraph to the ternary form, shall the uniformity be assumed because of and in order to avoid pointless differentiation, that one has given every ternary form:

$$ax^2 + by^2 + cz^2 + 2a'yz + 2b'xz + 2c'xy \tag{1}$$

through transposition or change of sign of indeterminate elements, as a result of which the form does not belong to the same class, a single form, that firstly $a \leq b \leq c$ holds, that secondly under the coefficients a', b', c', when not the case that all of them are nonzero and are negative, none has the negative sign, and thirdly, when $b = c$ holds, c' apart from the sign not greater than b', when $a = b$ holds, b' not greater than a', and lastly when $a = b = c$ holds, neither c' greater than b' nor b' greater than a' holds. As is easy to see these condition can only be satisfied in one way and their introduction gives the advantage that, as already without these conditions every ternary form corresponds with a completely determined parallelepiped, now to every parallelepiped also belongs an analytical expression the coefficients of which are also completely determined with regard to their sequence and their signs. This assumed, we mention the form (1) in which

$a \leq b \leq c$ also holds, a reduced one, when it corresponds with a reduced parallelepiped. There the diagonal of the area must not be smaller than the sides themselves, so one has:

$$a \pm 2c' + b \geq b,\ a \pm 2b' + c \geq c,\ b \pm 2a' + c \geq c.$$

One sets $\sigma = -1$, where a', b', c' are all three negative, otherwise $\sigma = 1$, so these conditions are synonymous with:

$$a \geq 2c'\sigma,\ a \geq 2b'\sigma,\ b \geq 2a'\sigma, \tag{2}$$

and only, when the equal sign holds, would one of the diagonals in the corresponding parallelogram be equal to a side. The conditions with regard to the diagonals of the parallelepiped result in:

$$a + b + c + 2a'\epsilon + 2b'\delta + 2c'\delta\epsilon \geq c \quad (\delta = \pm 1, \epsilon = \pm 1),$$

where the signs in $\delta = \pm$, $\epsilon = \pm 1$ are arbitrary. One look next at the case where none of the coefficients a', b', c' is negative, and take into account the four sign combinations, as well as that, when a and b are equal to one another then $b' < a'$, so one immediately sees that our inequality is by itself capable of always meet the condition contained, and that the limiting case of the equation in which the diagonals of the edge $\sqrt{c}$ become equal, only once and only then can it occur, when one of the quantities b', c' is equals to zero and when at the same time of the conditions (2) the two quantities b', c' of which are relating to one another, as well as the one which contains a', satisfies the limiting case of the equation. a', b', c' are negative, then our inequality is always fulfilled that the limiting case can not take place, except when $\delta = \epsilon = 1$, so that the consequently the new condition is established:

$$a + b + 2a' + 2b' + 2c' \geq 0, \tag{3}$$

where again the lower sign relates to the equality between a diagonal and the edge $\sqrt{c}$.

The condition just developed (2) and, when a', b', c' are negative, (3) above is are therefore fulfilled, that the inequality takes place in none of the inequalities of the limiting case, therefore in the class to which the form belongs it would not give a second one of these various ones with or without equality signs in the definition condition, here according to at the end of the last paragraph notice that the corresponding system of points can only be partitioned from a reduced parallelepiped. The matter stands differently when the upper signs do not take place in all conditions; there could then occur in the same class several reduced forms that can be derived from a given one. It is sufficient to demonstrate this for a main case. We choose for this the case where $b < c$.

Next one assumes $a > 2c'\sigma$, therefore the direction of the edge $\sqrt{c}$ can only be altered when there are namely in the plane of higher base still one or more points, the distance from the vertex of which amounts to $\sqrt{c}$. When ξ, η are 1 the one such points corresponding values of the element so would, when the third edge depends on it, all the coefficients except a', b' remain unchanged, these respectively change into $a' + c'\xi + b\eta$, $b' + a\xi + c'\eta$ as one is easily and almost convinced without calculation. Now from the specification made earlier are the values of ξ, η which meet the condition, when $a = 2b'\sigma$:

$$\xi = -\sigma, \quad \eta = 0;$$

when $b = 2a'\sigma$:

$$\xi = 0, \quad \eta = -\sigma;$$

when simultaneously $a = 2b', b = 2a', c' = 0$:

$$\xi = -1, \quad \eta = -1;$$

and when a', b', c' are negative and the equation $a + b + 2a' + 2b' + 2c' = 0$ complied with:

$$\xi = 1, \quad \eta = 1.$$

Corresponding to these four assumptions one has consequently transformed a', b' into:

$$a' - c'\sigma,\ b' - a\sigma (= -b');\ -a',\ \ b' - c'\sigma;\ -a',\ \ -b';\ a' + b + c',\ a + b' + c'.$$

From the third case and generally from the assumption $c' = 0$ one can foresee, here this represents a new form which afterwards one has undertaken in the same one the change of sign stated in the beginning of this paragraph, apparently with the form from which one has derived, become identical. In each of the three remaining assumptions one obtains from application of the necessary sign change a new reduced form belonging to the same class (provided that it does not coincide with the original one), and one obtains two such forms when two of our assumptions hold at the same time. With this then the specification of the form is brought to an end, here apparently the simultaneity of all three assumptions can not take place. If, always under the assumption $b < c$, $a = 2c'\sigma$, one would have, provided that $a < b$, the two edges rotated in the base, for $a = b$ the first edge can also pass into the original position of the two first edges and this new position or both of these new positions of the first two edges must be associated with all the directions of the third one, the original one not excluded.

One can thereby easily remove the inconvenience that in singular cases several reduced forms can be associated in the same class, and thereby take away the exception that for such singular cases in general definition one still include the known secondary conditions which can be proved when one for instance set up the demand that the last coefficient c', provided that it is not fully fixed, maintains the smallest numerical value of which it is capable in the reduced forms of the class, and then likewise with regard to b'. For this to notify an example, we will examine it under the singular cases dealt with earlier where $b < c$, from the three conditions (2) none with the lower signs holds, however the three negative values a', b', c' satisfy the equation:

$$a + b + 2a' + 2b' + 2c' = 0.$$

From the previous observations c' is fixed, and there exists for this case only two reduced forms. a' and b' are the values of the forth and the fifth coefficients in one of them, therefore they are in the other $a'+b+c'$, $a+b'+c'$, or, here the last value are apparently positive, c' is negative and consequently, in order for the sign specification to be sufficient, z to transform into $-z$, rather $-(a'+b+c')$, $-(a+b'+c')$. As it is quite natural, these values suffice when one substitute them for a', b', again the equation:

$$a+b+2a'+2b'+2c'=0,$$

and from them come the values a', b' in the same manner, as they themselves are originated from a', b'. Here according to this the fifth coefficient admits only the two negative values b' and $-(a+b'+c')$, their sum equals $-a-c'$, therefore one see that when one further add to the definition conditions:

$$-b' \leqq \frac{1}{2}(a+c'),$$

the class would contain only one reduced form.

While we conclude the essay, we will still from our principles derive a beautiful theorem, found by *Seeber* through induction and demonstrated by *Gauss* in the announcement already often quoted. From this theorem in a reduced form the production of the first three coefficients is not larger than the doubled absolute value of the determinant.

Here the absolute value of the determinant is equal to the square of the volume of the parallelepiped corresponding to the form, thus consequently, from the expression employed in §5, 3 the inequality to be proved:

$$abc \leqq 2\Delta h,$$

where Δ represents the square of the base. One sets:

$$c=b+t,$$

where consequently t is not negative, draws off from the inequality obtained in §5, 3:

$$h \geqq c-\rho=b-\rho+t,$$

from which one has multiplied it with 2Δ, the equation:

$$abc=ab^2+abt,$$

thus one obtains:

$$2\Delta h-abc \geqq 2\Delta(b-\rho)-ab^2+(2\Delta-ab)t.$$

Here now from the inequality at the end of §4, $2\Delta(b-\rho)-ab^2$ is not negative and $2\Delta-ab \geqq \frac{1}{2}ab$ is positive, therefore the truth of the theorem becomes evident.

§ D.2 G. F. Voronoi, 1908 (I)

New applications of continuous parameters to the theory of the quadratic form

First Memoir

On some properties of the perfect positive quadratic forms

by Mr. Georges Voronoï in Warsaw

[*Journal für die reine und angewandte Mathematik*, V. 133, p. 97–178, 1908]

[translated by K N Tiyapan]

Introduction

Hermite had introduced in the theory of numbers a new and fruitful principle, namely: being given a set (x) of systems $(x_1, x_2, \ldots, x_n)$ for all the values of $x_1, x_2, \ldots, x_n$, one associates with the set (x) a set (R) composed of the domains in a manner such that by studying the set (R) one studies at the same time the set (x) .

Hermite has shown † numerous applications of the new principle to the generalisation of continuous fractions, to the study of algebraic units, etc.

The ideas of Hermite have been developed in the works of Mr.'s Zolotareff, Charve, Selling, Minkowski. ‡

I intend to publish a series of Mémoires in which I shall show new applications of the principle of Hermite to the various problems of the arithmetic theory of definite and indefinite quadratic forms.

---

† *Hermite.* Extraits de lettres de M. Ch. Hermite à M. Jacobi sur differents objets de la théorie des nombres. [Excerpts from letters of Mr. Ch. Hermite to Mr. Jacobi on various subjects in the theory of numbers] (This Journal V. 40, p. 261)

Hermite. Sur l'Introduction des variables continues dans la théorie des nombres. [On the introduction of the continuous variables in the theory of numbers] (This Journal V. 41, p. 191)

Hermite. Sur la théorie des formes quadratiques. [On the theory of quadratic forms] (This Journal V. 47, p 313)

‡ *Zolotareff.* On an indeterminate equation of the third degree (Petersbourg, 1869, in Russian.)

Zolotareff. Theory of complex integers with applications to the integral calculus. (Petersbourg, 1874, in Russian.)

Charve. De la réduction des formes quadratiques ternaires positives et de leur application aux irrationelles de troisième degré. [Of the reduction of positive ternary quadratic forms and of their application to the irrationals of third detree] (Suppl. to V. IX of Annales Scientifiques de l'Ecole Normale Supérieure, 1880)

Selling. Über die binären und ternären quadratischen Formen. [On the binary and ternary quadratic forms] (This Journal, V. 77, p. 143)

Minkowski. Geometrie der Zahlen. [Geometry of numbers] (Leipzig, 1896)

In this Mémoire, I study the properties of the minimum of positive quadratic forms and of their various representations by systems of integers.

Hermite has discovered an important property of the minimum M of positive quadratic forms $\sum a_{ij}x_ix_j$ in n variables and of the determinant D, namely:

$$M \leq \left(\frac{4}{3}\right)^{\frac{n-1}{2}} \sqrt[n]{D},$$

and he has demonstrated numerous applications of this formula.

In a letter to Jacobi, Hermite has said §:

"That which precedes sufficiently indicates an infinity of other analogous consequences which, all, will depend on the difficult study of an exact limit of the minimum of any definite form. Thereupon I then form only one conjecture. My first studies in the case of a form in n variables of the determinant D have given me the limit $\left(\frac{4}{3}\right)^{\frac{n-1}{2}} \sqrt[n]{D}$, I am inclined to presume, but without being able to demonstrate that the numerical coefficient $\left(\frac{4}{3}\right)^{\frac{n-1}{2}}$ has to be replaced by $\frac{2}{\sqrt[n]{n+1}}$"

Mr.'s Korkine and Zolotareff has under taken the study of the exact limit of the minimum of positive quadratic forms of the same determinant.

By indicating with $M(a_{ij})$ the minimum and with $D\ (a_{ij})$ the determinant of the form $\sum a_{ij}x_ix_j$, one will have the minimum

$$\mathcal{M}(a_{ij}) = \frac{M(a_{ij})}{\sqrt[n]{D(a_{ij})}}$$

of a positive quadratic form with determinant 1.

By virtue of the theorem of Hermite the function $\mathcal{M}(a_{ij})$ verifies the inequality ¶

$$\mathcal{M}(a_{ij}) \leq \left(\frac{4}{3}\right)^{\frac{n-1}{2}},$$

therefore it is bounded within the set (f) of all the positive quadratic forms of real coefficients.

Mr.'s Korkine and Zolotareff have demonstrated † that the function $\mathcal{M}(a_{ij})$ possesses many maxima in the set (f) which correspond to the various classes of equivalent positive quadratic forms.

The limit $\frac{2}{\sqrt[n]{n+1}}$ indicated by Hermite in the letter to Jacobi (source cited) is only a maximum value of the function $\mathcal{M}(a_{ij})$.

The binary and ternary positive quadratic forms possess a single maximum which is therefore, in this case, the exact limit of values of the function $\mathcal{M}(a_{ij})$.

Reckoning from the number of variables $n \geq 4$, one meets many maxima of the function $\mathcal{M}(a_{ij})$.

Mr.'s Korkine and Zolotareff have found many values of various maxima of the function $\mathcal{M}(a_{ij})$ which exceed the limit $\frac{2}{\sqrt[n]{n+1}}$ indicated by Hermite, but do not exceed the limit 2.

The study of the exact limit of the minimum of positive quadratic forms of the equal determinant comes down, after Mr.'s Korkine and Zolotareff, to the study of all the various classes of positive quadratic forms to which correspond the maximum values of the function $\mathcal{M}(a_{ij})$.

The maximum maximorum of values of the function $\mathcal{M}(a_{ij})$ is the largest value of the function $\mathcal{M}(a_{ij})$ which presents a numerical function as $\mu(n)$.

Mr.'s Korkine and Zolotareff have determined the following values of the function $\mu(n)$:

$$\mu(2) = \sqrt{\frac{4}{3}},\ \mu(3) = \sqrt[3]{2},\ \mu(4) = \sqrt[4]{4},\ \mu(5) = \sqrt[5]{8},.$$

They have called extreme the quadratic forms which yield to the function $\mathcal{M}(a_{ij})$ a maximum value.

The extreme quadratic forms enjoy an important property, namely:

I. Any extreme quadratic form is determined by the value of its minimum and by all the representations of the minimum.

Mr.'s Korkine and Zolotareff have determined all the classes of extreme forms in 2, 3, 4 and 5 vertices.

By studying these extreme forms, I have observed that they are all well defined by the property (I). There is only reckoning from positive forms in six variable which I have encountered positive quadratic forms which enjoyed the property (I) and are not of extreme forms.

I call "perfect" any positive quadratic form which enjoys the property (I).

I demonstrate that the set of all the perfect forms in n variables can be divided into classes the number of which is finite.

---

§ This Journal. V 40, p. 296

¶ Mr. Minkowski has demonstrated an upper limit of the function $\mathcal{M}(a_{ij})$

$$\mathcal{M}(a_{ij}) \leq n$$

much simpler than that from Hermite.

(*Minkowski.* Über die positiven quadratischen Formen und über kettenbruchähnliche Algorithmen. [On the positive quadratic forms and on continued fraction algorithm] This Journal V. 107, p. 291)

† *Korkine* and *Zolotareff.* Sur les formes quadratiques. [On the quadratic forms] Mathematische Annalen, V. VI, p. 366 and V. XI, p. 242

All extreme form being, by virtue of the property I, a perfect form, it results in that the function $\mu(n)$ presents the maximum of values of the function $\mathcal{M}(a_{ij})$ which correspond to the various classes of perfect forms.

I have established an algorithm for the search of various perfect forms by introducing a definition of contiguous perfect forms.

To that effect, I make correspond to the set (φ) of all the perfect forms in n variables a set (R) of domains in $\frac{n(n+1)}{2}$ dimensions determined with the help of linear inequalities.

The set (R) of domains in $\frac{n(n+1)}{2}$ dimensions presents a partition of the set (f) of all the positive quadratic forms in n variables.

Each domain R possesses in the set (R) a contiguous domain which is well determined by any one face in $\frac{n(n+1)}{2} - 1$ dimensions of the domain R.

I demonstrate that the domain R corresponding to the perfect form $\varphi(x_1, x_2, \ldots, x_n)$ being determined by the linear inequalities

$$\sum p_{ij}^{(k)} a_{ij} \geq 0, \quad (k = 1, 2, \ldots, \sigma)$$

one will have σ perfect forms defined by the equalities

$$\varphi_k(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho_k \Psi_k(x_1, x_2, \ldots, x_n), \qquad (k = 1, 2, \ldots, \sigma) \tag{1}$$

where

$$\Psi_k(x_1, x_2, \ldots, x_n) = \sum p_{ij}^{(k)} x_i x_j,$$

provided that the positive parameter ρ_k $(k = 1, 2, \ldots, \sigma)$ presents the smallest value of the function

$$\frac{\phi(x_1, x_2, \ldots, x_n) - M}{-\Psi(x_1, x_2, \ldots, x_n)}$$

where $\Psi(x_1, x_2, \ldots, x_n) < 0$ and M is minimum of the form $\phi(x_1, x_2, \ldots, x_n)$.

I call "contiguous to the perfect form $\phi(x_1, x_2, \ldots, x_n)$" the perfect forms (1).

Any substitution in integer coefficients and with determinant ±1 belonging to the group g of substitutions which do not change the form ϕ permute only the forms (1). One can, therefore, divide the forms (1) into classes of equivalent forms with the help of substitutions of the group g. By choosing one form in each class, one will have a system of perfect forms contiguous to the perfect form ϕ which can replace the system (1).

By proceeding in this manner, one can obtain a system complete of representatives of various classes of perfect forms.

The corresponding domains will form complete system of representatives of various classes of the set (R).

I have remarked that a similar system

$$R, R_1, R_2, \ldots, R_{\tau-1} \tag{2}$$

of domains of the set (R) can serve towards the reduction of positive quadratic forms.

I call reduced any positive quadratic form belonging to one of the domains (2).

It results from this definition:

I. Any positive quadratic form can be transformed into an equivalent reduced form, with the help of a substitution which presents a product of substitutions belonging to a series of substitutions

$$S_1, S_2, \ldots, S_m$$

which depend only on the choice of the system (2).

II. Two reduced forms can be equivalent only provided that the corresponding substitution belonged to a series of substitutions the number of which is finite.

The weak point of the new method of reduction of positive quadratic forms, demonstrated in this Mémoire, consists in that the number of substitutions which transform into itself the domains of the set (R) or their faces is, in general, very large.

The application of the general theory demonstrated in this Mémoire to the numerical examples will be particularly facilitated if one knew how to solve the following problem:

Being given a group G of substitutions which transform into itself a domain R, one would like to partition this domain into equivalent parts the number of which will be equal to the number of substitutions of the group G and on condition that the number of faces in $\frac{n(n+1)}{2} - 1$ dimensions of domains obtained be the smallest possible.

I show in this Mémoire the solution of the problem introduced in two cases: $n = 2$ and $n = 3$.

From the number of variable $n \geq 4$, I do not know any practical solution of the problem posed.

First Part

General theory of perfect positive quadratic forms and domains which correspond to them.

Definition of perfect quadratic forms.

1 Let

$$\phi(x_1, x_2, \ldots, x_n) = \sum a_{ij} x_i x_j \tag{1}$$

be any positive quadratic form. By indicating with

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1s}, l_{2s}, \ldots, l_{ns}) \tag{2}$$

the various representations of the minimum M of the form $\sum a_{ij} x_i x_j$, one will have the equalities

$$\sum a_{ij} l_{ik} l_{jk} = M, \quad (k = 1, 2, \ldots, s) \tag{3}$$

One will not consider in the following the two systems

$$(l_{1k}, l_{2k}, \dots, l_{nk}) \quad \text{and} \quad (-l_{1k}, -l_{2k}, \dots, -l_{nk}), \quad (k = 1, 2, \dots, s)$$

as different and one will arbitrarily choose one of these systems.

On the ground of the supposition made, one will have the inequality

$$\sum a_{ij} x_i x_j > M$$

provided that a system $(x_1, x_2, \dots, x_n)$ of integer values of variables $x_1, x_2, \dots, x_n$ did not belong to the series (2), excluding the system $x_1 = 0, x_2 = 0, \dots, x_n = 0$.

By considering the equalities (3) as the equations which serve to determine $\frac{n(n+1)}{2}$ coefficients of the quadratic form $\sum a_{ij} x_i x_j$, one will have only two cases to examine:

1.) there exist a finite number of solutions of equations (3),

2.) the equations (3) admit only a single system of solutions.

2 Let us examine the first case,

Let us suppose that there exists an infinite number of solutions of equations (3).

One will find in this case an infinite number of values of parameters

$$p_{ij} = p_{ji}, \quad (i = 1, 2, \dots, n; j = 1, 2, \dots, n)$$

verifying the equations

$$\sum p_{ij} l_{ik} l_{jk} = 0, \quad (k = 1, 2, \dots, s) \tag{4}$$

the values $p_{ij} = 0$, $i = 1, 2, \dots, n; j = 1, 2, \dots, n$ being excluded.

By indicating

$$\Psi(x_1, x_2, \dots, x_n) = \sum p_{ij} x_i x_j,$$

let us consider the set of positive quadratic forms determined by the equality

$$f(x_1, x_2, \dots, x_n) = \varphi(x_1, x_2, \dots, x_n) + \rho\Psi(x_1, x_2, \dots, x_n), \tag{5}$$

the parameter ρ being arbitrary.

For a quadratic form determined by the equality (5) to be positive, it is necessary and sufficient that the corresponding value of the parameter ρ be continuous in a certain interval

$$-R' < \rho < R.$$

It can turn out that $R = +\infty$, in this case the lower limit $-R'$ will be finite. By replacing in the equality (5) the form $\Psi(x_1, x_2, \dots, x_n)$ by the form $-\Psi(x_1, x_2, \dots, x_n)$, that which is permitted by virtue of (4), one will have the interval

$$-R < \rho < R',$$

therefore one can suppose that the upper limit R is finite.

The corresponding quadratic form, determined by the equality

$$f(x_1, x_2, \dots, x_n) = \varphi(x_1, x_2, \dots, x_n) + R\Psi(x_1, x_2, \dots, x_n),$$

will not be positive, but it will not have negative values either; one concludes that least for a system $(\xi_1, \xi_2, \dots, \xi_n)$ of real values of variables $x_1, x_2, \dots, x_n$ the form $f(x_1, x_2, \dots, x_n)$ attains in its value the smallest which is zero, and it follows that the system $(\xi_1, \xi_2, \dots, \xi_n)$ verifies the equation

$$\frac{\partial f}{\partial \xi_i} = \frac{\partial \varphi}{\partial \xi_i} + R\frac{\partial \Psi}{\partial \xi_i} = 0. \quad (i = 1, 2, \dots, n)$$

By eliminating from these equations $\xi_1, \xi_2, \dots, \xi_n$ one obtains the equation

$$D(R) = \begin{vmatrix} a_{11} + RP_{11}, & a_{12} + RP_{12}, & \dots, & a_{1n} + RP_{1n} \\ a_{21} + RP_{21}, & a_{22} + RP_{22}, & \dots, & a_{2n} + RP_{2n} \\ & \dots & & \\ a_{n1} + RP_{n1}, & a_{n2} + RP_{n2}, & \dots, & a_{nn} + RP_{nn} \end{vmatrix} = 0.$$

The smallest positive root of this equation presents the value of R searched for.

3 Let us examine the set (f) of positive quadratic forms determined by the equality (5) with condition

$$0 < \rho < R. \tag{6}$$

Theorem. To the set (f) belongs a quadratic form $\varphi_1(x_1, x_2, \dots, x_n)$ which is well determined by the following conditions:

1. all the representations of the minimum M of the form $\varphi(x_1, x_2, \dots, x_n)$ are also representations of the minimum M of the form $\varphi_1(x_1, x_2, \dots, x_n)$,

2. the form $\varphi_1(x_1, x_2, \dots, x_n)$ moreover possesses at least another representation of the minimum M.

Let us indicate by $M(\rho)$ the minimum and by $D(\rho)$ the determinant of the quadratic form $f(x_1, x_2, \dots, x_n)$ defined by the equality (5) with condition (6).

By virtue fo the theorem by Hermite, one will have the inequality

$$M(\rho) \leq \mu(n) \sqrt[n]{D(\rho)}. \tag{7}$$

We have demonstrated that $D(R) = 0$, it results in that a value of the parameter ρ can be chosen in the interval (6) such that the inequality

$$\mu(n)\sqrt[n]{D(\rho)} < M$$

holds. One will have, because of (7),

$$M(\rho) < M. \tag{8}$$

Let us indicate by $(l_1, l_2, \ldots, l_n)$ a representation of the minimum $M(\rho)$ of the form $f(x_1, x_2, \ldots, x_n)$ verifying the inequality (8).

One will have

$$\varphi(l_1, l_2, \ldots, l_n) + \rho\Psi(l_1, l_2, \ldots, l_n) < M, \tag{9}$$

and as a result

$$\varphi(l_1, l_2, \ldots, l_n) > M \text{ and } \Psi(l_1, l_2, \ldots, l_n) < 0. \tag{10}$$

This posed, let us find the smallest value of the function

$$\frac{\varphi(x_1, x_2, \ldots, x_n) - M}{-\Psi(x_1, x_2, \ldots, x_n)} \tag{11}$$

determined with condition

$$\Psi(x_1, x_2, \ldots, x_n) \leq 0. \tag{12}$$

To that effect, let us examine the inequality

$$\frac{\varphi(x_1, x_2, \ldots, x_n) - M}{-\Psi(x_1, x_2, \ldots, x_n)} \leq \frac{\varphi(l_1, l_2, \ldots, l_n) - M}{-\Psi(l_1, l_2, \ldots, l_n)}.$$

By virtue of (9), (10) and (12), one will have

$$\varphi(x_1, x_2, \ldots, x_n) + \rho\Psi(x_1, x_2, \ldots, x_n) < M.$$

The quadratic form $\varphi(x_1, x_2, \ldots, x_n) + \rho\Psi(x_1, x_2, \ldots, x_n)$ being positive, there exists only a limited number of integer values of $x_1, x_2, \ldots, x_n$ verifying this inequality. Among these systems are found all the systems which give back to the function (11) the smallest value determined with condition (12).

Let us indicate by

$$\left(l'_1, l'_2, \ldots, l'_n\right), \left(l''_1, l''_2, \ldots, l''_n\right), \ldots, \left(l^{(r)}_1, l^{(r)}_2, \ldots, l^{(r)}_n\right)$$

all the representations of the positive minimum ρ_1 of the function (11).

By declaring

$$\varphi_1(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho_1\Psi(x_1, x_2, \ldots, x_n),$$

one obtains the positive quadratic form $\varphi_1(x_1, x_2, \ldots, x_n)$ the minimum M of which is represented by the systems (2) and (13), this is that which one will demonstrate without trouble.

4 With the help of the procedure previously shown, one will determine a series of positive quadratic forms

$$\varphi, \varphi_1, \varphi_2, \cdots \tag{14}$$

which enjoy the following property: by indicating with s_k the number of representations of the minimum of the form $\varphi_k (k = 1, 2, \ldots)$, one will have the inequalities

$$s < s_1 < s_2 < \cdots \tag{15}$$

A similar series of positive quadratic forms of n variables can not be extended indefinitely, this is that which we will demonstrate with the help of the following lemma.

Lemma. The number of various representations of the minimum of a positive quadratic form in n variables does not exceed $2^n - 1$.

Let us indicate by $(l_1, l_2, \ldots, l_n)$ and $(l'_1, l'_2, \ldots, l'_n)$ any two representations of the minimum M of the positive quadratic form $\sum a_{ij}x_i x_j$.

Let us suppose that by declaring

$$l'_i = l_i + 2t_i, \quad (i = 1, 2, \ldots, n) \tag{16}$$

the number $t_1, t_2, \ldots, t_n$ would be integer.

As

$$\sum a_{ij} l'_i l'_j = M \quad \text{and} \quad \sum a_{ij} l_i l_j = M,$$

by virtue of (16), it becomes

$$\sum a_{ij} l_i t_j + \sum a_{ij} t_i t_j = 0$$

One will present this equality under the form

$$\sum a_{ij}(l_i + t_i)(l_j + t_j) + \sum a_{ij} t_i t_j = \sum a_{ij} l_i l_j. \tag{17}$$

By noticing that

$$\sum a_{ij} t_i t_j \geq \sum a_{ij} l_i l_j,$$

one finds, by virtue of (17),

$$\sum a_{ij}(l_i+t_i)(l_j+t_j)\leq 0,$$

therefore it is necessary that

$$\sum a_{ij}(l_i+t_i)(l_j+t_j)=0,$$

and consequently

$$l_i+t_i=0. \quad (i=1,2,\dots,n)$$

Because of (16), one obtains

$$l'_i=-l_i. \quad (i=1,2,\dots,n)$$

This posed, let us divide the set (X) of all the systems $(x_1,x_2,\dots,x_n)$ of integer values of $x_1,x_2,\dots,x_n$ into 2^n classes, with regard to the modulo 2.

We have demonstrated that two different representations of the minimum M of the form $\sum a_{ij}x_ix_j$ will not belong to the same class; neither will any representation of the minimum M belong to the class made up of systems $(x_1,x_2,\dots,x_n)$ satisfying the conditon

$$x_i\equiv 0(\text{mod}2), \quad (i=1,2,\dots,n)$$

therefore the number of various representations of the minimum of a positive quadratic form can not be greater than 2^n-1.

5

We have demonstrated that the series (14) of positive quadratic forms satisfying the condition (15) can not be extended indefinitely, therefore the series (14) will be terminated by a form φ_k which enjoys the following property: the form φ_k is determined by the representations of its minimum.

Definition. One will call perfect any positive quadratic form which is determined by the representations of its minimum.

Let us suppose that the form (1) be perfect, one will have in this case only a single system of solutions of equations (3).

On the ground of the supposition made, the equations

$$\sum p_{ij}l_{ik}l_{jk}=0, \quad (k=1,2,\dots,s)$$

admit only a single system of solutions

$$p_{ij}=p_{ji}=0. \quad (i=1,2,\dots,n; j=1,2,\dots,n)$$

By effecting the solution of equations (3), one obtains the equalities

$$a_{ij}=\alpha_{ij}M, \quad (i=1,2,\dots,n; j=1,2,\dots,n)$$

where the coefficients α_{ij} are rational.

It results in that the perfect form $\frac{\varphi}{M}$ is of rational coefficients. In the following one will not consider as different the perfect forms of proportional coefficients.

Fundamental properties of perfect quadratic forms.

6

Let

$$\varphi(x_1,x_2,\dots,n)=\sum a_{ij}x_ix_j$$

be a perfect quadratic form. Let us suppose that all the different representations of the minimum of the perfect form φ make up the series

$$(l_{11},l_{21},\dots,l_{n1}),(l_{12},l_{22},\dots,l_{n2}),\dots,(l_{1s},l_{2s},\dots,l_{ns}). \tag{1}$$

By choosing any n systems in this series, let us examine the determinant

$$\begin{vmatrix} l_{11}, & l_{12}, & \dots, l_{1n} \\ l_{21}, & l_{22}, & \dots, l_{2n} \\ & \dots & \\ l_{n1}, & l_{n2}, & \dots, l_{nn} \end{vmatrix}=\pm\omega. \tag{2}$$

All the determinants that one can form this way can not cancel each other out. By supposing the contrary, one will have s equations of the form

$$l_{ik}=\sum_{r=1}^{n-1} l_{ir}u_r^{(k)}, \quad (i=1,2,\dots,n; k=1,2,\dots,s) \tag{3}$$

One will choose a system of $\frac{n(n+1)}{2}$ parameters $p_{ij}=p_{ji}$ verifying $\frac{n(n+1)}{2}$ equations

$$\sum p_{ij}l_{ir}l_{jt}=0, \quad (r=1,2,\dots,n-1; t=1,2,\dots,n-1)$$

and by virtue of (3), one will have

$$\sum p_{ij}l_{ik}l_{jk}=0, \quad (k=1,2,\dots,s)$$

which is impossible.

The numerical value ω of the determinant (2) can not exceed a fixed limit. To demonstrate this, let us effect a transformation of the perfect form φ with the help of a substitution

$$x_i = \sum_{r=1}^{n} l_{ir} x'_r; \quad (i = 1, 2, \ldots, n) \tag{4}$$

one will obtain a form

$$\varphi'(x'_1, x'_2, \ldots, x'_n) = \sum a'_{ij} x'_i x'_j,$$

where

$$a'_{ii} = M. \quad (i = 1, 2, \ldots, n) \tag{5}$$

By indicating with D' the determinant of the form φ', one will have the inequality

$$a'_{11} a'_{22} \cdots a'_{nn} \geq D',$$

by virtue of the known property of positive quadratic forms.

Considering (5), one obtains

$$M^n \geq D'. \tag{6}$$

By indicating with D the determinant of the form φ, one will have, because of (2) and (4),

$$D' = D\omega^2,$$

therefore the inequality (6) reduces to the one here:

$$D\omega^2 \leq M^n.$$

By virtue of the theorem by Hermite, one has the inequality

$$M \leq \mu(n) \sqrt[n]{D};$$

it follows that

$$\omega \leq [\mu(n)]^{\frac{n}{2}}. \dagger \tag{7}$$

7 Any perfect form will obviously be transformed into a form, also perfect, with the help of all linear substitution of integer coefficients and of determinant ±1.

One concludes this that there exists a finite a finite number of equivalent perfect forms.

The set (φ) of all the perfect forms in n variables can be divided into different classes provided that each class be made up of all the equivalent perfect forms.

Theorem. The number of different classes of perfect forms in n variables is finite.

Let us indicate by

$$\lambda_k = l_{1k}x_1 + l_{2k}x_2 + \ldots + l_{nk}x_n \quad (k = 1, 2, \ldots, s)$$

s linear forms

$$\lambda_1, \lambda_2, \ldots, \lambda_s \tag{8}$$

which correspond to the systems (1) of representations of the minimum of the form Ψ.

One establishes this way a uniform correspondence between a perfect form φ and the system (8) of linear forms.

Let us suppose that one had transformed the perfected form φ with the help of a substitution S by integer coefficients and with determinant ±1, one will obtain an equivalent perfect form φ'. Let us indicate by

$$\lambda'_1, \lambda'_2, \ldots, \lambda'_s \tag{9}$$

the corresponding system of linear forms.

One will easily demonstrate that the substitution T, adjointed to the substitution S §, will transform the system (8) into a system (9).

One concludes that a certain reduction of perfect forms can be effected with the help of the reduction of corresponding systems of linear forms.

The reduction of the system (8) comes down, by virtue of (7), to the reduction of any n linear forms

$$\lambda_1, \lambda_2, \ldots, \lambda_n \tag{10}$$

---

† See the Mémoire of Mr.'s Korkine and Zolotareff *sur les formes quadratiques positives.* (Mathematische Annalen V. XI, p. 256)

§ The substitution S being defined by the equalities

$$x_i = \sum_{k=1}^{n} \alpha_{ik} x'_k, \quad (i = 1, 2, \ldots, n)$$

one calls "substitution adjointed to S" the substitution T which is determined by the equalities

$$\sum_{k=1}^{n} \alpha_{ik} x_k = x'_i. \quad (i = 1, 2, \ldots, n)$$

belonging to the system (8) and with determinant $\pm\omega$ which does not cancel each other out.

One will determine with the help of the known method a substitution T which will transform the linear forms (10) of integer coefficients into linear forms,

$$\lambda'_1, \lambda'_2, \ldots, \lambda'_n \tag{11}$$

satisfying to the following conditions

$$\begin{cases} \lambda'_k = p_{k,k}x' + p_{k+1,k}x'_{k+1} + \ldots + p_{n,k}x'_n, & (k = 1, 2, \ldots, n) \\ p_{11}p_{22}\cdots p_{nn} = \omega \text{ and } p_{kk} > 0, & (k = 1, 2, \ldots, n) \\ 0 \leq p_{k+i,k} < p_{kk}. & (i = 1, 2, \ldots, n-k; k = 1, 2, \ldots, n) \end{cases}$$

The coefficients of forms (11) being integers, as a result they do not exceed fixed limits.

The substitution T will transform the system (8) into a system

$$\lambda'_1, \lambda'_2, \ldots, \lambda'_s \tag{12}$$

of linear forms. By examining successively the determinants of forms

$$(\lambda'_k, \lambda'_2, \ldots, \lambda'_n), (\lambda'_1, \lambda'_k, \ldots, \lambda'_n), \ldots, (\lambda'_1, \lambda'_2, \ldots, \lambda'_k), \ (k = n+1, n+2, \ldots, s)$$

one will demonstrate that the numerical values of coefficients of all the linear forms (12) do not exceed fixed limits.

The number of similar systems of linear forms in integer coefficients being limited, it results in that the number of different classes of perfect forms is also limited.

On the domains determined with the help of linear inequalities

8

We have seen in Number 7 that the study of perfect forms can be brought back to the study of certain systems of linear forms.

One will acquire a new basis to these studies by making correspond to each perfect quadratic form in n variables a domain in $\frac{n(n+1)}{2}$ dimensions determined with the help of linear inequalities.

One will address first the general problem by studying the properties of domains determined with the help of linear inequalities. ‡

Let us consider a system of linear inequalities

$$p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m \geq 0, \quad (k = 1, 2, \ldots, \sigma)$$

in any real coefficients.

One will call point (x) any system $(x_1, x_2, \ldots, x_m)$ of real values of variables $x_1, x_2, \ldots, x_m$ and one will indicate

$$y_k(x) = p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m. \quad (k = 1, 2, \ldots, \sigma)$$

One will call "domain" the set R of points verifying the inequalities

$$y_k(x) \geq 0. \ (k = 1, 2, \ldots, \sigma) \tag{1}$$

Let us suppose that to the domain R belonged to points verifying the inequalities

$$y_k(x) > 0, \quad (k = 1, 2, \ldots, \sigma)$$

one will call such points interior to the domain R, and the domain R will be said to be of m dimensions.

It can be the case that the domain R does not possess interior points. One will demonstrate in this case all the points belonging to the domain R verify at least one equation

$$y_k(x) = 0,$$

the indice h being a value $1, 2, \ldots, \sigma$.

It is important to have a criteria with the help of which one could recognise whether a domain determined by the help of inequalities (1) will be in m dimensions or not.

Fundamental principle. For a domain determined with the help of inequalities (1) to be of m dimensions, it is necessary and sufficient that the equation

$$\sum_{k=1}^{\sigma} \rho_k y_k(x) = 0 \tag{Ξ}$$

did not reduce into an identity so long as the parameters $\rho_2, \rho_2, \ldots, \rho_\sigma$ *are positive or zero, the values* $\rho_1 = 0, \rho_2 = 0, \ldots, \rho_\sigma = 0$ *being excluded.*

The principle introduced, considered from a certain point of view, is evident, but one arrive at the rigorous demonstration of this principle only with the help of the in depth study of domains determined with the help of linear inequalities.

For more simplicity, one will examine in that which follows only domains satisfying the following conditions: the equations

$$y_k(x) = 0 \ \ (k = 1, 2, \ldots, \sigma) \tag{2}$$

can not be verified by any point, the point $x_1 = 0, x_2 = 0, \ldots, x_m = 0$ being excluded.

It is easy to demonstrate that the general case will always come down to the case examined.

9

---

‡ See: *Minkowski.* Geometrie der Zahlen [Geometry of the numbers], No. 19, p. 39.

Definition. One will call edge of the domain R determined with the help of inequalities (1) the set of points belonging to the domain R and verifying the equations

$$y_k(x) = 0, \quad (k = 1, 2, \ldots, r \text{ where } r < \sigma)$$

provided that these equations defined the values of $x_1, x_2, \ldots, x_m$ to an immediate common factor.

By indicating with $(\xi_1, \xi_2, \ldots, \xi_m)$ a point of the edge considered, one will determine all the points of the edge with the help of equalities

$$x_i = \rho\xi_i, \quad (i = 1, 2, \ldots, n)$$

ρ being an arbitrary positive parameter.

This results in that each edge of the domain R is well determined by any point belonging to it.

Let us suppose that the domain R possesses s edges characterised by the points

$$(\xi_k) = (\xi_{1k}, \xi_{2k}, \ldots, \xi_{mk}). \quad (k = 1, 2, \ldots, s)$$

By declaring

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik}, \quad (i = 1, 2, \ldots, m) \tag{3}$$

where

$$\rho_k \geq 0, \quad (k = 1, 2, \ldots, s) \tag{4}$$

one obtains a point (x) belonging to the domain R, the positive or zero parameters $\rho_1, \rho_2, \ldots, \rho_s$ being arbitrary.

10 *Fundamental theorem. Let us suppose that the inequalities (1) which define the domain R satisfy the condition (Ξ).*

The domain R will be of m dimensions and each point belonging to it will be determined by the equalities (3) with condition (4).

The theorem introduced is well known in the case $m = 2$ and $m = 3$.

We will demonstrate that by supposing that the theorem be true in the case of $m - 1$ variables, the theorem will again be true in the case of m variables.

Let us examine first the various inequalities of the system (1). It can be the case that many among them could be put under the form

$$y_h(x) = \sum_{k=1}^{s} \rho_k^{(h)} y_k(x) \quad \text{where} \quad \rho_k^{(h)} \geq 0. \quad (k = 1, 2, \ldots, s; \rho_h^{(h)} = 0)$$

One will call such inequalities dependent and one will exclude them from the system (1).

Let us suppose that the system (1) contained only independent inequalities.

Their number ρ, on the ground of the supposition (2) made, will not be less than m.

This posed, let us examine a set P_h of points belonging to the domain R and verifying an equation

$$y_h(x) = 0, \tag{5}$$

the indice h having a value $1, 2, \ldots, \sigma$.

One will call "face of the domain R" the domain P_h.

On the ground of the supposition made, the face P_h will be in $m - 1$ dimensions.

To demonstrate this, let us make correspond to any point (x) verifying the equation (5) a point (u) in $m - 1$ coordinates $(u_1, u_2, \ldots, u_{m-1})$ by declaring

$$x_i = \sum_{j=1}^{m-1} \alpha_{ij} u_j. \quad (i = 1, 2, \ldots, m)$$

The system of inequalities (1) will be transformed into a system

$$\eta_k(u) \geq 0 \quad (k = 1, 2, \ldots, \sigma; k \neq h) \tag{7}$$

of inequalities in $m - 1$ variables $u_1, u_2, \ldots, u_{m-1}$.

Let us suppose that one knew how to reduce the equation

$$\sum_{k=1}^{\sigma} \rho_k \eta_k(u) = 0 \text{ where } \rho_h = 0 \text{ and } \rho_k \geq 0 \quad (k = 1, 2, \ldots, \sigma) \tag{8}$$

into an identity. By virtue of (6), one will obtain the identity

$$\sum_{k=1}^{\sigma} \rho_k y_k(x) = \rho y_h(x) \quad \text{where} \quad \rho_h = 0.$$

One can not suppose that $\rho > 0$, since otherwise the inequality

$$y_h(x) \geq 0$$

would be dependent and on the ground of the supposition made would not belong to the system (1).

By supposing that $\rho \leq 0$, one will admit $\rho_h = -\rho$ and one will obtain the identity

$$\sum_{k=1}^{\sigma} \rho_k y_k(x) = 0 \text{ where } \rho_k \geq 0, \quad (k = 1, 2, \ldots, \sigma)$$

which is contrary to the hypothesis.

We have supposed that the theorem introduced be true in the case of $m-1$ variables. As the equation (8) can not be reduced into an identity, one concludes that the system of inequalities (7) defines a domain $\mathcal{B}_h$ in $m-1$ dimensions. Moreover, by indicating with

$$(u_{11}, u_{21}, \ldots, u_{m-1,1}), (u_{12}, u_{22}, \ldots, u_{m-1,2}), \ldots, (u_{1t}, u_{2t}, \ldots, u_{m-1,t}) \tag{9}$$

the points which characterised t edges of the domain $\mathcal{B}_h$, one will determine any point (u) of this domain by the equalities

$$u_i = \sum_{k=1}^{t} \rho_k u_{ik} \text{ where } \rho_k \geq 0, \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m-1) \tag{10}$$

One will make correspond to the points (9) the points

$$(\xi_r) = (\xi_{1r}, \xi_{2r}, \ldots, \xi_{mr}), \quad (r = 1, 2, \ldots, t) \tag{11}$$

by determining them with the help of equalities (6) and (9).

The points obtained (11) characterise t edges of the domain R belonging to the face P_h. Any point (x) belonging to the face P_h will be determined, on the grounds of (6) and (10), by the equalities

$$x_i = \sum_{k=1}^{t} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0. \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m) \tag{12}$$

Let us notice that all the points (11) verify the equation

$$y_h(x) = 0 \tag{13}$$

and satisfy the conditions

$$y_k(x) \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

One obtains thus the equalities

$$y_h(\xi_r) \geq 0 \quad (r = 1, 2, \ldots, t; k = 1, 2, \ldots, \sigma) \tag{14}$$

The face P_h being in $m-1$ dimensions, the equalities (14) would define the coefficients of the equation (13) to a close by common factor.

11 Let us suppose that one had determined this way all the faces

$$P_1, P_2, \ldots, P_\sigma \tag{15}$$

in $m-1$ dimensions of the domain R.

Let us suppose that the points

$$(\xi_k) = (\xi_{1k}, \xi_{2k}, \ldots, \xi_{mk}) \quad (k = 1, 2, \ldots, s) \tag{16}$$

characterise the various edges of the domain R belonging to the various faces (15).

By indicating

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0, \quad (k = 1, 2, \ldots, s; i = 1, 2, \ldots, m) \tag{17}$$

one obtains a set of points which all belong to the domain R.

I say that any point (x) belonging to the domain R can be determined with the help of equalities (17).

One can suppose that the point (x) does not belong to any one of the faces (15), since any point belonging to them can be determined with the help of equalities (12).

By supposing that one had the inequalities

$$y_k(x) > 0, \quad (k = 1, 2, \ldots, \sigma)$$

let us arbitrarily choose a point (ξ_r) among those of the series (16) and let us admit

$$x_i' = x_i - \rho \xi_{ir} \text{ where } \rho > 0. \quad (i = 1, 2, \ldots, m) \tag{18}$$

So long as the parameter ρ is sufficiently small, one will also have

$$y_k(x') > 0. \quad (k = 1, 2, \ldots, \sigma)$$

By making the parameter increase in a continuous manner, one will determine with the help of equalities (18) a point (x') verifying an equation

$$y_h(x') = 0$$

and satisfying the condition

$$y_k(x') \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

The point obtained (x') belongs to the face P_h, therefore one can declare

$$x'_i = \sum_{k=1}^{t} \rho'_k \xi_{ik} \text{ where } \rho'_k \geq 0. \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

By virtue of (18), it becomes

$$x_i = \rho \xi_{ir} + \sum_{k=1}^{t} \rho'_k \xi_{ik} \text{ where } \rho > 0,\ \rho'_k \geq 0.\ (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

It remains to demonstrate that the domain F is in m dimensions.

Let us notice that all the points determined by the equalities (17) with condition

$$\rho_k > 0 \quad (k = 1, 2, \ldots, s)$$

are interior to the domain R.

In effect, all the points (16) verify the inequalities

$$y_h(\xi_k) \geq 0.\ (k = 1, 2, \ldots, s; h = 1, 2, \ldots, \sigma) \tag{19}$$

By multiplying these inequalities by ρ_k, let us make the sum of inequalities obtained; one will have, because of (17),

$$y_h(x) = \sum_{k=1}^{s} \rho_k y_h(\xi_k) \geq 0. \quad (h = 1, 2, \ldots, \sigma)$$

By virtue of (19), one will have the inequality

$$y_h(x) > 0, \quad (h = 1, 2, \ldots, s) \tag{20}$$

so long as the numbers $y_h(\xi_1), y_h(\xi_2), \ldots, y_h(\xi_s)$ do not cancel each other out.

One can not suppose that the equalities

$$y_h(\xi_k) = 0 \quad (k = 1, 2, \ldots, s)$$

holds, because otherwise all the equations

$$y_1(x) = 0, y_2(x) = 0, \ldots, y_\sigma(x) = 0$$

would be of proportional coefficients, which is contrary to the hypothesis; therefore one will have the inequalities (20), and it follows that the domain R is of m dimensions.

We have demonstrated that the condition (Ξ) is sufficient for the domain R to be of m dimensions. It is easy to demonstrate that this condition is necessary.

12

We have defined in Number 10 the faces in $m - 1$ dimensions of the domain R. This definition can be generalised.

Definition. One will call face in μ dimensions of the domain R $(\mu = 1, 2, \ldots, m-1)$ a domain $P(\mu)$ formed from points belonging to the domain R and verifying a system of equations

$$y_k(x) = 0, \quad (k = 1, 2, \ldots, \tau) \tag{21}$$

provided that these equations define a domain in μ dimensions composed of points which, all, do not verify any other equation $y_{\tau+1}(x) = 0, \ldots, y_\sigma(x) = 0$.

Let us choose among the points (16) all those which verify the equations (21).

By indicating with

$$\xi_k = (\xi_{1k}, \xi_{2k}, \ldots, \xi_{mk}), \quad (k = 1, 2, \ldots, t)$$

one will declare

$$x_i = \sum_{k=1}^{t} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0.\ (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m) \tag{22}$$

It is easy to demonstrate that any point (x) belonging to the face $P(\mu)$ can be determined with the help of equalities (22).

Corollary. Each face of the domain R is a set of points determined by the equalities (22) provided that any point belonging to it could not be determined by the equalities

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik} \quad \text{where} \quad \rho_k \geq 0, \quad (k = 1, 2, \ldots, s; i = 1, 2, \ldots, m)$$

unless all the parameters $\rho_{t+1}, \rho_{t+2}, \ldots, \rho_s$ do not cancel each other.

13

Any point belonging to the domain R either is interior to the domain R or is interior to a face of that domain.

Let us suppose that the point (x) be interior to a face $P(\mu)$ of the domain R which is formed from all the points determined by the equalities (22).

I argue that one can always determine the point (x) by the equalities (22) provided that

$$\rho_k > 0. \quad (k = 1, 2, \ldots, t)$$

To demonstrate this, let us indicate

$$\rho_i = \sum_{k=1}^{t} \xi_{ik}. \quad (i = 1, 2, \ldots, m)$$

The point (α) is interior to the face $P(\mu)$.

By admitting

$$x_i' = x_i - \rho\alpha_i \text{ where; } \rho > 0, \quad (i = 1, 2, \ldots, m) \tag{23}$$

one obtains a point (x_i') which will be interior to the face $P(\mu)$ so long as the parameter ρ will be sufficiently small; it follows that

$$x_i' = \sum_{k=1}^{t} \rho_k' \xi_{ik} \quad \text{where} \quad \rho_k' \geq 0. \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

By virtue of (23), one obtains

$$x_i = \sum_{k=1}^{t} (\rho + \rho_k')\xi_{ik}, \quad (i = 1, 2, \ldots, m)$$

and by making

$$\rho + \rho_k' = \rho_k, \quad (k = 1, 2, \ldots, t)$$

one will have

$$x_i = \sum_{k=1}^{t} \rho_k \xi_{ik} \quad \text{where} \quad \rho_k > 0. \quad (k = 1, 2, \ldots, t; i = 1, 2, \ldots, m)$$

Let us notice that by making $\mu = m$ and $t = s$, one will indicate with the symbol $P(m)$ the domain R; one concludes that any point (x) which is interior to the domain R can be determined by the equalities

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik} \text{ where } \rho_k > 0. \quad (k = 1, 2, \ldots, s; i = 1, 2, \ldots, m)$$

14 *On the correlative domains.*

Definition. Let us suppose that a domain R be determined with the help of inequalities

$$p_{1k}x_1 + p_{2k}x_2 + \ldots + p_{mk}x_m \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

One will call correlative to the domain R the domain $\mathcal{R}$ which is formed from all the points (x) determined by the equalities

$$x_i = \sum_{k=1}^{\sigma} \rho_k p_{ik} \text{ where } \rho_k \geq 0. \ (k = 1, 2, \ldots, \sigma; i = 1, 2, \ldots, m) \tag{1}$$

I say that the domain $\mathcal{R}$ will be in m dimensions, if the domain R does not possess points verifying the equations

$$p_{1k}x_1 + p_{1k}x_2 + p_{mk}x_m = 0, \quad (k = 1, 2, \ldots, \sigma)$$

the point $x_1 = 0, x_2 = 0, \ldots, x_m = 0$ being excluded.

In effect, if all the points of the domain $\mathcal{R}$ verified the same equation

$$\xi_1 x_1 + \xi_2 x_2 + \ldots + \xi_m x_m = 0,$$

one would have the equalities

$$\xi_1 p_{1k} + \xi_2 p_{2k} + \ldots + \xi_m p_{mk} = 0, \quad (k = 1, 2, \ldots, \sigma)$$

by virtue of (1), which is contrary to the hypothesis.

Theorem. By supposing that the domain R be formed from all the points (x) determined by the equalities

$$x_i = \sum_{k=1}^{s} \rho_k \xi_{ik} \text{ where } \rho_k \geq 0, \ (k = 1, 2, \ldots, s; i = 1, 2, \ldots, m) \tag{2}$$

one will define the correlative domain $\mathcal{R}$ with the help of inequalities.

$$\xi_{1k}x_1 + \xi_{2k}x_2 + \ldots + \xi_{mk}x_m \geq 0. \ (k = 1, 2, \ldots, s) \tag{3}$$

Let us indicate by $\mathcal{R}'$ thedomain determined with the help of inequalities (3).

On the ground of the supposition made, all the points

$$(\xi_{11}, \xi_{21}, \ldots, \xi_{m1}), (\xi_{12}, \xi_{22}, \ldots, \xi_{m2}), \ldots, (\xi_{1s}, \xi_{2s}, \ldots, \xi_{ms})$$

characterise the edges of the domain R, and one will have the inequalities

$$p_{1h}\xi_{1k} + p_{2h}\xi_{2k} + \ldots + p_{mh}\xi_{mk} \quad (h = 1, 2, \ldots, \sigma; h = 1, 2, \ldots, \sigma). \tag{4}$$

We have seen in Number 10 that each face P_h in $m-1$ dimensions of the domain R is characterised by the points

$$(\xi_{11}, \xi_{21}, \dots, \xi_{m1}), (\xi_{12}, \xi_{22}, \dots, \xi_{m2}), \dots, (\xi_{1t}, \xi_{2t}, \dots, \xi_{mt})$$

which verify the equation

$$y_h^x = 0 \tag{5}$$

of the face P_h. One obtains the equalities

$$p_{1h}\xi_{1k} + p_{2h}\xi_{2k} + \dots + p_{mh}\xi_{mk} = 0 \quad (k = 1, 2, \dots, t)$$

which define the coefficients $p_{1h}, p_{2h}, \dots, p_{mh}$ of the equation (5) to an immediate common factor.

One concludes, by virtue of the definition established in Number 9, that the point $(p_{1h}, p_{2h}, \dots, p_{mh})$ characterises an edge of the domain $\mathcal{R}'$.

By attributing with the indice h the values $1, 2, \dots, \sigma$, one obtains a series

$$(p_{11}, p_{21}, \dots, p_{m1}), (p_{12}, p_{22}, \dots, p_{m2}), \dots, (p_{1\sigma}, p_{2\sigma}, \dots, p_{m\sigma})$$

of points which characterise different edges of the domain $\mathcal{R}'$.

I argue that the domain $\mathcal{R}'$ does not possess other edges. To demonstrate this, let us suppose that a point $p_1, p_2, \dots, p_m$ characterises an edge of the domain $\mathcal{R}'$.

One will have the equalities

$$p_1\xi_{1h} + p_2\xi_{2h} + \dots + p_m\xi_{mh} = 0, \quad (h = 1, 2, \dots, t) \tag{6}$$

which define the coefficients $p_1, p_2, \dots, p_m$ to a nearby common factor, and one will have the inequalities

$$p_1\xi_{1h} + p_2\xi_{2h} + \dots + p_m\xi_{mh} \geq 0. \quad (k = 1, 2, \dots, s) \tag{7}$$

Let (x) be any point of the domain R. One will determine the point (x) with the help of equalities (2). By multiplying the inequalities (7) with ρ_k and by making the sum of inequalities obtained, one will have, because of (2),

$$p_1x_1 + p_2x_2 + \dots + p_mx_m \geq 0.$$

One concludes that the inequalities

$$-p_1x_1 - p_2x_2 - \dots - p_mx_m \geq 0 \text{ and } p_{1k}x_1 + p_{2k}x_2 + \dots + p_{mk}x_m \geq 0,$$

define a domain which is not in m dimensions.

By virtue of the fundamental theorem of Number 10, one will determine in this case positive values or zeros of parameters $\rho, \rho_1, \dots, \rho_\sigma$ which reduce the equation

$$-\rho(p_1x_1 + p_2x_2 + \dots + p_mx_m) + \sum_{k=1}^{\sigma} \rho_k(p_{1k}x_1 + p_{2k}x_2 + \dots + p_{mk}x_m) = 0$$

into an identity.

It follows that

$$p_i = \sum_{k=1}^{\sigma} \frac{\rho_k}{\rho} p_{ik} \text{ where } \frac{\rho_k}{\rho} \geq 0. \quad (k = 1, 2, \dots, \sigma; i = 1, 2, \dots, m)$$

By substituting (6), one will have

$$\sum_{k=1}^{\sigma} \frac{\rho_k}{\rho}(\xi_{1h}p_{1k} + \xi_{2h}p_{2k} + \dots + \xi_{mh}p_{mk}) = 0. \quad (h = 1, 2, \dots, t)$$

By virtue of (4), one finds

$$\frac{\rho_k}{\rho}(\xi_{1h}p_{1k} + \xi_{2h}p_{2k} + \dots + \xi_{mh}p_{mk}) = 0. \ (h = 1, 2, \dots, t; k = 1, 2, \dots, \sigma)$$

Let us suppose that $\frac{\rho_k}{\rho} > 0$, then

$$\xi_{1h}p_{1k} + \xi_{2h}p_{2k} + \dots + \xi_{mh}p_{mk} = 0, \quad (h = 1, 2, \dots, t)$$

therefore the coefficients $p_1, p_2, \dots, p_m$, by virtue of (6), are proportional to the coefficients $p_{1k}, p_{2k}, \dots, p_{mk}$; it follows that the points $(p_{1k}, p_{2k}, \dots, p_{mk})$ and $(p_1, p_2, \dots, p_m)$ characterise the same edge of the domain $\mathcal{R}'$.

By virtue of the fundamental theorem in Number 10, all the points of the domain $\mathcal{R}'$ will be determined by the equality (1), this results in that the domains $\mathcal{R}$ and $\mathcal{R}'$ coincide.

Corollary. Let us suppose that a face $P(\mu)$ in μ dimensions of the domain R be determined by the equations

$$p_{1k}x_1 + p_{2k}x_2 + \dots + p_{mk}x_m = 0, \quad (k = 1, 2, \dots, \tau)$$

and that any point (x) belonging to the face $P(\mu)$ be determined by the equalities

$$x_i = \sum_{k=1}^{t} \rho_k\xi_{ik} \text{ where } \rho_k \geq 0. \quad (k = 1, 2, \dots, t; i = 1, 2, \dots, m)$$

The correlative domain $\mathcal{R}$ will possess a corresponding face $\mathcal{B}(m-\mu)$ in $m-\mu$ dimensions determined by the equations

$$\xi_{1k}x_1 + \xi_{2k}x_2 + \ldots + \xi_{mk}x_m = 0 \quad (k = 1, 2, \ldots, t)$$

and any point (x) belonging to the face $\mathcal{B}(m-\mu)$ will be determined by the equalities

$$x_i = \sum_{k=1}^{\tau} \rho_k p_{ik} \text{ where } \rho_k \geq 0. \quad (k = 1, 2, \ldots, \tau; i = 1, 2, \ldots, m)$$

Definition of domains of quadratic forms corresponding to the various perfect forms

15 Let us consider any one perfect quadratic form φ.

Let us suppose that all the representations of the minimum of the form φ make up the series

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1s}, l_{2s}, \ldots, l_{ns}). \tag{1}$$

By indicating

$$\lambda_k = l_{1k}x_1 + l_{2k}x_2 + \ldots + l_{nk}x_n, \quad (k = 1, 2, \ldots, s) \tag{2}$$

one corresponds to the series (1) a series of linear forms

$$\lambda_1, \lambda_2, \ldots, \lambda_s.$$

Let us consider a domain R of quadratic forms determined by the equality

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{s} \rho_k \lambda_k^2$$

with condition that

$$\rho_k \geq 0. \quad (k = 1, 2, \ldots, s)$$

One will say that the domain R correspond to the perfect form φ.

Let us notice that the domain R is in $\frac{n(n+1)}{2}$ dimensions.

By supposing the contrary let us suppose that all the quadratic forms belonging to the domain R verifies a linear equation

$$\Psi(f) = \sum p_{ij}a_{ij} = 0.$$

On the ground of the established definition, one will have the equalities

$$\Psi(\lambda_k^2) = 0 \quad (k = 1, 2, \ldots, s)$$

or, that which comes to the same thing, because of (2),

$$p_{ij}l_{ik}l_{jk} = 0 \quad (k = 1, 2, \ldots, s)$$

which is impossible, the form φ being perfect.

On the ground of what has been said in Number 9–14, the domain R possesses s edges characterised by the quadratic forms

$$\lambda_1^2, \lambda_2^2, \ldots, \lambda_s^2. \tag{3}$$

Let us suppose that one had determined all the faces

$$P_1, P_2, \ldots, P_\sigma$$

in $\frac{n(n+1)}{2} - 1$ dimensions of the domain R.

Each face P_k can be determined by two methods:

1. All the quadratic forms belonging to the face P_k verify an equation

$$\Psi_k(f) = \sum p_{ij}^{(k)}a_{ij} = 0$$

which can be determined in such a way that the inequality

$$\Psi_k(f) > 0$$

held so long as the form f belonging to the domain R is exterior to the face P_k.

2. By choosing among the quadratic forms (3) these

$$\lambda_1^2, \lambda_2^2, \ldots, \lambda_t^2$$

which verify the equation (4), one will determine all the quadratic forms belonging to the face P_k by the equalities

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{t} \rho_k \lambda_k^2,$$

where

$$\rho_k \geq 0. \quad (k = 1, 2, \ldots, t)$$

By virtue of the theorem of Number 14, the domain R can be considered as a set of points verifying the inequalities

$$\Psi_k(f) \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

On the extreme quadratic forms

16

Let us indicate by $M(a_{ij})$ the minimum and by $D(a_{ij})$ the determinant of a positive quadratic form $\sum a_{ij}x_ix_j$. The positive quadratic form $\frac{1}{\sqrt[n]{D(a_{ij})}}\sum a_{ij}x_ix_j$ will be of determinant 1 and will possess the minimum

$$\frac{M(\alpha_{ij})}{\sqrt[n]{D(a_{ij})}} = \mathcal{M}(a_{ij}).$$

Let us examine the various value of the function $\mathcal{M}(a_{ij})$ which is well determined in the set (f) of all the positive quadratic forms in n variables.

Definition. One will call extreme ‡ *a positive quadratic form $\sum a_{ij}x_ix_j$ which enjoys the property that the corresponding value of the function $\mathcal{M}(a_{ij})$ is minimum.*

Let us notice that the function $\mathcal{M}(a_{ij})$ does not change its value when one replaces quadratic form $\sum a_{ij}x_ix_j$ by a form of proportional coefficients.

By attributing to the coefficients of the extreme form $\sum a_{ij}x_ix_j$ variations

$$\epsilon_{ij} = \epsilon_{ji} \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

satisfying the condition

$$|\epsilon_{ij}| < \epsilon, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{1}$$

ϵ being an arbitrary positive parameter, let us examine the corresponding value of the function $\mathcal{M}(a_{ij})$.

On the ground of the definition established, one can determine the parameter ϵ such that the inequality

$$\mathcal{M}(a_{ij} + \epsilon_{ij}) < \mathcal{M}(a_{ij}) \tag{2}$$

held with condition (1) and so long as the coefficients ϵ_{ij} are not proportional to the coefficients

$$a_{ij}. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

17

Theorem. For a quadratic form $\sum a_{ij}x_ix_j$ to be extreme, it is necessary and sufficient that it be perfect and that its adjointed form $\sum \frac{\partial D(a_{ij})}{\partial a_{ij}}x_ix_j$ be interior to the domain corresponding to the form $\sum a_{ij}x_ix_j$.

Let us indicate by

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1s}, l_{2s}, \ldots, l_{ns}) \tag{3}$$

the various representations of the minimum $M(a_{ij})$ of the form $\sum a_{ij}x_ix_j$.

Let us consider a quadratic form $\sum(a_{ij} + \rho\epsilon_{ij})x_ix_j$, the parameter ρ being arbitrary. One can determine an interval

$$-\delta < \rho < \delta \text{ where } 0 < \delta < 1 \tag{4}$$

such that all the representations of the minimum of the form $\sum(a_{ij} + \rho\epsilon_{ij})x_ix_j$ are found among the systems (3) so long as the variations ϵ_{ij} satisfy the condition (1).

By indicating with

$$M' = \sum(a_{ij} + \rho\epsilon_{ij})l_{ik}l_{jk} \text{ and } M = \sum a_{ij}l_{ik}l_{jk} \tag{5}$$

the minima of forms $\sum(a_{ij} + \rho\epsilon_{ij})x_ix_j$ and $\sum a_{ij})x_ix_j$ and with D' and D their determinants, one will have

$$\mathcal{M}(a_{ij} + \rho\epsilon_{ij}) = \frac{\sum(a_{ij} + \rho\epsilon_{ij})l_{ik}l_{jk}}{\sqrt[n]{D'}}, \quad \mathcal{M}(a_{ij}) = \frac{\sum a_{ij}l_{ik}l_{jk}}{\sqrt[n]{D}}.$$

By virtue of (2), one obtains the inequality

$$\frac{\sum(a_{ij} + \rho\epsilon_{ij})l_{ik}l_{jk}}{\sqrt[n]{D'}} < \frac{\sum a_{ij}l_{ik}l_{jk}}{\sqrt[n]{D}}$$

or, that which comes to the same thing,

$$\rho\sum \epsilon_{ij}l_{ik}l_{jk} < M\left(\sqrt[n]{\frac{D'}{D}} - 1\right). \tag{6}$$

This declared, let us suppose that the form $\sum a_{ij}x_ix_j$ be not perfect.

One will determine in this case the variations ϵ_{ij} such that the equalities

$$\sum \epsilon_{ij}l_{ik}l_{jk} = 0,$$

held. By virtue of (6), one will obtain the inequality

$$D' > D.$$

---

‡ See the Mémoire of Mr.'s Korkine and Zolotareff, Sur les formes quadratiques [On the quadratic forms], Mathematische Annalen V. VI, p. 368

By developping the determinant D' into a series, one will have the inequality

$$\rho \sum \epsilon_{ij} \frac{\partial D}{\partial a_{ij}} + \frac{\rho^2}{2} \sum \epsilon_{ij}\epsilon_{kh} \frac{\partial^2 D}{\partial a_{ij} \partial a_{kh}} + \ldots > 0. \tag{7}$$

The parameter ρ being arbitrary satisfying the condition (4), it is necessary tht

$$\sum \epsilon_{ij} \frac{\partial D}{\partial a_{ij}} = 0.$$

Mr.'s Korkine and Zolotareff have demonstrated ‡ that in this case one will always have the inequality

$$\sum \epsilon_{ij}\epsilon_{kh} \frac{\partial^2 D}{\partial a_{ij} \partial a_{kh}} < 0,$$

therefore the inequality (7) is impossible.

18 We have demonstrated that the form $\varphi = \sum a_{ij} x_i x_j$ has to be perfect.

Let us suppose that the domain R corresponding to the perfect form φ be determined by σ inequalities

$$\Psi_r(f) = \sum p_{ij}^{(r)} a_{ij} \geq 0. \quad (r = 1, 2, \ldots, \sigma)$$

On the grounds of these inequalities, one will have

$$\Psi_r(\lambda_k^2) = \sum p_{ij}^{(r)} l_{ik} l_{jk} \geq 0. \;\; (k = 1, 2, \ldots, s; r = 1, 2, \ldots, \sigma) \tag{8}$$

Let us declare

$$\epsilon_{ij} = t p_{ij}^{(r)} \;\text{ where }\; t > 0. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

By virtue of (6), one will have

$$\rho t \sum p_{ij}^{(r)} l_{ik} l_{jk} < M \left(\sqrt[n]{\frac{D'}{D}} - 1 \right). \tag{9}$$

Let us attribute to the parameter ρ a positive value satisfying the condition (4), by virtue of (8) and (9) there will arrive

$$D' > D.$$

By developing the determinant D' into a series, one obtains the inequality

$$\rho t \sum p_{ij}^{(r)} \frac{\partial D}{\partial a_{ij}} + \frac{(\rho t)^2}{2} \sum p_{ij}^{(r)} p_{ih}^{(r)} \frac{\partial^2 D}{\partial a_{ij} \partial a_{kh}} + \ldots > 0.$$

The positive parameter ρ being as small as one wish, it follows that

$$\sum p_{ij}^{(r)} \frac{\partial D}{\partial a_{ij}} > 0. \quad (r = 1, 2, \ldots, \sigma)$$

It is thus demonstrated that the form $\sum \frac{\partial D}{\partial a_{ij}} x_i x_j$, adjointed to the form φ, is interior to the domain R.

I argue that in this case the perfect form φ will be extreme.

By supposing the contrary, let us suppose that the inequality

$$\mathcal{M}(a_{ij} + \epsilon_{ij}) \geq \mathcal{M}(a_{ij}) \tag{10}$$

be verified by any one system of variations ϵ_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$ satisfying the condition (1) however small the parameter ϵ may be.

By virtue of (10), one obtains

$$\sum \epsilon_{ij} l_{ik} l_{jk} \geq M \left(\sqrt[n]{\frac{D'}{D}} - 1 \right); \;\; (k = 1, 2, \ldots, s) \tag{11}$$

the inequality obtained has to hold whatever may te the value of the index $k = 1, 2, \ldots, s$.

By indicating

$$\eta_{ij} = a_{ij} \left(\sqrt[n]{\frac{D}{D'}} - 1 \right) + \epsilon_{ij} \sqrt[n]{\frac{D}{D'}}, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{12}$$

let us examine the quadratic form

$$\varphi_0(x_1, x_2, \ldots, x_n) = \sum (a_{ij} + \eta_{ij}) x_i x_j. \tag{13}$$

By virtue of (12) the form φ_0 is of determinant D.

---

‡ Mathematische Annalen, V. XI, p. 250

By choosing the parameter ϵ sufficiently small, one can suppose that

$$|\eta_{ij}| < \eta, (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{14}$$

η being a positive parameter as small as one would like.

By virtue of (5), (11) and (12), one obtains

$$\sum \eta_{ij} l_{ik} l_{jk} \geq 0. \quad (k = 1, 2, \ldots, s) \tag{15}$$

By developping the determinant D of the form (13) into series, one will find

$$D + \sum \eta_{ij} \frac{\partial D}{\partial a_{ij}} + R_2 = D. \tag{16}$$

In this equality the remainder R_2 verifies an inequality

$$|R_2| < \eta^2 P,$$

P being a positive number not depending on the parameter η so long as $\eta < 1$.

By virtue of (16), one obtains

$$\left| \sum \eta_{ij} \frac{\partial D}{\partial a_{ij}} \right| < \eta^2 P. \tag{17}$$

We have suppose that the quadratic form $\sum \frac{\partial D}{\partial a_{ij}} x_i x_j$, adjointed to the form φ, be interior to the domain R. On the ground of that which has been said in Number 13, one will determine the form $\sum \frac{\partial D}{\partial a_{ij}} \cdot x_i x_j$ with the help of the equality

$$\sum \frac{\partial D}{\partial a_{ij}} x_i x_j = \sum_{k=1}^{s} \rho_k \lambda_k^2 \tag{18}$$

where

$$\rho_k > 0. \quad (k = 1, 2, \ldots, s) \tag{19}$$

The equality (18) can be replaced by the following ones:

$$\frac{\partial D}{\partial a_{ij}} = \sum_{k=1}^{s} \rho_k l_{ik} l_{jk}. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

By multiplying these equations by η_{ij} and by adding up the equalities obtained, one will have

$$\sum \eta_{ij} \frac{\partial D}{\partial a_{ij}} = \sum_{k=1}^{s} \rho_k \sum \eta_{ij} l_{ik} l_{jk}. \tag{20}$$

By virtue of (15), (17) and (19), one obtains the inequalities

$$0 \leq \sum \eta_{ij} l_{ik} l_{jk} < \eta^2 \frac{P}{\rho_k}; \quad (k = 1, 2, \ldots, s)$$

therefore one can admit

$$\sum \eta_{ij} l_{ik} l_{jk} = \tau_k \eta^2, \quad (k = 1, 2, \ldots, s) \tag{21}$$

and the positive numbers or zeros τ_k $(k = 1, 2, \ldots, s)$ will not exceed fixed limits which do not depend on the parameter η.

After the definition of perfect forms, the equations (21) admit only a single system of solutions. By effecting this solution of equations (21), one obtains

$$\eta_{ij} = \tau_{ij} \eta^2 \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

where

$$|\tau_{ij}| < T, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

T being a positive number which does not depend on the parameter η; therefore one will have the inequalities

$$|\eta_{ij}| < \eta^2 T. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{22}$$

This stated, let us take any one positive fraction ϑ and declare

$$\eta = \frac{\vartheta}{T}.$$

By virtue of (14), one will have

$$|\eta_{ij}| < \frac{\vartheta}{T}, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

and because of (22), it will become

$$|\eta_{ij}| < \frac{\vartheta^2}{T}. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

By admitting

$$\eta_{ij} < \frac{\vartheta^2}{T},$$

one will have, because of (22),

$$|\eta_{ij}| < \frac{\vartheta^4}{T}, \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

and so on.

One will obtain in this manner the inequalities

$$|\eta_{ij}| < \frac{\vartheta^{2^k}}{T} \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; k = 0, 1, 2, \ldots)$$

it follows that

$$\eta_{ij} = 0. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

By virtue of (12), one obtains

$$\epsilon_{ij} = a_{ij} \left(\sqrt[n]{\frac{D'}{D}} - 1 \right); \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

therefore the coefficients ϵ_{ij} are proportional to the coefficients a_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$, which is contrary to the hypothesis.

Properties of the set of domains corresponding to the various perfect forms in n variables.

19

Any perfect form φ will be transformed by an equivalent perfect form φ' with the help of any substitution S of integer coefficients and of determinant ±1.

Let us indicate by R and R' the domains corresponding to the perfect forms φ and φ' and by T the substitution adjointed to the substitution S.

One will easily demonstrate that the domain R will be transformed into an equivalent domain R' with the help of the substitution T.

One concludes that there exists a finite number of domains equivalent to the domain R.

Let us indicate by (R) the set of all the domains corresponding to the various perfect forms in n variables.

The set (R) can be divided into classes of equivalent domains.

On the ground of that which has been previously said, the number of different classes of the set (R) is equal to the number of classes of perfect forms in n variables.

20

Theorem. Let us suppose that a quadratic form f be interior to a face $P(\mu)$ in μ dimensions of the domain R $(\mu = 1, 2, \ldots, \frac{n(n+1)}{2})$.

The form f will belong only to the domains of the set (R) which are contiguous through the face $P(\mu)$.

Let us suppose that the domain R be characterised by the quadratic form

$$\lambda_1^2, \lambda_2^2, \ldots, \lambda_s^2 \tag{1}$$

and that the face $P(\mu)$ in μ dimensions of the domain R be characterised by the quadratic forms

$$\lambda_1^2, \lambda_2^2, \ldots, \lambda_t^2. \tag{2}$$

In the case $\mu = \frac{n(n+1)}{2}$, one will admit $t = s$, and the symbol $P\left(\frac{n(n+1)}{2}\right)$ will indicate the domain R.

The quadratic form f being interior to the face $P(\mu)$, one will have the equality

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{t} \rho_k \lambda_k^2 \text{ where } \rho_k > 0. \ (k = 1, 2, \ldots, t) \tag{3}$$

Let us suppose that the same form f belonged to another domain R' of the set (R).

Let us suppose that the domain R' be characterised by the quadratic forms

$${\lambda'_1}^2, {\lambda'_2}^2, \ldots, {\lambda'_\sigma}^2 \tag{4}$$

and that the form f be interior to the face $P'(\nu)$ of the domain R' characterised by the quadratic forms

$${\lambda'_1}^2, {\lambda'_2}^2, \ldots, {\lambda'_\tau}^2. \tag{5}$$

One will have, on the ground of the supposition made,

$$f(x_1, x_2, \ldots, x_n) = \sum_{h=1}^{\tau} \rho'_h {\lambda'_h}^2 \text{ where } \rho'_h > 0. \ (h = 1, 2, \ldots, \tau) \tag{6}$$

This declared, let us indicate by φ and φ' the perfect forms corresponding to the domains R and R' and suppose, for more simplicity, that the minimum of forms φ and φ' be M.

By indicating with the symbol (f, f') the result

$$(f, f') = \sum a_{ij} a'_{ij},$$

from two quadratic forms

$$f(x_1, x_2, \dots, x_n) = \sum a_{ij} x_i x_j \text{ and } f'(x_1, x_2, \dots, x_n) = \sum a'_{ij} x_i x_j,$$

let us examine two results (f, φ) and (f, φ').

By virtue of (13), one obtains

$$(f, \varphi) = \sum_{k=1}^{t} \rho_k (\varphi, \lambda_k^2) \text{ and } (f, \varphi') = \sum_{k=1}^{t} \rho_k (\varphi', \lambda_k^2). \tag{7}$$

By virtue of (6), one obtains

$$(f, \varphi) = \sum_{h=1}^{\tau} \rho'_h (\varphi, {\lambda'_h}^2) \text{ and } (f, \varphi') = \sum_{h=1}^{\tau} \rho'_h (\varphi', {\lambda'_h}^2). \tag{8}$$

Let us notice that

$$(\varphi, \lambda_k^2) = M \text{ and } (\varphi', \lambda_k^2) \geq M; \quad (k = 1, 2, \dots, s) \tag{9}$$

$$(\varphi, {\lambda'_h}^2) \geq M \text{ and } (\varphi', {\lambda'_h}^2) = M. \quad (h = 1, 2, \dots, \sigma) \tag{10}$$

From equalities (7), one derives

$$(f, \varphi') - (f, \varphi) = \sum_{k=1}^{t} \rho_k \left[(\varphi', \lambda_k^2) - (\varphi, \lambda_k^2) \right], \tag{11}$$

and by virtue of (3) and (9) there comes

$$(f, \varphi') - (f, \varphi) \geq 0.$$

From equalities (8), one derives

$$(f, \varphi') - (f, \varphi) = \sum_{h=1}^{\tau} \rho'_h \left[(\varphi', {\lambda'_h}^2) - (\varphi, {\lambda'_h}^2) \right], \tag{12}$$

and by virtue of (6) and (10), one will have

$$(f, \varphi') - (f, \varphi) \leq 0.$$

It follows that

$$(f, \varphi') = (f, \varphi),$$

and the equalities (11) and (12) give

$$(\varphi', \lambda_k^2) = (\varphi, \lambda_k^2), \quad (k = 1, 2, \dots, t)$$

$$(\varphi', {\lambda'_h}^2) = (\varphi, {\lambda'_h}^2). \quad (h = 1, 2, \dots, \tau)$$

By virtue of (9) and (10), there arrive

$$(\varphi, {\lambda'_h}^2) = M, \quad (h = 1, 2, \dots, \tau) \tag{13}$$

$$(\varphi', \lambda_k^2) = M. \quad (k = 1, 2, \dots, t) \tag{14}$$

By virtue of equalities (13), the quadratic forms (5) are found among those of the series (1). By virtue of (14), the quadratic forms (2) are found among those of the series (4).

I argue that in this case the series (2) and (5) contain the same forms.

To demonstrate this, let us suppose that all the forms belonging to the face $P(\mu)$ verify the equations

$$\Psi_1(f) = 0, \ \Psi_2(f) = 0, \dots, \ \Psi_r(f) = 0$$

and that any form belonging to the domain R verifies the inequalities

$$\Psi_1(f) \geq 0, \ \Psi_2(f) \geq 0, \dots, \ \Psi_r(f) \geq 0. \tag{15}$$

By virtue of (6), one will have

$$\rho'_1 \Psi_i({\lambda'_1}^2) + \rho'_2 \Psi_i({\lambda'_2}^2) + \dots + \rho'_\tau({\lambda'_\tau}^2) = 0, \quad (i = 1, 2, \dots, r)$$

and because of (15), one finds

$$\Psi_i({\lambda'_h}^2) = 0; \quad (i == 1, 2, \dots, r; h = 1, 2, \dots, \tau)$$

therefore all the forms of the series (5) belong to the series (2).

In the same way, one will demonstrate that all the forms of the series (2) belong to the series (5).

One concludes that the faces $P(\mu)$ and $P'(\nu)$ coincide, therefore the domains R and R' are contiguous through the face $P(\mu)$.

Corollary. A quadratic form which is interior to a domain of the set (R) *can not belong to any other domain of that set.*

21 *Theorem. Let us suppose that to a face $P(\mu)$ of the domain R belong positive quadratic forms. In this case, the number of domains of the set (R) contiguous through the face $P(\mu)$ is finite.*

Let us indicate by

$$R, R_1, R_2, \ldots$$

the domains of the set (R) contiguous through the face $P(\mu)$. Let

$$\varphi, \varphi_1, \varphi_2, \ldots$$

be the corresponding perfect forms having the minimum M.

On the ground of the supposition made, one positive quadratic form f will be interior to the face $P(\mu)$.

We have demonstrated in the previous number that

$$(f, \varphi) = (f, \varphi_1) = (f, \varphi_2) = \ldots \tag{16}$$

It is easy to demonstrate that the number of perfect forms having the minimum M and verifying the equalities (16) is finite.

Algorithm for the search for the domain of the set (R) contiguous to another domain by a face in $\frac{n(n+1)}{2} - 1$ dimensions

22 Let

$$\varphi(x_1, x_2, \ldots, x_n) = \sum a_{ij} x_i x_j$$

be a perfect form having the minimum M the various representation of which make up a series

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1s}, l_{2s}, \ldots, l_{ns}). \tag{1}$$

Let us suppose that a face P of the domain R corresponding to the perfect form φ be determined by the equation

$$\Psi(f) = \sum p_{ij} a_{ij} = 0$$

and by the condition

$$\Psi(f) \geq 0$$

which is verified by any quadratic form belonging to the domain R.

Let us suppose that the face P be characterised by the quadratic forms

$$\lambda_1^2, \lambda_2^2, \ldots, \lambda_t^2 \tag{2}$$

where

$$\lambda_k = l_{1k} x_1 + l_{2k} x_2 + \ldots + l_{nk} x_n. \quad (k = 1, 2, \ldots, s)$$

On the ground of the supposition made, one will have the equalities

$$\sum p_{ij} l_{ik} l_{jk} = 0 \ \ (k = 1, 2, \ldots, t) \tag{3}$$

which define the coefficients P_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$ to an immediate common factor.

Let us suppose that the face P could belong to the other domains of the set (R). Let us indicate by R' a similar domain. Let φ' be the perfect form corresponding to the domain R'.

By virtue of the supposition made, the quadratic form (2) belong to the domains R and R'. It results in that the systems

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, [\ldots,] l_{n2}), \ldots, (l_{1t}, l_{2t}, \ldots, l_{nt}) \tag{4}$$

corresponding to the forms (2) represent the minimum of forms φ and φ'.

Let us suppose, for more simplicity, that the forms φ and φ' had the minimum M. One will have the equalities

$$\sum a_{ij} l_{ik} l_{jk} = M \ \text{ and } \ \sum a'_{ij} l_{ik} l_{jk} = M, \quad (k = 1, 2, \ldots, t) \tag{5}$$

by putting

$$\varphi'(x_1, x_2, \ldots, x_n) = \sum a'_{ij} x_i x_j.$$

From equation (5), one gets

$$\sum (a'_{ij} - a_{ij}) l_{ik} l_{jk} = 0, \quad (k = 1, 2, \ldots, t)$$

and by virtue of (3), it becomes

$$a'_{ij} = a_{ij} + \rho p_{ij}. \ \ (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{6}$$

Let us indicate

$$\Psi(x_1, x_2, \ldots, x_n) = \sum p_{ij} x_i x_j.$$

By virtue of (16), one obtains

$$\varphi'(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho \Psi(x_1, x_2, \ldots, x_n). \tag{7}$$

This stated, let us choose in the series (1) a system $(l_{1h}, l_{2h}, \ldots, l_{nh})$ which does not belong to the series (4). As

$$\varphi(l_{1h}, l_{2h}, \ldots, l_{nh}) = M \quad \varphi'(l_{1h}, l_{2h}, \ldots, l_{nh}) \geq M$$

and

$$\Psi(l_{1h}, l_{2h}, \ldots, l_{nh}) > 0,$$

one deduces from the equality

$$\varphi'(l_{1h}, l_{2h}, \ldots, l_{nh}) = \varphi(l_{1h}, l_{2h}, \ldots, l_{nh}) + \rho\Psi(l_{1h}, l_{2h}, \ldots, l_{nh})$$

the inequality

$$\rho \geq 0.$$

The supposition $\rho = 0$ being obviously impossible, one obtains

$$\rho > 0,$$

and it follows that

$$\varphi'(l_{1h}, l_{2h}, \ldots, l_{nh}) > M.$$

Let us indicate by

$$(l'_1, l'_2, \ldots, l'_n), (l''_1, l''_2, \ldots, l''_n), \ldots, (l_1^{(r)}, l_2^{(r)}, \ldots, l_n^{(r)}) \tag{8}$$

all the representations of the minimum of the perfect form φ' which are not found in the series (4). By virtue of (7), one will have

$$\varphi'(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) = \varphi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) + \rho\Psi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) \qquad (\mathrm{k} = 1, 2, \ldots, \mathrm{r})$$

which results in

$$\varphi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) > M \ \text{ and } \ \Psi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) < 0. \qquad (k = 1, 2, \ldots, r) \tag{9}$$

The value of the parameter ρ will have for expression

$$\rho = \frac{\varphi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)}) - 1}{-\Psi(l_1^{(k)}, l_2^{(k)}, \ldots, l_n^{(k)})}. \quad (k = 1, 2, \ldots, r)$$

Let us examine any one value of the function

$$\frac{\varphi(x_1, x_2, \ldots, x_n) - M}{\Psi(x_1, x_2, \ldots, x_n)} \tag{10}$$

determined with the condition

$$\Psi(x_1, x_2, \ldots, x_n) < 0. \tag{11}$$

I argue that one will have the inequality

$$\frac{\varphi(x_1, x_2, \ldots, x_n) - M}{\Psi(x_1, x_2, \ldots, x_n)} \geq \rho.$$

Let us suppose the contrary. By supposing that

$$\frac{\varphi(x_1, x_2, \ldots, x_n) - M}{\Psi(x_1, x_2, \ldots, x_n)} < \rho,$$

one will find, because of (11),

$$\varphi(x_1, x_2, \ldots, x_n) + \rho\Psi(x_1, x_2, \ldots, x_n) < M$$

or, that which comes to the same thing because of (7),

$$\varphi'(x_1, x_2, \ldots, x_n) < M,$$

which is contrary to the hypothesis.

We have arrived at the following important result:

There exists only a single domain R' contiguous to the domain R through the face P. The corresponding perfect form φ' will be determined by the equality (7) provided that the parameter ρ presents the smallest positive value of the function (10).

Let us notice that by virtue of (3) and (9), all the quadratic forms belonging to the domain R' verify the inequality

$$\Psi(f) \leq 0.$$

One concludes that the domains R and R' are found from two opposite sides of the plane in $\frac{n(n+1)}{2} - 1$ dimensions determined by the equation

$$\Psi(f) = 0.$$

23 The smallest positive value of the function (10) can be obtained with the help of operations the number of which is finite.

The whole problem is reduced to the preliminary study of a system $(l_1, l_2, \ldots, l_n)$ of integers verifying the inequality

$$\Psi(l_1, l_2, \ldots, l_n) < 0$$

and satisfying the condition that the quadratic form

$$\varphi_0(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho_0 \Psi(x_1, x_2, \ldots, x_n),$$

where one has admitted

$$\rho_0 \frac{\varphi(l_1, l_2, \ldots, l_n) - M}{-\Psi(l_1, l_2, \ldots, l_n)}$$

be positive.

One will determine in this case all the systems $(x_1, x_2, \ldots, x_n)$ of integers verifying the inequality

$$\varphi(x_1, x_2, \ldots, x_n) \geq M \tag{12}$$

the number of which is finite, and one will find among these systems all those which define the smallest value of the function (10).

Let us indicate by R, as we have done in Number 2, the upper limit of values of the parameter ρ.

The problem is reduced to the study of a system $(l_1, l_2, \ldots, l_n)$ of integers verifying the inequality

$$\varphi(l_1, l_2, \ldots, l_n) + R\Psi(l_1, l_2, \ldots, l_n) < M. \tag{13}$$

It can turn out that the equation

$$\varphi(x_1, x_2, \ldots, x_n) + R\Psi(x_1, x_2, \ldots, x_n) = 0$$

will be verified by integers, one will determine them with the help of equations

$$\frac{\partial \varphi}{\partial x_i} + R\frac{\partial \Psi}{\partial x_i} = 0. \quad (i = 1, 2, \ldots, n)$$

In the case where these equations can not be verified by any one system of integers, one will study the values of linear forms

$$\frac{\partial \varphi}{\partial x_i} + R\frac{\partial \Psi}{\partial x_i} \quad (i = 1, 2, \ldots, n)$$

and one will determine as many as one wish of the systems of integers verifying the inequality (13).

By supposing that a system of integers $(l_1, l_2, \ldots, l_n)$ verifying the inequality (13) were determined, one can look for the smallest positive value ρ of the function (10) with the help of the following procedure.

The inequality (12) can be put under the form

$$\varphi(x_1, x_2, \ldots, x_n)\left(1 - \frac{\rho_0}{R}\right) + \frac{\rho_0}{R}\left[\varphi(x_1, x_2, \ldots, x_n) + R\Psi(x_1, x_2, \ldots, x_n)\right] \leq M,$$

and as

$$\varphi(x_1, x_2, \ldots, x_n) + R\Psi(x_1, x_2, \ldots, x_n) \geq 0,$$

it becomes

$$\varphi(x_1, x_2, \ldots, x_n)\left(1 - \frac{\rho_0}{R}\right) \leq M,$$

or differently

$$\varphi(x_1, x_2, \ldots, x_n) \leq M\frac{R}{R - \rho_0}.$$

Among the systems of integers verifying this inequality one will find all the systems (8) searched for.

Algorithm for the search for the domain of the set (R) to which belongs an arbitrary positive quadratic form.

24 *Theorem. Any positive quadratic form belongs to at least one domain of the set (R).*

Let

$$f(x_1, x_2, \ldots, x_n) = \sum a_{ij} x_i x_j$$

be any one positive quadratic form.

Let us choose any one domain R from the set (R).

Let us suppose that the form f, did not belong to the domain (R).

In that case all the linear inequalities which defined the domain R will not be verified. Let us suppose that one had the inequality

$$\Psi(f) = \sum p_{ij} a_{ij} < 0. \tag{1}$$

Let us indicate by R_1 the domain contiguous to the domain R through the face in $\frac{n(n+1)}{2} - 1$ dimensions determined by the equation

$$\Psi(f) = 0.$$

By indicating with φ and φ_1 the contiguous perfect forms corresponding to the domains R and R_1, one will have, as we have seen in Number 22,

$$\varphi_1(x_1, x_2, \ldots, x_n) = \varphi(x_1, x_2, \ldots, x_n) + \rho\Psi(x_1, x_2, \ldots, x_n) \tag{2}$$

where $\rho > 0$ and $\Psi(x_1, x_2, \ldots, x_n) = \sum p_{ij} x_i x_j$.

Let us examine two results (f, φ) and (f, φ_1). By virtue of (2), one will have

$$(f, \varphi_1) = (f, \varphi) + \rho(f, \Psi),$$

and as, because of (1),

$$(f, \Psi) = \sum p_{ij} a_{ij} < 0,$$

it becomes

$$(f, \varphi) > (f, \varphi_1).$$

Let us suppose that by proceeding in this manner one obtains a series of domains

$$R, R_1, R_2, \dots. \tag{3}$$

By indicating with

$$\varphi, \varphi_1, \varphi_2, \dots \tag{4}$$

the series of corresponding perfect forms, one will have the inequalities

$$(f, \varphi) > (f, \varphi_1) > (f, \varphi_2) > \cdots$$

so long as the form f did not belong to the domains (3).

By noticing that all the perfect forms (4) possess the same minimum M, one will easily demonstrate that the number of perfect forms (4) verifying the integrality

$$(f, \varphi) < P$$

is bounded, whatever may be the value of the positive parameter P.

One concludes that the series of domains (3) will necessarily be terminated by a domain R_m to which belonged the form f considered.

Study of a complete system of domains representing the various classes of the set (R).

25 Let R be any one domain of the set (R). Let us suppose that one had determined all the domains

$$R, R_1, R_2, \dots, R_\sigma \tag{1}$$

contiguous to the domain R through the various faces in $\frac{n(n+1)}{2} - 1$ dimensions, then let us suppose that one had determined all the domains contiguous to the domains (1) and so on.

I say that by proceeding in this manner one will come across any domain of the set (R) arbitrarily chosen.

For example, if one wish to arrive at a domain $R^{(0)}$, one will choose a positive quadratic form f which is interior to the domain $R^{(0)}$ and one will proceed as we have done in Number 24. One will determine this way a series of domains

$$R, R', R'', \dots, R^{(\mu)}, R^{(0)}$$

which are successively contiguous through faces in $\frac{n(n+1)}{2} - 1$ dimensions.

We have seen in Number 19 that the set (R) can be divided into classes of equivalent domains the number of which is finite.

Let us find a system of domains representing the various classes of the set (R).

By starting from the domain R, we have determined all the domains

$$R_1, R_2, \dots, R_\sigma$$

contiguous to the domain R. By not considering equivalent domains as being different, let us choose among the domain (1) those which are not one to one equivalent and are not equivalent to the domain R. Let us suppose that one had obtained the series

$$R, R_1, R_2, \dots, R_{\mu-1} \tag{2}$$

of domains which are not one to one equivalent.

One will study in the same way the domains contiguous to the domains $R, R_1, R_2, \dots, R_{\mu-1}$ and one will extend the series (2) by adding to it new domains

$$R_\mu, R_{\mu+1}, \dots, R_{\nu-1}$$

which are not one to one equivalent and are not equivalent to the domains (2).

By proceeding in this way, one will always obtain a series

$$R, R_1, R_2, \dots, R_{\tau-1} \tag{3}$$

which enjoys the following property: the domain belonging to the series (3) are not one to one equivalent and all the domains contiguous to the domains (3) are equivalent to them.

The series (3) obtained presents a complete system of representations of various classes of the set (R).

26 The study of the series (3) can be facilitated particularly by the help of substitutions which transform into itself the domains of the set (R).

Let us suppose that the domain R corresponding to a perfect form φ be determined by the inequalities

$$\sum p_{ij}^{(k)} a_{ij} \geq 0. \quad (k = 1, 2, \dots, \sigma)$$

By declaring

$$\Psi_k(x_1, x_2, \dots, x_n) = \sum p_{ij}^{(k)} x_i x_j, \quad (k = 1, 2, \dots, \sigma)$$

one will determine, as we have seen in Number 22, by the equalities

$$\varphi_k = \varphi + \rho_k \Psi_k \quad (k = 1, 2, \ldots, \sigma) \tag{4}$$

σ perfect forms $\varphi_1, \varphi_2, \ldots, \varphi_\sigma$. One will call them contiguous to the perfect form φ.

Let us indicate by g the group of substitutions which do not change the perfect form φ.

The perfect forms $\varphi_1, \varphi_2, \ldots, \varphi_\sigma$ being well determined by the perfect form φ, one concludes that all the substitutions of the group g will only permute the forms $\varphi_1, \varphi_2, \ldots, \varphi_\sigma$.

By not considering as different the forms in proportional coefficients, one can say, by virtue of (4), that the group g will only permute the quadratic forms

$$\Psi_1, \Psi_2, \ldots, \Psi_\sigma. \tag{5}$$

Let us suppose that one had chosen in this series the forms

$$\Psi_1, \Psi_2, \ldots, \Psi_{\mu-1} \tag{6}$$

which enjoyed the following properties: each form of the series (5) will be transformed into a form of the series (6) with the help of a substitution belonging to the group g, the forms (6) can not be transformed one to one with the help of substitutions of the group g.

The perfect forms

$$\varphi_k = \varphi + \rho_k \Psi_k \quad (k = 1, 2, \ldots, \mu - 1)$$

can replace the perfect forms (4), therefore one will determine only the values of parameters $\rho_1, \rho_2, \ldots, \rho_{\mu-1}$.

The corresponding domains

$$R_1, R_2, \ldots, R_{\mu-1}$$

can replace the domains (1).

It can come to pass that among the domains $R, R_1, R_2, \ldots, R_{\mu-1}$ are found equivalent domains, one will recognise this with the help of particular methods.

On a reduction method of positive quadratic forms.

27

Definition. One will call reduced any positive quadratic form belonging to any one domain

$$R, R_1, R_2, \ldots, R_{\tau-1} \tag{1}$$

of a complete system of representations of various classes of the set (R).

Let us suppose that one had determined all the substitutions

$$S_1, S_2, \ldots, S_m \tag{2}$$

which transform the domains contiguous with the domains (1) through the faces in $\frac{n(n+1)}{2} - 1$ dimensions into these domains here.

Let f be any one positive quadratic form which is not reduced. One will determine with the help of the algorithm shown in Number 24 a series of domains

$$R, R', R'', \ldots, R^{(h)}$$

successively contiguous. Let us suppose that the domain $R^{(h)}$ be the first one which does not belong to the series (1).

With the help of a substitution S' which is found among those of the series (2), one will transform the domain $R^{(h)}$ into a domain R_k belonging to the series (1).

By transforming the form f with the help of the substitution S' into an equivalent form f', one will determine with the help of the form f' a new series of domains

$$R_k, R'_k, \ldots, R_k^{(t)}$$

and so on.

One will determine in this way a series of substitutions

$$S', S'', \ldots, S^{(\lambda)}$$

which, all, are found among those of the series (2) and the product

$$S = S'S'' \cdots S^{(\lambda)}$$

of which presents a substitution S with the help of which the form f will be transformed into a reduced form.

28

Let us suppose now that two reduced forms f and f' be equivalent.

If one of these forms, for example f, is interior to the domain R_k, the form f' will also be interior to it. One concludes that the form f can be transformed into a form f' only with the help of a substitution which transforms the domain R_k into itself.

Let us suppose that the reduced equivalent forms f and f' be interior to the faces in μ dimensions of domains (1).

In this case one will declare supplementary conditions for the reduced forms f and f'. After having determined all the faces in μ dimensions of domains (1), one will choose a complete system of representatives of these various classes. Let us suppose that this system be formed by the faces in μ dimensions

$$P_1(\mu), P_2(\mu), \ldots, P_\nu(\mu). \tag{3}$$

Any positive quadratic form which is interior to a face in μ dimensions of a domain of the set (R) will be equivalent to a form which is interior to the faces (3), one will call it reduced.

Two reduced positive quadratic forms which are interior to the faces (3) will be equivalent only provided that they be interior to the same face and that the substitution which transforms one of them into another one also transforms this face into itself.

We have arrived at the following result:

A reduced quadratic form can be transformed into another reduced form or into itself only with the help of a substitution which transforms into itself a domain or a face of domains belonging to the series (1).

Second Part

Some applications of the general theory to the study of perfect quadratic forms.

On the principal perfect form

29 We will not consider as different the quadratic forms of proportional coefficients, therefore one can arbitrarily choose the minimum value of a positive quadratic form.

In that which follows, one will study only the perfect quadratic forms whose minimum is 1. One will indicate by D the determinant of these forms.

Among the various perfect forms, one form

$$\varphi = x_1^2 + x_2^2 + \ldots + x_n^2 + x_1x_2 + x_1x_3 + \ldots + x_{n-1}x_n \ ^{\ddagger}$$

where

$$a_{ii} = 1,\ (i = 1,2,\ldots,n),\ a_{ij} = \frac{1}{2},\ (i = 1,2,\ldots,n; j = 1,2,\ldots,n; i \neq j) \text{ and } D = \tfrac{n+1}{2^n}$$

One will call principal the perfect form φ.

The perfect form φ possesses $\frac{n(n+1)}{2}$ representations of the minimum 1, which define $\frac{n(n+1)}{2}$ linear forms

$$\lambda_1 = x_1,\ \lambda_2 = x_2, \ldots,\ \lambda_n = x_n,\ \lambda_{n+1} = x_1 - x_2,\ \lambda_{n+2} = x_1 - x_3, \ldots, \lambda_{\frac{n(n+1)}{2}} = x_{n-1} - x_n.$$

The domain R corresponding to the perfect form φ is made up of all the quadratic forms determined by the equality

$$\sum a_{ij}x_ix_j = \sum_{k=1}^{\frac{n(n+1)}{2}} \rho_k\lambda_k^2 \text{ where } \rho_k \geq 0. \quad (k = 1,2,\ldots,\frac{n(n+1)}{2})$$

From this equality one obtains

$$\rho_k = a_{1k} + a_{2k} + \ldots + a_{nk} \text{ so long as } k = 1,2,\ldots,n,$$

$$\rho_k = -a_{ij} \text{ so long as } k > n; \quad (i = 1,2,\ldots,n; j = 1,2,\ldots,n; i \neq j)$$

therefore the domain R will be determined by the following inequalities:

$$\begin{cases} a_{1k} + a_{2k} + \ldots + a_{nk} \geq 0, \quad (k = 1,2,\ldots,n) \\ -a_{ij} \geq 0. \quad (i = 1,2,\ldots,n; j = 1,2,\ldots,n; i \neq j) \end{cases} \tag{1}$$

By virtue of (1), the perfect form φ possesses $\frac{n(n+1)}{2}$ contiguous perfect forms which are determined by the equalities

$$\begin{cases} \varphi_k = \varphi + \rho_k x_k(x_1, x_2, \ldots, x_n), \quad (k = 1,2,\ldots,n) \\ \varphi_k = \varphi - \rho_k x_i x_j, \quad \left(k = n+1, n+2, \ldots, \frac{n(n+1)}{2}, i = 1,2,\ldots,n; j = 1,2,\ldots,n; i \neq j\right) \end{cases} \tag{2}$$

30 Let us find equivalent forms among the perfect forms contiguous to the perfect form φ.

To this effect, let us determine the group g of substitutions which do not change the form φ.

Let us examine, in the first place, the form adjointed to the form φ.

One will easily demonstrate that the coefficients of the form adjointed to φ are proportional to those of the form

$$\omega = \lambda_1^2 + \lambda_2^2 + \ldots + \lambda_{\frac{n(n+1)}{2}}^2. \tag{3}$$

One concludes, by virtue of the theorem of Number 17, that the principal perfect form φ is extreme.

The quadratic form ω will have for expression

$$\omega = nx_1^2 + nx_2^2 + \ldots + nx_n^2 - 2x_1x_2 - 2x_1x_3 - \ldots - 2x_{n-1}x_n$$

where

$$a_{ii} = n,\ (i = 1,2,\ldots,n) \quad a_{ij} = -1. (i = 1,2,\ldots,n; j = 1,2,\ldots,n; i \neq j)$$

Let us find all the representations of the minimum of the form ω.

The linear forms

$$x_1, x_2, \ldots, x_n, x_1 + x_2 + \ldots + x_n \tag{4}$$

characterise $n+1$ representations of the value n of the form ω.

I say that the form ω has the minimum n and all the representations of the minimum of the form ω are characterised by the linear form (4).

The form φ has been given for the first time by Zolotareff in a Mémoire titled: On an indeterminate equation of the third degree (in Russian)

To demonstrate this, let us examine any one value $\omega(x_1, x_2, \dots, x_n)$ of the form ω. By supposing that none of the numbers $x_1, x_2, \dots, x_n$ becomes zero, one will have by virtue of (13)

$$\omega(x_1, x_2, \dots, x_n) > n,$$

the system $x_1 = 1,\ x_2 = 1, \dots, x_n = 1$ being excluded.

Let us suppose that any one of the numbers $x_1, x_2, \dots, x_n$ does not cancel out and that

$$x_{k+1} = 0,\quad x_{k+2} = 0, \dots,\quad x_n = 0;$$

one obtains, by virtue of (3),

$$\omega(x_1, x_2, \dots, x_k, 0, \dots, 0) = (n - k + 1)(x_1^2 + x_2^2 + \dots + x_k^2) + \sum (x_k - x_h)^2,$$

and it follows that

$$\omega(x_1, x_2, \dots, x_n) \geq k(n - k + 1),$$

therefore

$$\omega(x_1, x_2, \dots, x_n) > n \quad \text{so long as} \quad k \geq 2.$$

This stated, let us indicate by G the group of substitutions which transform into itself the domain R. By virtue of (3), any substitution of the group G does not change the form ω.

The group g being adjointed to the group G, one concludes that each substitution of the group g will only permute te linear forms (4) by changing the sign of a few among them.

By noticing that

$$x_1^2 + x_2^2 + \dots + x_n^2 + (x_1 + x_2 + \dots + x_n)^2 = 2\varphi,$$

one concludes that the group g is composed of all the substitutions which permute the forms

$$x_1^2 + x_2^2 + \dots + x_n^2 + (x_1 + x_2 + \dots + x_n)^2.$$

Let us indicate

$$x_0 = -x_1 - x_2 - \dots - x_n \text{ and } x_0' = -x_1' - x_2' - \dots - x_n', \tag{5}$$

and let $k_0, k_1, \dots, k_n$ be any one permutation of numbers $0, 1, 2, \dots, n$.

By posing

$$x_i = e_i x_{k_i}' \quad \text{where} \quad e_i = \pm 1,\ \ (i = 0, 1, 2, \dots, n) \tag{6}$$

one will have

$$x_0 + x_1 + \dots + x_n = e_0 x_{k_0}' + e_1 x_{k_1}' + \dots + e_n x_{k_n}',$$

and as, because of (5),

$$x_0 = x_1 + \dots + x_n = 0 \quad \text{and} \quad x_0', x_1', \dots, x_n' = 0,$$

it is necessary that

$$e_0 = e_1 = \dots = e_n;$$

therefore the equalities (6) reduce to the one here:

$$x_i = e x_{k_i}'. \quad (i = 0, 1, 2, \dots, n; e = \pm 1) \tag{7}$$

The number of substitutions defined by the formulae obtained is equal to $2 \cdot 1 \cdot 2 \cdots (n + 1)$. By not considering as different two substitutions of opposite coefficients, one will say that the group g is composed of $(n_1)!$ different substitutions. §

With the help of substitutions (7), one can transform any perfect form (2) contiguous to the principal form φ into another form contiguous to the form φ, arbitrarily chosen.

We have arrived at the following important result.

All the perfect forms contiguous to the principal perfect form are equivalent.

31 Let us choose one form among those of the series (2). Let us declare

$$\varphi_1 = \varphi - \rho x_1 x_2.$$

All the perfect form contiguous to the form φ are equivalent to the form φ_1.

Let us find the corresponding value of the parameter ρ.

As we have seen in Number 22, the value searched for of ρ presents the smallest value of the function

$$\frac{\varphi(x_1, x_2, \dots, x_n) - 1}{x_1 x_2} \tag{8}$$

determined with condition

$$x_1 x_2 > 0.$$

One will distinguish in the subsequent studies two cases:

$$1).\ n = 2 \quad \text{and} \quad 2).\ n \geq 3.$$

First case:

By comparing two $n = 2$ binary forms

$$\varphi = x_1^2 + x_2^2 + x_1 x_2 \quad \text{and} \quad \varphi_1 = x_1^2 + x_2^2 + x_1 x_2 - \rho x_1 x_2,$$

---

§ See: *Minkowski*, Zur Theorie der positiven quadratischen Formen [On the theory of the positive quadratic forms], (This Journal, V. 101, p. 200) one will notice that by making $\rho = 2$ one obtains the form

$$\varphi_1 = x_1^2 + x_2^2 - x_1 x_2$$

which is evidently equivalent to the perfect form φ, therefore the perfect form φ_1 is that which one has searched for.

Second case:

By making

$$x_1 = 1,\ x_2 = 1,\ x_3 = -1,\ x_4 = 0, \dots,\ x_n = 0,$$

one obtains a value of the function (8) which is equal to 1.

By making $\rho = 1$, one will present the form φ_1 under the following form:

$$\varphi_1 = \frac{1}{2}\left[(x_1, x_2, \dots, x_n)^2 + (x_1 - x_2)^2 + x_3^2 + \dots + x_n^2\right]. \tag{9}$$

It results in that the form φ_1 is positive. On the ground of that which has been said in Number 23, one will find now all the systems of integers veifying the inequality

$$\varphi_1(x_1, x_2, \dots, x_n) \leq 1.$$

By noticing that the inequality

$$\varphi_1(x_1, x_2, \dots, x_n) < 1$$

is impossible, because the positive form φ_1 has integer values which corresponds to the integer values of variables, one concludes that the form φ_1 is perfect.

With the help of the equality (9), one will easily determine all the presentations of the minimum of the perfect form φ_1.

On the binary and ternary perfect forms and on the domains which correspond to them.

32 The binary principal perfect form

$$\varphi = x_2 + xy + y_2, \quad D = \frac{3}{4}$$

possesses, as we have seen in Number 29, three contiguous perfect forms which are equivalent to the principal form.

One concludes that all the perfect binary forms constitute only a single class of forms equivalent to the principal form.

The domain $\mathcal{R}$ corresponding to the principal form is made up of binary forms (a, b, c) which are determined by the equality

$$ax^2 + 2bxy + cy^2 = \rho x^2 + \rho' y^2 + \rho''(x - y)^2$$

where

$$\rho \geq 0, \quad \rho' \geq 0, \quad \rho'' \geq 0$$

It follows that the domain R is determined by the inequalities

$$\rho = a + b \geq 0, \quad \rho' = -b \geq 0, \quad \rho'' = c + b \geq 0.$$

By calling reduced the positive binary forms verifying these inequalities, as we have done in Number 27, one will establish a well known method of reduction, due to Mr. Selling. ¶

It results in that the domain R^0 is determined by the inequalities

$$\rho = c - a \geq 0, \quad \rho' = a + 2b \geq 0, \quad \rho'' = -b \geq 0.$$

The inequalities obtained only differ from famous conditions of reduction of positive binary quadratic forms due to Lagrange by the choice of the sign of the coefficient b, that which one can arbitrarily make in the method of Lagrange. ‡

33 Let us examine now the ternary perfect forms.

The principal perfect form

$$\varphi = x_2 + y^2 + z^2 + yz + zx + xy, \quad D = \frac{1}{2}$$

possesses six contiguous perfect forms which, all, are equivalent to the perfect form

$$\varphi_1 = x_2 + y^2 + z^2 + yz + zx$$

which we have found in Number 31.

The substitution

$$x = -x', \ y = y', \ z = -y' - z'$$

transforms the form φ_1 into principal form.

One concludes that all the ternary perfect forms form only a single class.

¶ *Selling.* Über die binären und ternären quadratischen Formen. [On the binary and ternary quadratic forms] (This Journal, V. 77, p. 143)

‡ See: *Lagrange.* Recherches d'Arithmétique. [Studies in arithmetic] (Oeuvres de Lagrange published by Serret, V. III, p. 698)

Gauss. Disquisitiones arithmeticae, art. 171. (Werke, V. I.)

Lejeune Dirichlet. Vorlesungen über Zahlentheorie [Letcures on number theory], published by Dedekind, (Braunschweig 1894, §64, p. 155)

The domain R corresponding to the principal form is made up of all the ternary quadratic forms $\begin{pmatrix} a & a' & a'' \\ b & b' & b'' \end{pmatrix}$ which are determined by the equality

$$ax^2 + a'y^2 + a''z^2 + 2byz + 2b'zx + 2b''xy = \rho_1' + \rho_2 y^2 + \rho_3 z^2 + \rho_4 (y-z)^2 + \rho_5 (z-x)^2 + \rho_6 (x-y)^2.$$

The domain R is determined by the inequalities

$$\begin{matrix} \rho_1 = a + b' + b'' \geq 0, & \rho_2 = a' + b'' + b \geq 0, & \rho_3 = a'' + b + b' \geq 0, \\ \rho_4 = -b \geq 0, & \rho_5 = -b' \geq 0, & \rho_6 = -b'' \geq 0. \end{matrix}$$

By calling reduced the positive ternary quadratic forms belonging to the domain R, one will establish a method of reduction due to Mr. Selling.

The domain R can be partitioned into 24 equivalent parts which can be transformed one into another with the help of 24 substitution adjoined to those which do not change the principal form.

One of these parts, the domain $\mathcal{R}$, will be composed of all the ternary quadratic forms determined by the equality

$$ax^2 + a'y^2 + a''z^2 + 2byz + 2b'zx + 2b''xy = \rho_1 x^2 + \rho_2 y^2 + \rho_3 z^2 + \rho_4 (y-z)^2 + \rho_5 \Psi + \rho_{[6]} \omega$$

where

$$\Psi = x^2 + y^2 + z^2 (y-z)^2 + (z-x)^2, \; \omega = x^2 + y^2 + z^2 (y-z)^2 + (z-x)^2 + (x-y)^2.$$

One will determine the domain $\mathcal{R}$ with the help of inequalities

$$\begin{matrix} \rho_1 = a + 2b' + b'' \geq 0, & \rho_2 = a' + b + b' + b'' \geq 0, & \rho_3 = a'' + b + b' + b'' \geq 0, \\ \rho_4 = -b + b' \geq 0, & \rho_5 = -b' + b'' \geq 0, & \rho_6 = -b'' \geq 0. \end{matrix}$$

The domain $\mathcal{R}$ enjoys the following properties:

1. Any positive ternary quadratic form is equivalent to at least one form belonging to the domain $\mathcal{R}$.
2. Two ternary quadratic forms which are interior to the domain $\mathcal{R}$ can not be equivalent.

By effecting the transformation of the domain $\mathcal{R}$ with the help of all the substitutions of integer coefficients and of determinant ±1, one will make up the set $(\mathcal{R})$ of domains.

Each domain $\mathcal{R}$ belonging to the set $(\mathcal{R})$ possesses six domain contiguous by faces in 5 dimensions.

The domain $\mathcal{R}$ will be transformed into contiguous domain with the help of the following substitutions:

$$S_1 = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad S_2 = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}, \quad S_3 = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{pmatrix},$$
$$S_4 = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad S_5 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad S_6 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Each substitution of this series transforms into itself a corresponding face in 5 dimensions of the domain $\mathcal{R}$ and permutes two domains of the set $(\mathcal{R})$ which are contiguous through this face.

This results in a method for the search for the substitution which transforms a given form into a form belonging to the domain $\mathcal{R}$. This method is analogous to that which has been shown in Number 27.

By calling reduced any positive ternary quadratic form belonging to the domain $\mathcal{R}$, one will establish a new method of reduction of positive ternary quadratic forms which can be considered as a generalisation of the method of reduction of Lagrange.

34 *On the perfect form* $x_1^2 + x_2^2 + \ldots + x_n^2 + x_1 x_3 + x_1 x_4 + \ldots + x_{n-1} x_n$.

Let us examine the perfect form

$$\varphi_1 = x_1^2 + x_2^2 + \ldots + x_n^2 + x_1 x_3 + x_1 x_4 + \ldots + x_{n-1} x_n$$

obtained in Number 31. One has admitted

$$a_{ii} = 1, \; (i = 1, 2, \ldots, n), \; a_{12} = 0, \; a_{ij} = \frac{1}{2}. \qquad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

It results in that

$$D = \frac{1}{2^{n-2}}.$$

By supposing that $n \geq 4$, one will have $n^2 - n$ representations of the minimum of the form φ_1 the number of which is greater than $\frac{n(n+1)}{2}$.

These representations of the minimum of the form φ_1 will be characterised by the linear forms

$$\left\{ \begin{array}{l} \lambda_1 = x_1, \lambda_2 = x_2, \ldots, \lambda_n = x_n, \lambda_{n+1} = x_1 - x_3, \ldots, \lambda_{\frac{n(n+1)}{2}-1} = x_{n-1} - x_n, \; \lambda_{\frac{n(n+1)}{2}} = x_1 + x_2 - x_3, \ldots, \\ \lambda_{\frac{n(n+1)}{2}+n-3} = x_1 + x_2 - x_n, \lambda_{\frac{n(n+1)}{2}+n-2} = x_1 + x_2 - x_3 - x_4, \ldots, \lambda_{n^2-n} = x_1 + x_2 - x_{n-1} - x_n. \end{array} \right. \tag{1}$$

The domain R_1 corresponding to the perfect form φ_1 is made up of forms determined by the equality

$$f(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n^2-n} \rho_k \lambda_k^2 \text{ where } \rho_k \geq 0. \quad (k = 1, 2, \ldots, [n^2 - n])$$

Let us find the linear inequalities which define the domain R_1.
The number of these inequalities is so large in deed for $n = 4$.
One will overcome the difficulties which result by the help of a particular method.

35 Let us find the group g_1 of substitutions which do not change the form φ_1.
To this effect, let us introduce in our studies a quadratic form ω determined by the equality

$$\omega = \frac{2}{n-1}(\lambda_1^2 + \lambda_2^2 + \ldots + \lambda_{n^2-n}^2).$$

After the reductions, one obtains

$$\omega(x_1, x_2, \ldots, x_n) = nx_1^2 + nx_2^2 + 4x_3^2 + \ldots + 4x_n^2 + 2(n-2)x_1x_2 - 4x_1x_3 - \ldots - 4x_1x_n - 4x_2x_3 - \ldots - 4x_2x_n.$$

One can give in the form ω_2 the following expression:

$$\omega(x_1, x_2, \ldots, x_n) = (x_1 - x_2)^2 + (x_1 + x_2)^2 + (x_1 + x_2 - 2x_3)^2 + \ldots + (x_1 + x_2 - 2x_n)^2.$$

It is easy to demonstrate that the form added to the perfect form φ_1 has coefficients which are proportional to those of the form ω.
It follows that the perfect form φ_1 is extreme.
Let us observe that the linear form

$$x_1 + x_2 + \ldots + x_n,\ x_1 - x_2,\ x_3,\ x_4, \ldots,\ x_n$$

characterise n minimum 4 representations of the form ω. In the case $n \geq 5$, other representations of the minimum of the form ω do not exist; in the case $n = 4$, one obtains 12 representations of the minimum of the form ω.
By noticing that

$$\varphi_1 = \frac{1}{2}\left[(x_1 + x_2 + \ldots + x_n)^2 + (x_1 - x_2)^2 + x_3^2 + \ldots + x_n^2\right],$$

one can say that the group g_1, in the case $n \geq 5$, is composed of all the permutations of the forms

$$(x_1 + x_2 + \ldots + x_n)^2,\ (x_1 - x_2)^2,\ x_3^2, \ldots,\ x_n^2.$$

In the case $n = 4$, one will determine by this method only divisor of the group g_1.
By indicating

$$\begin{array}{llll} u_1 = x_1 + x^2 + \ldots + x_n, & u_2 = x_1 - x_2, & u_3 = x_3, & \ldots, \quad u_n = x_n, \\ u'_1 = x'_1 + x'_2 + \ldots + x'_n, & u'_2 = x'_1 - x'_2, & u'_3 = x'_3, & \ldots, \quad u'_n = x'_n \end{array}$$

let us declare

$$u_i = e_i u'_{k_i}, \quad (i = 1, 2, \ldots, n) \tag{2}$$

where $e_i = \pm 1$ $(i = 1, 2, \ldots, n)$ and the indices $k_1, k_2, \ldots, k_n$ present any one permutation of numbers $1, 2, \ldots, n$.
Each system of equalities (2) defines a substitution of the group g_1.
One concludes that the group g_1 is composed of $2^{n-1}n!$ different substitutions, in the case $n \geq 5$.

36 Let us suppose that the domain R_1 be determined by the inequalities

$$\sum p_{ij}^{(k)} a_i \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

By indicating

$$\Psi_k(x_1, x_2, \ldots, x_n) = \sum p_{ij}^{(k)} x_i x_j, \quad (k = 1, 2, \ldots, \sigma)$$

one will determine, as we have seen in Number 22, σ perfect forms

$$\varphi_1^{(k)} = \varphi_1 + \rho_k \Psi_k \quad (k = 1, 2, \ldots, \sigma) \tag{3}$$

contiguous to the perfect form φ_1.
All the substitutions of the group g_1 will make only one permutation of forms

$$\Psi_1, \Psi_2, \ldots, \Psi_\sigma. \tag{4}$$

Let us effect the transformation of forms (3) and (4) with the help of the sustitution

$$x_1, x_2, \ldots, x_n = x'_1,\ x_1 - x_2 = x'_2,\ x_3 = x'_3, \ldots,\ x_n = x'_n. \tag{5}$$

The series (4) will be transformed into a series

$$\Psi'_1, \Psi'_2, \ldots, \Psi'_\sigma.$$

Let us indicate by **g** a group of substitutions

$$x'_i = e_i x''_{k_i}, \quad (i = 1, 2, \ldots, n)$$

where $e = \pm 1$ $(i = 1, 2, \ldots, n)$ and $k_1, k_2, \ldots, k_n$ present a permutation of numbers $1, 2, \ldots, n$. Each substitution of the group **g** makes only one permutation of forms (6), and to a similar substitution corresponds a substitution of the group g_1.

By indicating

$$\Psi'_k(x'_1, x'_2, \ldots, x'_n) = \sum P^{(k)}_{ij} x'_i x'_j, \quad (k = 1, 2, \ldots, \sigma)$$

one will determine with the help of inequalities

$$\sum P^{(k)}_{ij} a_i a_j \geq 0, \quad (k = 1, 2, \ldots, \sigma) \tag{7}$$

a domain $\mathcal{R}$.

The form φ_1 will be transformed into a form

$$\frac{1}{2}({x'_1}^2 + {x'_2}^2 + \ldots + {x'_n}^2),$$

with the help of the substitution (5), and any system $(x_1, x_2, \ldots, x_n)$ of integers $x_1, x_2, \ldots, x_n$ will be replaced by a system $(x'_1, x'_2, \ldots, x'_n)$ of number, also integer, $x'_1, x'_2, \ldots, x'_n$ satisfying the condition

$$x'_1 + x'_2 + \ldots + x'_n \equiv 0 (\mathrm{mod}\ 2). \tag{8}$$

It results in that the linear forms (1) which correspond to the various representations of the minimum of the form φ_1 will be replaced by the forms

$$x'_i + x'_j \text{ and } x'_i - x'_j \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

which characterise the various representations of the minimum 2 of the quadratic form ${x'_1}^2 + {x'_2}^2 + \ldots + {x'_n}^2$, in the set (X') of all the systems $(x'_1, x'_2, \ldots, x'_n)$ of integers $x'_1, x'_2, \ldots, x'_n$ satisfying the condition (8).

One concludes that the edges of the domain $\mathcal{R}$ will be characterised by the quadratic form

$$(x'_i + x'_j)^2 \quad \text{and} \quad (x'_i - x'_j)^2. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

By virtue of (7), one obtains the inequalities

$$P^{(k)}_{ii} + 2P^{(k)}_{ij} + P^{(k)}_{jj} \geq 0 \text{ and } P^{(k)}_{ii} - 2P^{(k)}_{ij} + P^{(k)}_{jj} \geq 0. \quad (\mathrm{k} = 1, 2, \ldots, \sigma; \mathrm{i} = 1, 2, \ldots, \mathrm{n}; \mathrm{j} = 1, 2, \ldots, \mathrm{n}; \mathrm{i} \neq \mathrm{j}) \tag{9}$$

Let us examine any one form

$$\Psi'(x'_1, x'_2, \ldots, x'_n) = \sum P_{ij} x'_i x'_j \tag{10}$$

belonging to the series (6). By virtue of (9), one will have

$$P_{ii} + 2P_{ij} + P_{jj} \geq 0 \text{ and } P_{ii} - 2P_{ij} + P_{jj} \geq 0. \quad (\mathrm{i} = 1, 2, \ldots, \mathrm{n}; \mathrm{j} = 1, 2, \ldots, \mathrm{n}; \mathrm{i} \neq \mathrm{j}) \tag{11}$$

Among these conditions one will find t quantities which define the coefficients of the form (10) to an immediate common factor. All these equalities will be of the form

$$P_{kk} - 2e_{kh} P_{kh} + P_{hh} = 0 \text{ where } e_{kh} = \pm 1. \tag{12}$$

Let us suppose that there exists a combination of values of k and h satisfying the conditions

$$P_{kk} + 2P_{kh} + P_{hh} > 0 \text{ and } P_{kk} - 2P_{kh} + P_{hh} > 0. \tag{13}$$

By noticing that the coefficient P_{kh} does not enter the other inequalities (11), one concludes that the coefficient P_{kh} remains undetermined.

For all the coefficients of the form (10) to be determined by the conditions (12) to an immediate common factor, it is necessary, the coefficient P_{kh} being independent of other coefficients, that all the coefficients which remain cancel out.

By virtue of inequalities (13), this supposition is impossible, therefore the inequality (12) has to hold for all the values of indices k and h.

One obtains $\frac{n(n-1)}{2}$ conditions

$$P_{kk} - 2e_{kh} P_{kh} + P_{hh} = 0 \text{ where } e_{kh} = \pm 1. \quad (\mathrm{k} = 1, 2, \ldots, \mathrm{n}; \mathrm{h} = 1, 2, \ldots, \mathrm{n}; \mathrm{k} \neq \mathrm{h}) \tag{14}$$

which serve to determine the coefficients P_{kh} in functions of coefficients

$$P_{11}, P_{22}, \ldots, P_{nn}. \tag{15}$$

The coefficients $P_{11}, P_{22}, \ldots, P_{nn}$ can not be independent, and will be connected by at least $n-1$ equations of the form (12). Therefore, in at least $n-1$ case, one will have the equations of the form

$$P_{kk} \pm 2P_{kh} + P_{hh} = 0. \tag{16}$$

To make short we will call these equations double.

This stated, let us suppose, in the first place, that there exists at least one coefficient among those of the series (15) which does not enter in the double equations (16). One can suppose, to fix the ideas, that P_{11} be such a coefficient.

The coefficient P_{11} being independent, all the coefficients $P_{22}, \ldots, P_{nn}$ will cancel each other and, by virtue of (14), the coefficients

$$P_{23}, P_{24}, \ldots, P_{n-1,n}$$

will also cancel one another out.

The coefficient P_{11} is used for determining the coefficients $P_{12}, P_{13}, \ldots, P_{1n}$ with the help of equations (14) which take the form

$$P_{11} - 2e_{1k} P_{1k} = 0; \quad (k = 2, 3, \ldots, n)$$

it follows that

$$2P_{1k} = e_{1k}P_{11}. \quad (k = 2, 3, \ldots, n)$$

As, on the ground of the supposition made,

$$P_{11} + 2e_{1k}P_{1k} > 0, \quad (k = 2, 3, \ldots, n)$$

it is necessary that

$$P_{11} > 0,$$

and one can declare

$$P_{11} = 1.$$

The form (10) is determined by the equalities obtained, and one will have

$$\Psi'(x_1', x_2', \ldots, x_n') = {x_1'}^2 + e_{12}x_1'x_2' + \ldots + e_{1n}x_1'x_n'. \tag{17}$$

By replacing the variables

$$e_{12}x_2', \ e_{13}x_3', \ldots, \ e_{1n}x_n'$$

by the variables $x_2', \ldots, x_n'$, one will replace the form (17) by the form

$$\Psi'(x_1', x_2', \ldots, x_n') = x_1'(x_1', x_2', \ldots, x_n').$$

Let us suppose, in the second place, that all the coefficients (15) enter in the double equations (16).

At least one of the coefficients (15) is not zero. Let us suppose that $P_{kk} \neq 0$. Following the hypothesis, the coefficient P_{kk} enters in at least one double equation

$$P_{kk} \pm 2P_{kh} + P_{hh} = 0.$$

It follows that

$$P_{kh} = 0 \text{ and } P_{kk} + P_{hh} = 0,$$

therefore the coefficients P_{kk} and P_{hh} are of opposite signs. Let us suppose, to fix the ideas, that

$$P_{11} = -1. \tag{18}$$

By examining the inequalities

$$P_{11} \pm 2P_{1k} + Pkk \geq 0, \quad (k = 2, 3, \ldots, n)$$

one deduces

$$Pkk > 0. \quad (k = 2, 3, \ldots, n)$$

It results in that the double equation

$$P_{kk} \pm 2P_{kh} + P_{hh} = 0$$

has to be impossible so long as $k \geq 2$ and $h \geq 2$, therefore all the double equations will be of the form

$$P_{11} \pm 2P_{1k} + P_{kk} = 0. \quad (k = 2, 3, \ldots, n)$$

From these equations one gets, by virtue of (18),

$$P_{kk} = 1 \text{ and } P_{1k} = 0. \quad (k = 2, 3, \ldots, n) \tag{19}$$

By substituting the values obtained of coefficients $P_{11}, P_{22}, \ldots, P_{nn}$ in the equations

$$P_{kk} - 2e_{kh}P_{kh} + P_{hh} = 0 \text{ where } e_{kh} = \pm 1, \quad (k = 2, 3, \ldots, n; h = 2, 3, \ldots, n; k \neq h)$$

one obtains, because of (19),

$$P_{kh} = e_{kh} \quad \text{where} \quad e_{kh} = \pm 1. \ (k = 2, 3, \ldots, n; h = 2, 3, \ldots, n; k \neq h)$$

The form (10) will have for expression

$$\Psi'(x_1', x_2', \ldots, x_n') = -{x_1'}^2 + {x_2'}^2 + {x_3'}^2 + \ldots + {x_n'}^2 + 2e_{23}x_2'x_3' + 2e_{24}x_2'x_4' + \ldots + 2e_{n-1,n}x_{n-1}'x_n' \tag{20}$$

where

$$e_{23} = \pm 1, \ e_{24} = \pm 1, \ldots, \ e_{n-1,n} = \pm 1.$$

One obtains in this way $2^{\frac{(n-1)(n-2)}{2}}$ different forms. By permuting the variables and by changing their signs, one will particularly decrease the number of various forms determined by the formula (20).

37 With the help of results obtained, one can easily recognise whether a given quadratic form $\sum a_{ij}x_ix_j$ belongs to the domain $\mathcal{R}$ or not.

One will examine, in the first place, the sums

$$e_{1k}a_{1k} + e_{2k}a_{2k} + \ldots + e_{nk}a_{nk} \text{ where } e_{1k} = \pm 1, e_{2k} = \pm 1, \ldots, e_{nk} = \pm 1 \text{ and } e_{kk} = 1. \quad (k = 1, 2, \ldots, n)$$

All these sums have to be positive or zero. The inequalities

$$a_{kk} - |a_{1k}| - \ldots - |a_{k-1,k}| - |a_{k+1,k}| - \ldots - |a_{nk}| \geq 0, \quad (k = 1, 2, \ldots, n) \tag{21}$$

present the conditions necessary and sufficient for the inequalities

$$e_{1k}a_{1k} + e_{2k}a_{2k} + \ldots + e_{nk}a_{nk} \geq 0 \quad (k = 1, 2, \ldots, n)$$

to hold.

Let us examine, in the second place, the inequalities

$$-a_{11} + a_{22} + a_{33} + \ldots + a_{nn} + 2e_{23}a_{23} + 2e_{24}a_{24} + \ldots + 2e_{n-1,n}a_{n-1,n} \geq 0$$

where

$$e_{23} = \pm 1, \;\; e_{24} = \pm 1, \ldots, \;\; e_{n-1,n} = \pm 1,$$

These inequalities can be replaced by a single one

$$-a_{11} + a_{22} + a_{33} + \ldots + a_{nn} - 2|a_{23}| - 2|a_{24}| - \ldots - 2|a_{n-1,n}| \geq 0.$$

One will present this inequality under the form

$$a_{11} + a_{22} + \ldots + a_{nn} - 2|a_{12}| - 2|a_{13}| - \ldots - 2|a_{n-1,n}| \geq 2\left(a_{11} - |a_{12}| - \ldots - |a_{1n}|\right).$$

By permuting the variables, one obtains n inequalities

$$a_{11} + a_{22} + \ldots + a_{nn} - 2|a_{12}| - 2|a_{13}| - \ldots - 2|a_{n-1,n}| \geq 2\left(a_{kk} - |a_{1k}| - \ldots - |a_{nk}|\right). \text{ where } k = 1, 2, \ldots, n \quad (22)$$

We have arrived at the following result. One can easily recognise whether a given positive quadratic form f belong to the domain R_1 or not. To this effect, one will transform the form f by a form f' with the help of the substitution adjointed to the substitution (5) and one will examine $2n$ inequalities (21) and (22). For the form f to belong to the domain R_1, it is necessary and sufficient that the form f' verifies $2n$ inequalities (21) and (22).

38 Let us return now to the perfect forms (3) contiguous to the perfect form φ_1. We have seen that these forms will be transformed with the help of the substitution (5) into forms

$$\frac{1}{2}\left({x'_1}^2 + {x'_2}^2 + \ldots + {x'_n}^2\right) + \rho_k \Psi'_k(x'_1, x'_2, \ldots, x'_n). \quad (k = 1, 2, \ldots, \sigma)$$

The forms $\Psi'_1, \Psi'_2, \ldots, \Psi'_\sigma$ can be transformed with the help of substitutions belonging to the group **g** into forms

$$\begin{cases} 1).\;\; x'_3(-x'_1 - x'_2 + x'_3 + x'_4 + \ldots + x'_n), \\ 2).\;\; -{x'_2}^2 + {x'_1}^2 + {x'_3}^2 + \ldots + {x'_n}^2 - 2x'_1x'_3 - \ldots - 2x'_1x'_n + 2e_{34}x'_3x'_4 + \ldots + 2e_{n-1,n}x'_{n-1}x'_n, \end{cases} \quad (23)$$

where

$$e_{34} = \pm 1, \ldots, \;\; e_{n-1,n} = \pm 1.$$

The inverse substitution to the substitution (5):

$$x'_1 = x_1 + x_2 + \ldots + x_n, \;\; x'_2 = x_1 - x_2, \;\; x'_3 - x_3, \ldots, \;\; x'_n = x_n$$

will transform the forms (23) into forms

$$1).-2x_1x_3$$

$$2).4(x_1x_2 - \delta_{34}x_3x_4 - \ldots - \delta_{n-1,n}x_{n-1}x_n), \text{ where; } \delta_{34} = 0 \text{ or } 1, \ldots, \delta_{n-1,n} = 0 \text{ or } 1.$$

One concludes that all the perfect forms contiguous to the form φ_1 are equivalent to the following perfect forms

$$1).\varphi_1 - \rho x_1 x_3,$$

$$2).\varphi_1 + \rho(x_1x_2 - \delta_{34}x_3x_4 - \ldots - \delta_{n-1,n}x_{n-1}x_n),$$

where

$$\delta_{34} = 0 \text{ or } 1, \ldots, \;\; \delta_{n-1,n} = 0 \text{ or } 1.$$

Study of the perfect form $\varphi_1 - \rho x_1 x_3$.

39 The perfect form φ_1, possesses, as we have seen in Number 38, many contiguous perfect forms which are not equivalent.

One will determine in the following only a single perfect form

$$\varphi_2 = \varphi_1 - \rho x_1 x_3$$

contiguous to the perfect form φ_1.

We have demonstrated in Number 22 that the parameter ρ presents the smallest value of the function

$$\rho = \varphi_1 \frac{(x_1, x_2, \ldots, x_n) - 1}{x_1 x_3} \quad (1)$$

determined on condiion that

$$x_1 x_3 > 0. \quad (2)$$

By declaring

$$x_1 = 1, \;\; x_2 = 0, \;\; x_3 = 1, \;\; x_4 = -1, \;\; x_5 = 0, \ldots, \;\; x_n = 0,$$

one obtains the value of the function (1) which is equal to 1, therefore

$$0 < \rho \leq 1. \quad (3)$$

Let us effect the transformation of the function (1) with the help of the substitution

$$x_3 = x'_1, \;\; -x_1 + x_2 = x'_2, \;\; x_1 + x_2 + \ldots + x_n = -x'_3, \;\; x_4 = x'_4, \ldots, \;\; x_n = x'_n \quad (4)$$

one will have

$$\rho = \frac{{x'_1}^2 + {x'_2}^2 + \ldots + {x'_n}^2 - 2}{-x'_1(x'_1 + x'_2 + \ldots + x'_n)} \tag{5}$$

where, because of (2),

$$x'_1(x'_1 + x'_2 + \ldots + x'_n) < 0 \tag{6}$$

and, because of (4),

$$x'_1 + x'_2 + \ldots + x'_n \equiv 0 (\mathrm{mod}\ 2),$$

the variables $x'_1 + x'_2 + \ldots + x'_n$ being integers.

Let us indicate

$$f(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_n^2 + \rho x_1(x_1 + x_2 + \ldots + x_n).$$

By virtue of (5) and (6) the value looked for of ρ is defined by the conditions that the inequality

$$f(x_1, x_2, \ldots, x_n) < 2$$

is impossible, so long as the integers $x_1, x_2, \ldots, x_n$ verify the congruence

$$x_1 + x_2 + \ldots + x_n \equiv 0 (\mathrm{mod}\ 2), \tag{7}$$

and that there exists at least one system $(l_1, l_2, \ldots, l_n)$ verifying the equation

$$f(x_1, x_2, \ldots, x_n) = 2 \tag{8}$$

and the congruence (7).

The form f can be determined by the equality

$$f(x_1, x_2, \ldots, x_n) = \left(x_2 + \rho\frac{x_1}{2}\right)^2 + \left(x_3 + \rho\frac{x_1}{2}\right)^2 + \ldots + \left(x_n + \rho\frac{x_1}{2}\right)^2 + \left(1 + \rho - \frac{n-1}{4}\rho^2\right) x_1^2. \tag{9}$$

It follows that the form f will be positive, provided that

$$1 + \rho - \frac{n-1}{4}\rho^2 > 0,$$

and the upper limit R of values of ρ verifies the equation

$$1 + R - \frac{n-1}{4}R^2 = 1,$$

therefore

$$\frac{R = 2}{\sqrt{n} - 1.} \tag{10}$$

40 This presented, let us examine a system $(l_1, l_2, \ldots, l_n)$ of integers verifying the equation (8) and the congruence (7).

I say that there will be the inequalities

$$\left| l_i + \rho\frac{l_1}{2} \right| \leq 1. \quad (i = 2, \ldots, n) \tag{11}$$

In effect, if one suppose that

$$\left| l_k + \rho\frac{l_1}{2} \right| < 1,$$

one will determine $e_k = \pm 1$ such that the inequality

$$\left| l_k + 2e_k + \frac{l_1}{2} \right| < \left| l_k + \rho\frac{l_1}{2} \right|$$

holds, and one will present

$$l'_i = l_i \text{ and } l'_k = l_k = 2e_k. \quad (i = 1, 2, \ldots, n; i \neq k)$$

The condition (7) will be satisfied, and one will have, by virtue of (9), the inequality

$$f(l'_1, l'_2, \ldots, l'_n) < 2,$$

which is contrary to the hypothesis.

By examining the inequalities (11) and the form f with the help of the formula (9), one will easily demonstrate that among the system of integers verifying the equation (8) with condition (7) is found at least one system $(l_1, l_2, \ldots, l_n)$ satisfying the conditions

$$f(l_1, l_2, \ldots, l_n) = 2 \tag{12}$$

and

$$l_2 = l_3 + \delta,\ l_3 = l_4 = \ldots = l_n \text{ where } \delta = 0 \text{ or } \pm 1. \tag{13}$$

By virtue of (6), one will have the inequality

$$l_1\left[l_1 + \delta + (n-2)l_3\right] < 0.$$

One can suppose that

$$l_1 < 0, \tag{14}$$

and it follows that

$$l_1 + \delta + (n-2)l_3 > 0,$$

therefore, because of (13) and (14), it is necessary that

$$l_3 > 0.$$

I say that $l_3 = 1$. To demonstrate this, let us effect the transformation of the positive quadratic form $f(x_1, x_2, \dots, x_n)$ with the help of the substitution

$$x_1 = -x, \; x_2 = y, \; x_3 = x_4 = \ldots = x_n - z; \tag{15}$$

one will obtain a ternary positive form

$$F(x,y,z) = x^2 + y^2 + (n-2)z^2 - \rho x(-x + y + (n-2)z).$$

By virtue of the condition (7), the integers x, y, z verify the congruence

$$x + y + (n-2)z \equiv 0 (\text{mod } 2). \tag{16}$$

By indicating

$$u = -l_1, \quad v = l_2, \quad w = l_3,$$

one will have, because of (12), (13) and (15),

$$F(u,v,w) = 2,$$

and the condition (16) will be fulfilled.

The inequality

$$f(x,y,z) < 2$$

is impossible so long as the integers x, y, z verify the congruence (16).

Let us effect the transformation of the form $F(x,y,z)$ with the help of the substitution

$$x = x_{\prime} + y' + (n-2)z', \;\; y = x' - y', \;\; z = z'. \tag{17}$$

The set of systems (x,y,z) of integers verifying the congruence (16) will be replaced by the set of systems (x', y', z') of arbitrary integers.

Let us indicate by $F'(x', y', z')$ the transformed form. Let D and D' be the determinants of forms $F(x,y,z)$ and $F'(x', y', z')$. By virtue of (17), one will have

$$D' = 4D. \tag{18}$$

Let us notice that the number 2 presents the minimum of the form obtained $F'(x', y', z')$ determined in the set of all the systems (x', y', z') of integers, the system $(0,0,0)$ being excluded.

On the ground of the known theorem § on the limit of the minimum of a ternary positive quadratic form, one will have the inequality

$$2 \leq \sqrt[3]{2D'}.$$

It follows that

$$D' \geq 4,$$

and because of (18), one obtains

$$D \geq 1. \tag{19}$$

This presented, let us observe that the form $F(x,y,z)$ has the following values:

$$F(u,v,w) = 2, \;\; F(1,1,0) = 2, \;\; F(1,-1,0) = 2 + 2\rho.$$

By transforming the form $F(x,y,z)$ with the help of the substitution

$$\begin{pmatrix} u, & 1, & 1 \\ v, & 1, & -1 \\ w, & 0, & 0 \end{pmatrix}, \tag{20}$$

one obtains a form

$$F_0(x', y', z') = ax'^2 + a'y'^2 + a''z'^2 + 2by'z' + 2b'z'x' + 2b''x'y',$$

where

$$a = 2, \; a' = 2, \; a'' = 2 + 2\rho \;\text{ and }\; b = \rho. \tag{21}$$

---

§ See: *Gauss.* Werke, V. II, p. 192, Göttingen 1863.

Lejeune-Dirichlet. Über die Reduktion der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen. (This Journal, V. 40, p. 209)

Hermite. Sur la théorie des formes quadratiques ternaires. [On the theory of ternary quadratic forms] (This Journal, V. 40, p. 173)

The product $a \cdot a' \cdot a''$ in any positive ternary quadratic form $\begin{pmatrix} a, & a', & a'' \\ b, & b' & b'' \end{pmatrix}$ is, as one knows, always greater than the determinant of the form, unless the coefficients b, b', b'' do not simultaneously cancel one another out.

By indicating with D_0 the determinant of the form $F_0(x', y', z')$, one will have, because of (21),

$$D_0 < 4(2 + 2\rho),$$

and as, by virtue of (20),

$$D_0 = 4w^2 D,$$

it becomes

$$w^2 D < 2 + 2\rho.$$

By virtue of (3) and (19), one obtains the inequality

$$w^2 < 4,$$

therefore

$$w = 1.$$

41 By returning to the equalities (13), one obtains

$$l_1 = -u, \quad l_2 = \delta \;\text{ and }\; l_3 = 1, \;\; l_4 = 1, \ldots, \;\; l_n = 1,$$

where

$$u > 0 \quad \text{and} \quad \delta = 0, 1, 2.$$

By substituting the values found of $l_1, l_2, \ldots, l_n$ in the function (5), one will have

$$\rho = \frac{u^2 + \delta^2 + n - 4}{u(-u + \delta + n - 2)} \tag{22}$$

It remains to determine the smallest value of this function providing that

$$u > 0, \; -u + \delta + n - 2 > 0, \; u \equiv n + \delta (\text{mod } 2) \;\text{ and }\; \delta = 0, 1, 2. \tag{23}$$

Let us admit

$$u = \sqrt{n} - 1 + \alpha, \tag{24}$$

α being a real number.

The function (22) takes the form

$$\rho = \frac{2n + (2\alpha - 2)\sqrt{n} + \alpha^2 - 2\alpha + \delta^2 - 3}{\sqrt[n]{n} + (\alpha - 2)n + (\delta - 2\alpha)\sqrt{n} + 1 - \alpha^2 + \alpha\delta - \delta}.$$

The value searched for of ρ has to verify the inequality

$$\rho < R,$$

therefore because of (10), one will have

$$\frac{2}{\sqrt{n} - 1} - \rho > 0. \tag{25}$$

After the reductions, one obtains

$$\frac{2}{\sqrt{n} - 1} - \rho = \frac{(1 - \delta^2 + 2\delta - \alpha^2)\sqrt{n} + \delta^2 - 2\delta - 1 - \alpha^2 - 2\alpha + 2\alpha\delta}{(\sqrt{n} - 1)\left[n\sqrt{n} + (\alpha - 2)n + (\delta - 2\alpha)\sqrt{n} + 1 - \alpha^2 + \alpha\delta - \delta\right]}$$

and, because of (25), it becomes

$$(1 - \delta^2 + 2\delta - \alpha^2)\sqrt{n} + \delta^2 - 2\delta - 1 - \alpha^2 - 2\alpha + 2\alpha\delta > 0.$$

By noticing that

$$\delta^2 - 2\delta - 1 - \alpha^2 - 2\alpha + 2\alpha\delta \leq 0 \quad \text{so long as} \quad \delta = 0, 1, 2,$$

one obtains the inequality

$$1 - \delta^2 + 2\delta - \alpha^2 > 0.$$

By making $\delta = 0$ and 2, one will have

$$\alpha^2 < 1 \;\text{ as long as }\; \delta = 0 \;\text{ and }\; 2 \tag{26}$$

By making $\delta = 1$, one will have

$$\alpha^2 < 2 \quad \text{so long as} \quad \delta = 1. \tag{27}$$

Let us indicate by m a positive integer determined with the help of inequalities

$$\sqrt{n} - 1 \leq m < \sqrt{n}. \tag{28}$$

By declaring

$$n = m^2 + p, \tag{29}$$

one will have a positive integer p verifying the inequalities

$$0 < p \leq 2m + 1. \tag{30}$$

First case: p *is an odd number.*

By virtue of (23) and (29), one will have a congruence

$$u \equiv m^2 + p + \delta (\text{mod } 2),$$

p being an odd number; one can declare

$$u = m^2 + \delta + 1 + 2t. \tag{31}$$

By declaring

$$\sqrt{n} = m + \xi,$$

one will have

$$0 < \xi \leq 1, \tag{32}$$

because of (28). By virtue of (24), one obtains the equality

$$u = m - 1 + \xi + \alpha,$$

and because of (31), it becomes

$$\xi + \alpha = m^2 - m + 2 + 2t + \delta. \tag{33}$$

By supposing that $\delta = 0$ or 2, one obtains

$$\xi + \alpha \equiv 0 (\text{mod } 2).$$

By virtue of (26) and (32), it is necessary that

$$\xi + \alpha = 0,$$

therefore

$$u = m - 1 \text{ so long as } \delta = 0 \text{ and } \delta = 2.$$

By supposing that $\delta = 1$, one obtains, because of (33),

$$\xi + \alpha \equiv 1 (\text{mod } 2).$$

By virtue of (27) and (32), the integer $\xi + \alpha$ can have only two values

$$\xi + \alpha = \pm 1,$$

and it results in that

$$u = m \quad \text{or} \quad m - 2 \quad \text{as long as} \quad \delta = 1.$$

One obtains four values of the function (22):

$$\rho_1 = \frac{(m-1)^2+n-4}{(m-1)(n-m-1)}, \quad \rho_2 = \frac{m^2+n-3}{m(n-m-1)},$$
$$\rho_3 = \frac{(m-2)^2+n-3}{(m-2)(n-m+1)}, \quad \rho_4 = \frac{(m-2)^2+n}{(m-1)(n-m+1)}$$

among which is found the smallest value looked for of ρ.

By noticing that

$$\rho_1 - \rho_2 = \frac{p-3}{m(m-1)(n-m-1)},$$
$$\rho_4 - \rho_1 = \frac{2p+2}{(m-1)(n-m-1)(n-m+1)},$$
$$\rho_3 - \rho_4 = \frac{p+1}{(m-1)(m-2)(n-m+1)},$$

one obtains, because of (30),

$$\rho_1 < \rho_4 < \rho_3$$

and

$$\rho_2 \leq \rho_1 \quad \text{so long as} \quad p \geq 3,$$
$$\rho_1 < \rho_2 \quad \text{so long as} \quad p < 3.$$

There exists only a single odd value of p verifying the inequalities $0 < p < 3$, therefore one will have the inequality

$$\rho_1 < \rho_2 \quad \text{so long as} \quad p = 1.$$

We have arrived at the following result.

The smallest value of ρ *will have for expression*

$$\rho = \frac{m^2 + n - 3}{m(n - m - 1),} \tag{34}$$

provided that $n = m^2 + p$, *and the odd number* p *verifies the inequalities*

$$3 \leq p \leq 2m + 1.$$

In the case $n = m^2 + 1$, *the smallest value of* ρ *will be*

$$\rho = \frac{(m-1)^2 + n - 4}{(m-1)(n-m-1)}.$$

Second case: p is an even number.

One will have, because of (23), the inequality [sic]

$$u \equiv m^2 + \delta (\text{mod } 2).$$

By presenting

$$u = m^2 + \delta + 2t,$$

one will have the equalities

$$u = m - 1 + \xi + \alpha \quad \text{and} \quad \xi + \alpha = m^2 - m + 2t + \delta + 1.$$

By supposing that $\delta = 0$ or 2, one obtains

$$\xi + \alpha = 1,$$

and it follows that

$$u = m \quad \text{so long as} \quad \delta = 0 \quad \text{and} \quad 2.$$

By supposing that $\delta = 1$, one obtains

$$\xi + \alpha = 0 \quad \text{or} \quad \xi + \alpha = 2,$$

therefore

$$u = m + 1 \quad \text{or} \quad u = m + 1 \quad \text{so long as} \quad \delta = 1.$$

The smallest value of ρ is found among the following values of the function (22):

$$\rho_1 = \frac{m^2+n-4}{m(n-m-2)}, \quad \rho_2 = \frac{(m+1)^2+n-3}{(m+1)(n-m-2)},$$
$$\rho_3 = \frac{(m-1)^2+n-3}{(m-1)(n-m)}, \quad \rho_4 = \frac{m^2+n}{m(n-m)}.$$

By noticing that

$$\rho_2 - \rho_1 = \frac{2m + 4 - p}{m(m+1)(n-m-2)},$$
$$\rho_1 - \rho_4 = \frac{4m - 2p}{m(n-m)(n-m-2)},$$
$$\rho_4 - \rho_3 = \frac{2m - p}{m(m-1)(n-m)},$$

one obtains, because of (30),

$$\rho_3 \leq \rho_4 \leq \rho_1 < \rho_2.$$

We have arrived at the following result:

The smallest value of ρ *is expressed by the equality*

$$\rho = \frac{(m-1)^2 + n - 3}{(m-1)(n-m)}$$

provided that $n = m^2 + p$, *and the even number* p *verifies the inequalities*

$$0 < p < 2m + 1.$$

42 We have determined the value of the parameter ρ which defines the perfect form $\varphi_1 + \rho x_1 x_3$. The determinant D of this form, by virtue of (4) and (9), will have for expression

$$D = \frac{4 + 4\rho - (n-1)\rho^2}{2^n}. \tag{35}$$

The corresponding value of the function $\mathcal{M}(a_{ij})$ defined in Number 16 will be

$$\mathcal{M}(a_{ij}) = 2\sqrt[n]{\frac{1}{4 + 4\rho - (n-1)\rho^2}}.$$

By applying the formulae obtained to the case:

$$n = 4, 5, 6, 7, 8,$$

one obtains the same value of ρ

$$\rho = 1.$$

The corresponding perfect forms will be

$$x_1^2+x_2^2+\ldots,x_4^2+x_1x_4+\ldots+x_3x_4,\ D=\frac{5}{2^4},\ \mathcal{M}(a_{ij})=2\ \sqrt[4]{\frac{1}{5}};$$

$$x_1^2+x_2^2+\ldots,x_5^2+x_1x_4+\ldots+x_4x_5,\ D=\frac{4}{2^5},\ \mathcal{M}(a_{ij})=2\ \sqrt[5]{\frac{1}{4}};$$

$$x_1^2+x_2^2+\ldots,x_6^2+x_1x_4+\ldots+x_5x_6,\ D=\frac{3}{2^6},\ \mathcal{M}(a_{ij})=2\ \sqrt[6]{\frac{1}{3}};$$

$$x_1^2+x_2^2+\ldots,x_7^2+x_1x_4+\ldots+x_6x_7,\ D=\frac{2}{2^7},\ \mathcal{M}(a_{ij})=2\ \sqrt[7]{\frac{1}{2}};$$

$$x_1^2+x_2^2+\ldots,x_8^2+x_1x_4+\ldots+x_7x_8,\ D=\frac{1}{2^8},\ \mathcal{M}(a_{ij})=2.$$

One comes across all these perfect forms in the Mémoire of Mr.'s Korkine and Zolotareff: Sur les formes quadratiques. [On the quadratic forms] ‡

The formulae obtained give a mean for the study of various perfect forms which verify the inequality

$$\mathcal{M}(a_{ij})>2.$$

By making, for example, $n=12$, one will have

$$m=3 \quad \text{and} \quad p=3.$$

By virtue of (34), one obtains

$$\rho=\frac{3}{4},$$

therefore, because of (35),

$$D=\frac{13}{16}\cdot\frac{1}{2^{12}},$$

and it follows that

$$\mathcal{M}(a_{ij})=2\ \sqrt[12]{\frac{16}{13}}>2.$$

All the extreme forms studied by Mr.'s Korkine and Zolotareff do not give a function $\mathcal{M}(a_{ij})$ of values which exceed 2.

On the quadratic perfect forms and on the domains which correspond to them.

43 We have seen in Number 29 that to the quaternary principal perfect form

$$\varphi=x_1^2+x_2^2+x_3^2,x_4^2,x_1x_2,x_1x_3,x_1x_4,x_2x_3,x_2x_4,x_3x_4,\quad D=\frac{5}{2^4}$$

corresponds the domain R made up of forms

$$\rho_1\mathrm{x}_1^2+\rho_2\mathrm{x}_2^2+\rho_3\mathrm{x}_3^2+\rho_4\mathrm{x}_4^2+\rho_5(\mathrm{x}_1-\mathrm{x}_2)^2+\rho_{[6]}(\mathrm{x}_1-\mathrm{x}_3)^2+\rho_7(\mathrm{x}_1-\mathrm{x}_4)^2+\rho_8(\mathrm{x}_2-\mathrm{x}_3)^2+\rho_9(\mathrm{x}_2-\mathrm{x}_4)^2+\rho_{10}(\mathrm{x}_3-\mathrm{x}_4)^2.$$

All the perfect forms contiguous to the principal form φ are equivalent to the form

$$\varphi_1=x_1^2+x_2^2,x_3^2,x_4^2,x_1x_3,x_1x_4,x_2x_3,x_2x_4,x_3x_4,\quad D=\frac{1}{4}.$$

The corresponding domain R_1 is made up of forms

$$\rho_1x_1^2+\rho_2x_2^2+\rho_3x_3^2+\rho_4x_4^2+\rho_5(x_1-x_3)^2+\rho_6(x_1-x_4)^2+\rho_7(x_2-x_3)^2+\rho_8(x_2-x_4)^2+$$
$$\rho_9(x_3-x_4)^2+\rho_{10}(x_1+x_2-x_3)^2+\rho_{11}(x_1+x_2-x_4)^2,\rho_{12}(x_1+x_2-x_3-x_4)^2.$$

Let us examine the perfect forms contiguous to the perfect form φ_1.

We have demonstrated in Number 38 that all these forms are equivalent to the forms

$$1).\ \varphi_1-\rho x_1x_3,$$

$$2).\ \varphi_1+\rho(x_1x_2-\delta x_3x_4),\ \text{ where } \delta=0 \text{ or } 1.$$

Let us examine three perfect forms

$$1).\ \varphi_1+\rho x_1x_2,\quad 2).\ \varphi_1-\rho x_1x_3,\quad 3).\ \varphi_1+\rho(x_1x_2-x_3x_4).$$

1). By making $\rho=1$ in the form $\varphi_1+\rho x_1x_2$, one obtains the principal perfect form φ.

2). Let us notice that the form $\varphi_1-\rho x_1x_3$ is equivalent to the form $\varphi_1+\rho x_1x_2$.

In effect, the substitution

$$x_1=-x_1',\ x_2=x_3',\ x_3=x_2',\ x_4=x_1'+x_4'$$

does not change the form φ_1 and transforms the form x_1x_2 into the form $-x_1'x_3'$.

---

‡ Mathematische Annalen, V. VI, p. 367.

3). By making $\rho = 1$ in the form $\varphi_1 + \rho(x_1x_2 - x_3x_4)$, one obtains the form

$$x_1^2 + x_2^2, x_3^2 + x_4^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4$$

which is evidently equivalent to the perfect form φ_1.

One concludes that all the perfect forms contiguous to the perfect form φ_1 are equivalent to the forms φ and φ_1.

It follows that the set of all the quaternary perfect forms be divided into two classes represented by the perfect forms φ and φ_1.

The set (R) of domains corresponding to various quaternary perfect forms is made up of two classes, too, represented by the domains R and R_1.

On the perfect forms in five variables and on the domains which correspond to them.

44 We have determined two perfect forms in five variables

$$\varphi = x_1^2 + x_2^2 + \ldots + x_5^2 + x_1x_2 + x_1x_3 + \ldots + x_4x_5, \quad D = \frac{6}{2^5},$$

$$\varphi_1 = x_1^2 + x_2^2 + \ldots + x_5^2 + x_1x_3 + x_1x_4 + \ldots + x_4x_5, \quad D = \frac{4}{2^5}.$$

The corresponding domains R and R_1 will be composed of forms

$$R)\ \rho_1x_1^2 + \rho_2x_2^2 + \ldots + \rho_5x_5^2 + \rho_6(x_1 - x_2)^2 + \rho_7(x_1 - x_3)^2 + \ldots + \rho_{15}(x_4 - x_5)^2,$$

$$R_1)\ \rho_1x_1^2 + \rho_2x_2^2 + \ldots + \rho_5x_5^2 + \rho_6(x_1 - x_3)^2 + \ldots + \rho_{20}(x_1 + x_2 - x_4 - x_5)^2.$$

Examine the perfect forms contiguous to the perfect face φ_1. We have demonstrated in Number 38 that all these forms are equivalent to the forms

$$\begin{aligned} &1).\quad \varphi_1 - \rho x_1x_3, \\ &2).\quad \varphi_1 - \rho(x_1x_2 - \delta x_3x_4 - \delta' x_3x_5 - \delta'' x_4x_5). \end{aligned} \tag{1}$$

where

$$\delta = 0 \text{ or } 1, \quad \delta' = 0 \text{ or } 1, \quad \delta'' = 0 \text{ or } 1.$$

In the second case one obtains 8 perfect forms. By permuting the variables x_3, x_4, x_5 one will replace the forms (1) by 4 forms; thus all the perfect forms contiguous to the perfect form φ_1 are equivalent to the 5 following forms:

$$1).\ \varphi_1 + \rho x_1x_2, \quad 2).\ \varphi_1 + \rho x_1x_3, \quad 3).\ \varphi_1 + \rho(x_1x_2 - x_4x_5),$$

$$4).\ \varphi_1 + \rho(x_1x_2 - x_3x_5 - x_4x_5), \quad 5).\ \varphi_1 + \rho(x_1x_2 - x_3x_4 - x_3x_5 - x_4x_5).$$

1). By making $\rho = 1$ in the perfect form $\varphi_1 + \rho x_1x_2$, one obtains the perfect form φ.

2). We have seen in Number 42 that the perfect form $\varphi_1 - \rho x_1x_3$ is determined by the value $\rho = 1$ of the parameter ρ in the case $n = 5$. One obtains the form

$$\varphi_1' = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_1x_4 + x_1x_5 + \ldots + x_4x_5 \tag{2}$$

which will be transformed with the help of the substitution

$$x_1 = -x_2^2,\ x_2 = x_1' - x_2',\ x_3 = x_3',\ x_4 = x_2 + x_4',\ x_5 = x_2' + x_5'$$

into a perfect form φ_1.

3). In the form $\varphi_1 + \rho(x_1x_2 - x_4x_5)$, one will put $\rho = 0$ and one will obtain the form

$$x_1^2 + x_2^2 + \ldots + x_5^2 + x_1x_2 + x_1x_3 + \ldots + x_3x_5$$

which is evidently equivalent to the form φ_1.

4). In the form $\varphi_1 + \rho(x_1x_2 - x_3x_5 - x_4x_5))$, one will put $\rho = 1$ and one will obtain the form

$$x_1^2 + x_2^2 + \ldots + x_5^2 + x_1x_2 + x_1x_3 + \ldots + x_2x_5$$

which is evident to the perfect form (2)

5). It remains only to determine the perfect form:

$$\varphi_1 + \rho(x_1x_2 - x_3x_4 - x_3x_5 - x_4x_5). \tag{3}$$

By effecting the transformation with the help of the substitution

$$-x_1 + x_2 = x_1', \quad x_1 + x_2 + x_3 + x_4 + x_5 = x_2', \quad x_3 = x_3', \quad x_4 = x_4', \quad x_5 = x_5' \tag{4}$$

of the form

$$2\varphi_1 + 2\rho(x_1x_2 - x_3x_4 - x_3x_5 - x_4x_5),$$

one obtains the form

$$\mathrm{x}_1'^2 + \mathrm{x}_2'^2 + \mathrm{x}_3'^2 + \mathrm{x}_4'^2 + \mathrm{x}_5'^2 + \frac{\rho}{2}\left[-\mathrm{x}_1'^2 + \mathrm{x}_2'^2 + \mathrm{x}_3'^2 + \mathrm{x}_4'^2 + \mathrm{x}_5'^2 - 2\mathrm{x}_2'\mathrm{x}_3' - 2\mathrm{x}_2'\mathrm{x}_4' - 2\mathrm{x}_2'\mathrm{x}_5' - 2\mathrm{x}_3'\mathrm{x}_4' - 2\mathrm{x}_3'\mathrm{x}_5' - 2\mathrm{x}_4'\mathrm{x}_5'\right]. \tag{5}$$

By virtue of (4) the integer variables $x_1', x_2', x_3', x_4', x_5'$ verify the congruence

$$x_1' + x_2' + x_3' + x_4' + x_5' \equiv 0(\mathrm{mod}2). \tag{6}$$

By applying to the form (5) the method unveiled in Number 2, one will determine the value of the upper limit $R > 0$ of value $\frac{\varrho}{2}$ with the help of equations

$$\xi_1 - R\xi_1 = 0,\ \xi_2 + R(\xi_2 - \xi_3 - \xi_4 - \xi_5) = 0,\ \xi_3 + R(-\xi_2 + \xi_3 - \xi_4 - \xi_5) = 0,$$
$$\xi_4 + R(-\xi_2 - \xi_3 + \xi_4 - \xi_5) = 0,\ \xi_5 + R(-\xi_2 - \xi_3 - \xi_4 + \xi_5) = 0.$$

It results in that

$$\xi_2 = \xi_3 = \xi_4 = \xi_5,$$

and one obtains the equations

$$\xi_1(1 - R) = 0 \quad \text{and} \quad \xi_2(1 - 2R) = 0,$$

thus

$$R = \frac{1}{2}.$$

By declaring

$$x'_1 = 0,\ x'_2 = 1,\ x'_3 = 1,\ x'_4 = 1, x'_5 = 1, \tag{7}$$

one will satisfy the condition (6) and one will have the value $4 - 4\rho$ of the form (5).

By making

$$4 - 4\rho = 2,$$

one obtains

$$\rho = \frac{1}{2}.$$

It follows that the positive quadratic form

$$x_1'^2 + x_2'^2 + \ldots + x_5'^2 + \frac{1}{4}\left[-x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2 + x_5'^2 - 2x'_2x'_3 - \ldots - 2x'_4x'_5\right] \tag{8}$$

will have a value 2 corresponding to the system (7).

By virtue of that which has been discussed in Number 23, the smallest value of the form (8) will correspond to a system $(l_1, l_2, \ldots, l_5)$ verifying the inequality

$$l_1^2 + l_2^2 + l_3^2 + l_4^2 + l_5^2 \leq 2 \cdot \frac{R}{R - \frac{1}{4}} \text{ where } R = \frac{1}{2}.$$

One obtains the inequality

$$l_1^2 + l_2^2 + l_3^2 + l_4^2 + l_5^2 \leq 4.$$

It is easy to demonstrate that the system (7) is the only one verifying this inequality on condition (6), the systems which verify the inequality

$$-x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2 + x_5'^2 - 2x'_2x'_3 - 2x'_2x'_4 - 2x'_2x'_5 - 2x'_3x'_4 - 2x'_3x'_5 - 2x'_4x'_5 \geq 0$$

being excluded. By making $\rho = \frac{1}{2}$ in the form (3), one obtains the perfect form

$$\varphi_2 = x'_1 + x'_2 + \ldots + x'_5 + \frac{1}{2}x_1x_2 + x_1x_3 + \ldots + x_2x_5 + \frac{1}{2}x_3x_4 + \frac{1}{2}x_3x_5 + \frac{1}{2}x_4x_5, \quad D = \left(\frac{3}{2}\right)^4 \frac{1}{2^5}.$$

The corresponding domain R^2 is composed of forms

$$\rho_1x_1^2 + \rho_2x_2^2 + \ldots + \rho_5x_5^2 + \rho_6(x_1 - x_3)^2 + \ldots + \rho_{11}(x_2 - x_5)^2$$
$$+\rho_{12}(x_1 + x_2 - x_3 - x_4)^2 + \rho_{13}(x_1 + x_2 - x_3 - x_5)^2$$
$$+\rho_{14}(x_1 + x_2 - x_4 - x_5)^2 + \rho_{15}(-x_1 - x_2 + x_3 + x_4 + x_5)^2.$$

The number of parameter $\rho_1, \rho_2, \ldots, \rho_{15}$ being equal to the number of dimensions of the domain R_2, one will determine without trouble 15 inequalities which define the domain R_2.

45 We have demonstrated that all the perfect forms contiguous to the perfect form φ_1 are equivalent to the perfect forms φ, φ_1 and φ_2.

Choose the perfect forms contiguous to the perfect form φ_2.

To this effect let us notice, in the first place, that the perfect form φ_1 is contiguous to the perfect form φ_2, then observe that all the perfect forms contiguous to the form φ_2 are equivalent.

To demonstrate this, examine all the faces in 14 dimensions of the domain R_2.

The domain R_2 is characterised by 15 quadratic forms

$$\begin{cases} x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, (x_1 - x_3)^2, (x_1 - x_4)^2, (x_1 - x_5)^2, (x_2 - x_3)^2, \\ (x_2 - x_4^2), (x_2 - x_5)^2, (x_1 + x_2 - x_3 - x_4)^2, (x_1 + x_2 - x_3 - x_5)^2, \\ (x_1 + x_2 - x_4 - x_5)^2, (-x_1 - x_2 + x_3 + x_4 + x_5)^2. \end{cases} \tag{9}$$

Each face in 14 domains of the domain R_2 possesses 14 of these, and the form which remains can be called form opposite to the face.

One concludes that each face is well determined by the opposite face.

For the perfect forms contiguous to the perfect form φ_2 to be equivalent, it is necessary and sufficient that all the faces of the domain R_2 could be transformed one to one with the help of substitutions which do not change the domain R_2.

It would be easy to write all these substitutions, but one will proceed in another way, more speedy.

Let us observe that the face P belonging to the domain R_1 and R_2 is characterised by all the forms (9), the form $(-x_1 - x_2 + x_3 + x_4 + x_5)^2$ being excluded.

With the aid of substitution associated with the substitution (4), one will replace the forms (9) by the forms:

$$\begin{cases} (x_1' \pm x_2')^2, (x_1' \pm x_3')^2, (x_1' \pm x_4')^2, (x_1' \pm x_5')^2, \\ (x_2' + x_3')^2, (x_2' + x_4')^2, (x_2' + x_5')^2, (x_3' + x_4')^2, (x_3' + x_{[}5]')^2, \\ (x_4' + x_5')^2, (x_2' + x_3' + x_4' + x_5')^2. \end{cases} \tag{10}$$

By changing the sign of x_1' and by permuting the variables x_2', x_3', x_4', x_5', one will transform into itself the forms (10), and the form $(x_2' + x_3' + x_4' + x_5')^2$ will not change.

To each similar substitution corresponds a substitution which transforms into itself the domain R_2 and the face P of the domain R_2, and does not change the form $(-x_1 - x_2 + x_3 + x_4 + x_5)^2$.

By changing the sign of x_1' and by permuting x_2', x_3', x_4', x_5', one will transform the form $(x_1' + x_2')^2$ into forms

$$(x_1' \pm x_2')^2, (x_1' \pm x_3')^2, (x_1' \pm x_4')^2, (x_1' \pm x_5')^2$$

and one will transform the form $(x_2' + x_3')^2$ into forms

$$(x_2' + x_3')^2, (x_2' + x_4')^2, (x_2' + x_5')^2, (x_3' + x_4')^2, (x_3' + x_{[5]}')^2, (x_4' + x_5')^2.$$

Thus only the forms

$$(x_1' + x_2')^2,\ (x_2' + x_3')^2,\ (x_2' + x_3' + x_4' + x_5')^2 \tag{11}$$

remain to examine.

By returning to the forms (9), one obtains the forms corresponding to the forms (11).

$$x_2^2, x_3^2, (-x_1 - x_2 + x_3 + x_4 + x_5)^2. \tag{12}$$

It is demonstrated that all the forms (9) can be transformed into forms (12) with the help of substitutions which do not change the domain R_2.

With the help of substitutions

$$x_1 = x_2' - x_5',\ x_2 = x_3',\ x_3 = x_2',\ x_4 = x_1' + x_2',\ x_4 = x_1' + x_2' - x_4' - x_5',\ x_5 = -x_1' + x_3'$$

and

$$x_1 = x_1',\ x_2 = -x_1' - x_2' + x_3' + x_4' + x_5',\ x_3 = x_3',\ x_4 = x_4',\ x_5 = x_5',$$

one will transform the domain R_2 into itself, and the form x_2^2 will be transformed into forms x_3^2 and $(-x_1' - x_2' + x_3' + x_4' + x_5')^2$.

We have demonstrated that all the forms of the domain R_2 are equivalent. It results in, from that we have seen, that all the perfect forms contiguous to the perfect form φ_2 are equivalent to the perfect form φ_1.

One concludes that all the perfect forms in five variables constitute three different classes represented by the perfect forms φ, φ_1 and φ_2.

The set of domains (R) can be divided into three classes also, represented by the domains R, R_1, and R_2.

End of the first Mémoire.

§ D.3 G. F. Voronoi, 1908 (II)

New applications of continuous parameters to the theory of quadratic forms

Second Memoir

Research on the primitive parallelohedron

by Mr. Georges Voronoï in Warsaw

[*Journal für die reine und angewandte Mathematik*, V. 134, 1908]

[translated by K N Tiyapan]

The well known method of reduction for the binary, ternary and quaternary positive quadratic forms † rests upon a property of the positive quadratic form, to know:

Every positive quadratic form $\sum_{i=1}^{n}\sum_{i=1}^{n} a_{ij}x_ix_j$ *has n variables in the set E composing all of the systems* $(x_1, x_2, \ldots, x_n)$ *of integers of the variables* x_1, x_2, …, x_n *n consecutive minima*

$$M_1 \le M_2 \le \cdots \le M_n$$

determined at condition which the determinant ω *of a system*

$$(l_{11}, l_{21}, \ldots, l_{n1}), (l_{12}, l_{22}, \ldots, l_{n2}), \ldots, (l_{1n}, l_{2n}, \ldots, l_{nn}) \tag{1}$$

which represent these minima in the set E does not vanish.

In all the cases where one has

$$\omega = \pm 1$$

---

† *Lagrange,* Recherches d'Arithmétique [Studies in arithmetic] (*Oeuvres,* V. III, p. 695)

Gauß, Disquisitiones arithméticae (*Oeuvres,* V. I, art. 171, p. 146)

Lejeune-Dirichlet, Über die Reduktion der positiven quadratischen Formen mit drei unbestimmten ganzen Zahlen [On the reduction of the positive quadratic forms with three indeterminate integers] (*Oeuvres,* V. II, p. 41)

Minkowski, Sur la réduction des formes quadratiques positives quaternaires [On the reduction of the quaternary positive quadratic forms] (*Comptes Rendus des séances de l'Académie de Paris,* V. 96, p. 1205) one can transform the quadratic form $\sum\sum a_{ij}x_ix_j$ into an equivalent form by using a substitution

$$x_i = \sum_{k=1}^{n} l_{ik}x'_k \qquad (i = 1, 2, \ldots, n)$$

In the transformed form $\sum\sum a'_{ij}x_ix_j$, one will have

$$a'_{kk} = M_k \qquad (k = 1, 2, \ldots, n)$$

The form $\sum\sum a'_{ij}x_ix_j$ obtained is said to be reduced with respect to the consecutive minima.

The binary, ternary, and quaternary positive quadratic forms can be reduced with respect to the consecutive minima. [‡] The algorithm which one uses in doing the reduction of these forms is founded on the following theorem.

For a positive quadratic form

$$f(x_1, x_2, \ldots, x_n) = \sum\sum a_{ij}x_ix_j \quad (n = 2, 3, 4)$$

to be reduced with respect to the consecutive minima, it is necessary and sufficient that one has the inequalities

$$f(x_1, x_2, \ldots, x_{k-1}, 1, x_{k+1}, \ldots, x_n) \geq a_{kk} \quad (k = 1, 2, \ldots, n) \tag{2}$$

and

$$a_{11} \leq a_{22} \leq \ldots \leq a_{nn} \tag{3}$$

which is valid for integers of the variables

$$x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \quad (k = 1, 2, \ldots, n)$$

By letting

$$x_i = x'_i + \delta_i x'_k \quad \text{where } \delta_k = 0 \text{ and } i = 1, 2, \ldots, n \tag{4}$$

one will determine for the given form $f(x_1, x_2, \ldots, x_n)$ integers $\delta_1, \ldots, \delta_{k-1}, \delta_{k+1}, \ldots, \delta_n$ the condition of which the corresponding value $f(\delta_1, \ldots, \delta_{k-1}, 1, \delta_{k+1}, \ldots, \delta_n)$ would be smallest. By making successively $k = 1, 2, \ldots, n$ and repeating the procedure stated, one will always transform the given form with the aid of the substitution (4) into a form which is no different from the reduced form except by a permutation of the coefficients $(n = 2, 3, 4)$

The procedure stated in the general case can not be carried on indefinitely and one will always arrive at an equivalent quadratic form $\sum a'_{ij}x_ix_j$ which verifies the inequalities (2) and (3), but one does not know from the number of variables $n > 4$ whether the coefficients a'_{kk} $(k = 1, 2, \ldots, n)$ in the form obtained exhibit a system of consecutive minima, besides: one also does not know whether the reduction of every positive quadratic form with respect to the consecutive minima is possible.

One rids oneself of the described difficulty by changing the notation of system with n consecutive minima into nothing more than considering the systems (1) which verify the equation

$$\omega = \pm 1 .$$

This is the method known as Hermite method [†] which has recently been improved by Mr. Minkowski in the memoir titled *Diskontinuitätsbereich für arithmetische Aquivalenz. [Discontinuity domain for arithmetical equivalence]* [‡] in the set E, the quadratique $\sum\sum a_{ij}x_ix_j$ being positive and $\alpha_1, \alpha_2, , \alpha_n$ any arbitrary parameters.

In the case $n = 2$, the problem put forward has been solved by Lejeune-Dirichlet and by Hermite [§]

By reflecting upon the principles which have served as basis in these researches of these two illustrious geometers, I have observed that the problem introduced is intimately connected to the problem of the reduction of positive quadratic form.

In effect, Lejeune-Dirichlet and Hermite have demonstrated the following theorem.

The conditions necessary and sufficient for which the inequality

$$ax^2 + 2bxy + cy^2 + 2\alpha y + 2\beta y \geq 0$$

holds, for any integer values of x *and* y*, in general come down to six inequalities*

$$\left\{ \begin{aligned} & al^2 + 2blm + cm^2 \pm 2(\alpha l + \beta m) \geq 0, \\ & al'^2 + 2bl'm' + cm'^2 \pm 2(\alpha l' + \beta m') \geq 0, \\ & al''^2 + 2bl''m'' + cm''^2 \pm 2(\alpha l'' + \beta m'') \geq 0, \end{aligned} \right. \tag{5}$$

where the systems of integers

$$(l, m), (l', m') \text{ and } (l'', m'')$$

[‡] *Korkine* and *Zolotareff,* Sur les formes quadratiques positives. (*Mathematische Annalen,* V. 6, p. 336 and V. 11, p. 242)

[†] *Hermite*, Extraits de lettres a Jacobi sur différents objet de la théorie des nombres (This Journal, V. 40, p. 302)

[‡] This Journal, V. 129, p.220

[§] *Lejeune-Dirichlet*, Mémoire cited

Hermite, Sur la théorie des formes quadratique ternaires [On the theory of ternary quadratic forms] (This Journal, V. 40, p. 178) *depend only on coefficients of the quadratic form (a, b, c).*

By considering the parameters α and β as the Cartesian coordinates of a point (α, β) of the plane, one will determine by the inequalities (5) a hexagonal P which is formed by three pairs of parallel edges. The study of properties of the hexagon P plays an important role in the study of Lejeune-Dirichlet which has indicated two fundamental properties of the hexagon P.

I. There exists a group of translations of the hexagon P with the aid of which all the plane will be covered by the congruent hexagons.

II. Any binary positive quadratic form can be transformed by an equivalent form (a, b, c) satisfying the conditions

$$a - b \geq 0, b \geq 0, c - b \geq 0. \tag{6}$$

The hexagon P corresponding to the form (a, b, c), in the case

$$a - b > 0, b > 0, c - b > 0,$$

is characterised by the systems

$$(1,0), (0,1), (1,-1) \tag{7}$$

In the case $a - b = 0$, or $(b = 0)$, or $c - b = 0$, the hexagon P reduces itself into a parallelogram.

The inequalities (6) define a domain D of binary quadratic forms which is perfectly determined by the systems (7).

With the help of the substitution

$$x = x', y = -y',$$

one will transform the domain D by a domain D' defined by the inequalities

$$a + b \geq 0, -b \geq 0, c + b \geq 0 \tag{8}$$

which is characterised by the systems

$$(1,0), (0,1), (1,1)$$

One calls reduced by Selling['s method] the binary positive quadratic forms which verify the inequalities (8).‡

By effecting all the transformations of the domain D with the help of substitutions

$$x = px' + qy', y = p'x' + q'y'$$

of integer coefficients and of determinant ±1, one obtains a set (D) of domains of binary quadratic forms.

The set (D) of domains uniformly partitions the set of all the binary positive quadratic forms, that is to say: a form which is interior to any one domain D of the set (D) does not belong to any other domain of this set; a form which is interior to a face of the domain D belongs to only one other domain of the set (D) which is contiguous to the domain (D) by this face.

The results summarised have brought me to a new point of view on the problem of reduction of positive quadratic forms.

The problem of reduction of positive quadratic forms consist of a uniform partition of the set of positive quadratic forms with the help of domains of forms, determined using linear inequalities and enjoying the property that any substitution of integer coefficients and of determinant ±1 does not change the set (D) of these domains. By partitioning the set (D) into classes of equivalent domains and by choosing the representatives of all the classes

$$D, D_1, \ldots, D_{m-1}, \tag{9}$$

one will call reduced the quadratic forms which belong to these domains.

One could attach the supplementary condition to the domains (9) by demanding: 1). that $m = 1, 2$, 2). that the positive quadratic forms interior to the domain D are not equivalent and lastly, 3). that the number of linear inequalities which define the domain D be the smallest one possible.

I hope to return another time to the problem posed of the reduction of positive quadratic forms.

In this mémoire, I restrict myself to the study of domains of quadratic forms which one obtains by generalising the results shown in studies of Lejeune-Dirichlet and of Hermite for the positive quadratic forms in any number of variables.

The hexagon of Lejeune-Dirichlet can be replaced for the positive quadratic forms of n variables by a convex polyhedron of the analytical space in n dimensions.

For a positive quadratic form $\sum\sum a_{ij}x_ix_j$, the corresponding polyhedron R presents a set of points (α_i) verifying the inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i \geq 0 \tag{10}$$

in the set E. The polyhedron R can be determined with the help of independent inequalities

$$\sum\sum a_{ij}l_{ik}l_{jk} \pm 2\sum \alpha_i l_{ik} \geq 0, \ (k = 1, 2, \ldots, \tau)$$

the number 2τ of which does not exceed a limit

$$2\tau \leq 2(2^n - 1).$$

the systems of integers

$$\pm(l_{11}, l_{21}, \ldots, l_{n1}), \pm(l_{12}, l_{22}, \ldots, l_{n2}), \ldots, \pm(l_{1\tau}, l_{2\tau}, \ldots, l_{n\tau}) \tag{11}$$

---

‡ *Selling,* Über die binären und ternären quadratischen Formen. [On the binary and ternary quadratic forms] (This Journal, V. 77, p.143)

define by the corresponding equations

$$\sum\sum a_{ij}l_{ik}l_{jk} \pm 2\sum \alpha_i l_{ik} = 0$$

2τ faces in $n-1$ dimensions of the polyhedron R. As these faces partition themselves into τ pairs of parallel faces, I call parallelohedron the polyhedron R corresponding to any positive quadratic form.

The systems (11) enjoy many important properties.

1. For a system $(l_1, l_2, \ldots, l_n)$ to belong to the series (11), it is necessary and sufficient that two systems $(l_1, l_2, \ldots, l_n)$ and $(-l_1, -l_2, \ldots, -l_n)$ are the only representations of the minimum of the form $\sum\sum a_{ij}x_ix_j$ in the set composed of all the systems of integers which are congruent to the system $(l_1, l_2, \ldots, l_n)$ by relation to the modulus 2, the system $l_1 = 0, l_2 = 0, \ldots, l_n = 0$ being excluded.

2. Among the systems (11) are found all the representations of the arithmetical minimum of the positive quadratic form $\sum\sum a_{ij}x_ix_j$.

3. among the systems (11) are found all the systems (1) which represent n consecutive minima of the form $a_{ij}x_ix_j$.

4. All the determinants which one can form of any n systems belonging to the series (11) do not exceed in numerical value a limit $n!$.

By designating by the symbol S_ν the number of faces in ν dimensions $(\nu = 0, 1, 2, \ldots, n-1)$ of a parallelohedron R, I have found that

$$S_\nu \leq (n+1-\nu)\Delta^{(n=\nu)}(m^n)_{m=1}. \quad (\nu = 0, 1, 2, \ldots, n-1)$$

By making $\nu = 0$ in this inequality, one obtains

$$S_0 \leq (n+1)!\,,$$

therefore the number of vertices of a parallelohedron R does not exceed a limit $(n+1)!$. By making $\nu = n-1$, one obtains

$$S_{n-1} \leq 2(2^n - 1)\,.$$

I demonstrate in this memoir that there exist parallelohedra, the symbol S_ν for which are expressed by the formula

$$S_\nu = (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}. \quad (\nu = 0, 1, 2, \ldots, n-1)$$

All these parallelohedra are primitive.

The notation of positive parallelohedra plays an important role in my studies.

I have arrived at the notation of primitive parallelohedra by observing that the parallelohedra possess Property I of hexagons of Lejeune-Dirichlet, in knowing:

I. There exists a group of transformations of a parallelohedron R with the help of which one uniformly fills the analytical space in n dimensions by the congruent parallelohedra.

Designate by (R) the set of parallelohedra which are defined by the inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i \geq \sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i\,,$$

$l_1, l_2, \ldots, l_n$ being arbitrary integers. Ay system (l_I) of integers characterise a parallelohedron of the set (R).

I demonstrate that the set (R) of parallelohedra corresponding to the various systems (l_i) of integers unformly fills the space in n dimensions.

The corresponding group of translations of the parallelohedron R defined by the inequalities (10) is composed of vectors $[\lambda_i]$ which are determined by the equalities

$$\lambda_i = -\sum_{k=1}^{n} a_{ik}l_k, \quad (i = 1, 2, \ldots, n)$$

$l_1, l_2, \ldots, l_n$ being arbitrary integers.

Any vertex (α_i) of parallelohedra of the set (R) belongs to at least $n+1$ parallelohedra. I call simple a vertex (α_i) which belongs only to $n+1$ parallelohedra of the set (R) and I establish a notion of primitive parallelohedron as follows:

One call primitive parallelohedron, a parallelohedron the vertices of which are simple.

All the parallelohedra which are not primitive are called nonprimitive. From this point of view, the hexagon of Lejeune-Dirichlet presents a primitive parallelohedron and each parallelogram is a nonprimitive parallelohedron in two dimensions.

Any nonprimitive parallelohedron is a boundary of primitive parallelohedra and can be considered as a case of degeneracy of primitive parallelohedra.

I divide the primitive parallelohedra into various types by characterising a type of primitive parallelohedra by a set (L) of simplexes correlative to the various vertices of parallelohedra which belong to the set (R).

An identical vertex (α_i) is determined by $n+1$ equations

$$\sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_i l_{ik} = A. \quad (k = 0, 1, 2, \ldots, n)$$

In $n+1$ systems of integers

$$(l_{1k}, l_{2k}, \ldots, l_{nk}), \quad (k = 0, 1, 2, \ldots, n)$$

I make a simplex L correspond by defining it as a set of points which are determined by the equations

$$x_i = \sum_{k=0}^{n} \vartheta_k l_{ik}, \text{ where } \sum_{k=0}^{n} \vartheta_k = 1 \text{ and } \vartheta_k \geq 0.$$

$$(k = 0, 1, 2, \dots, n, i = 1, 2, \dots, n)$$

The set (L) of simplexes which are correlative to the vertices of the set (R) of primitive parallelohedra enjoys important properties.

1. The set (L) of simplexes uniformly partition the space of n dimensions.

2. By effecting the various translations of a simplex of the set (L) the length of vector $[l_i]$ which are determined by the arbitrary integers $l_1, l_2, \dots, l_n$, one obtains a class of congruent simplexes which belong to the set (L).

3. The number of incongruent simplexes of the set (L) is finite.

Property II of hexagons of Lejeune-Dirichlet for the primitive parallelohedra can be generalised as follows:

II. All the quadratic forms which define the primitive parallelohedra belonging to the type characterised by the set (L) of simplexes are interior to a domain of quadratic form in $\frac{n(n+1)}{2}$ dimensions defined by linear inequalities.

I obtains the linear inequalities which define a domain D of quadratic forms corresponding to a set (L) of simplexes by examining the incongruent edges of primitive parallelohedra belonging to the type characterised by the set (L) of simplexes.

An vertex (α_i) of primitive parallelohedra of the set (R) belongs to $n+1$ edges $[\alpha_i, \alpha_{ik}]$ of these parallelohedra $(k = 0, 1, 2, \dots, n)$.

By putting

$$\alpha_{ik} - \alpha_i = p_{ik}\rho_k, \quad (i = 1, 2, \dots, n; k = 0, 1, 2, \dots, n)$$

one can determine the positive parameter ρ_k, of such manner that the numbers $p_{1k}, p_{2k}, \dots, p_{nk}$ are integers and do not possesses common divisor. I demonstrate that the parameter ρ_k expressed by a linear function

$$\rho_k = \sum\sum p_{ij}^{(k)} a_{ij} \tag{12}$$

of coefficients of the given quadratic form $\sum\sum a_{ij}x_ix_j$, the coefficients

$$p_{ij}^{(k)} = p_{jk}^{(k)}, \quad (i = 1, 2, \dots, n; j = 1, 2, \dots, n)$$

being rational.

I call regulator of the edge $[\alpha_i, \alpha_{ik}]$, the function ρ_k determined by the formula (12); the system (p_{ik}) is called characteristic of the edge

$$[\alpha_i, \alpha_{ik}]. \quad (k = 0, 1, 2, \dots, n)$$

As the edge $[\alpha_i, \alpha_{ik}]$ is correlative to a face P_k of $n-1$ dimensions of the simplex L which is correlative to the vertex (α_i), I call the function (12) *regulator of the face P_k* and the system $\pm(P_{ik})$ *characteristic of the face P_k of the simplex L $(k = 0, 1, 2, \dots, n)$*

By designating by

$$\rho_k \text{ and } \pm(p_{ik}), \quad (k = 1, 2, \dots, \sigma)$$

the regulators and the characteristics of all the incongruent faces in $n-1$ dimensions of the set (L) of simplexes, I demonstrate the following important theorem:

The domain of quadratic forms which is characterised by the set (L) of simplexes is defined by the linear inequalities

$$\rho_k = \sum\sum p_{ij}^{(k)} a_{ij} \geq 0. \quad (k = 1, 2, \dots, \sigma)$$

All the domains of quadratic forms which I have studied in this memoir possess a remarkable property: they are simple domains, that is to say the number of independent inequalities which define them is equal to $\frac{n(n+1)}{2}$.

Another coincidence has attracted my attention for a long time: that is the relation which exists within the results shown in this memoir and those which have been obtained in my first memoir titled: "On some properties of perfected positive quadratic forms" † I have observed that the set of characteristics $\pm(p_{ik}), k = 1, 2, \dots, \sigma$ is nothing but the set of all the representations of the minimum of a perfect quadratic form φ. Thee domain D either coincides well with the domain R corresponding to the perfect form φ, or presents well a group of this domain.

Despite all my effort, I have not succeeded in discovering the tie which attaches the two problems shown and which seem to be so different, abstraction made of a remarkable formula

$$\sum\sum a_{ij}x_ix_j = \frac{1}{(n-1)!}\sum_{k=1}^{\sigma} \rho_k\omega_k(p_{1k}x_1 + p_{2k}x_2 + \cdots + p_{nk}x_n)^2$$

which supplies the expression of an arbitrary quadratic form $\sum\sum a_{ij} \cdot x_ix_j$ in function of the regulators ρ_k $(k = 1, 2, \dots, \sigma)$ which are determined by the formula (12).

In this formula ω_k $(k = 1, 2, \dots, \sigma)$ are positive integers which depend only on corresponding faces of simplexes of the set (L).

To the various types of primitive parallelohedra corresponds a set (D) of domains of quadratic forms. The set (D) uniformly partitions the set of all the positive quadratic forms in n variables.

I show in this mémoire an algorithm, by the aid of which one can determine all the domains of forms which are contiguous to a domain of the set (D) by the faces in $\frac{n(n+1)}{2} - 1$ dimensions. This algorithm comes down to a certain reconstruction of the set (L) of simplexes by another set (L').

---

† This Journal, V. 133, p. 97

The set (D) of domains of forms transforms into itself by all the substitutions of integer coefficients and of determinant ±1. By dividing the set (D) into classes of equivalent domains, one obtains with the aid of the algorithm shown the representatives

$$D, D_1, \ldots, D_{m-1}$$

of various classes of domains belonging to the set (D).

By calling reduced the quadratic forms which belong to the domains obtained, one establishes a new method of reduction of positive quadratic forms.

I have applied the general theory shown to the study of two types of primitive parallelohedra of the space in n dimensions which correspond to the principal domain of quadratic forms and to the domains which are contiguous to the principal domain by the faces in $\frac{n(n+1)}{2} - 1$ dimensions. The principal domain is defined by the inequalities

$$\sum_{k=1}^{n} a_{ik} \geq 0, \quad (i = 1, 2, \ldots, n)$$

$$-a_{ij} \geq 0. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n; i \neq j)$$

I study in detail the parallelohedra of the space in 2, 3 and 4 dimensions.

In the space in 2 dimensions, there is only one type of primitive parallelohedra, provided that one does not consider as different the equivalent types; it is the hexagon of Lejeune-Dirichlet.

The set (D) of domains is composed in this case of a single class, the representative of which is the principal domain defined by the inequalities (8).

In the space in 2 dimensions, there is only one single space of primitive parallelohedra – it is the parallelogram.

In the space in 3 dimensions, there is only one single type of primitive parallelohedra – it is a polyhedron of 14 faces, 8 of which are hexagonal and 6 of which are parallelogrammatic.

The set (D) of domains is composed in this case of a single class, the representative of which is the principal domain. By calling reduced a ternary positive quadratic form $ax^2 + a'y^2 + a''z^2 + 2byz + 2b'zx + 2b''xy$ which belongs to the principal domain determined with the help of inequalities

$$a + b' + b'' \geq 0, a' + b'' + b \geq 0, a'' + b + b' \geq 0, -b \geq 0, -b' \geq 0, -b'' \geq 0,$$

one will arrive at the method of reduction of ternary positive quadratic forms due to Selling. ‡

In the space in 3 dimensions, there are 4 spaces of primitive parallelohedra, they are : 1). the parallelepiped, 2). the prism of hexagonal base, 3). the parallelegrammatic dodecahedron and 4). the dodecahedron in 4 hexagonal faces and 8 parallelogrammatic faces.

In the space of 4 dimensions, there are three types of primitive parallelohedra. The set (D) of domains is composed of three classes of domains of quaternary quadratic forms.

I have determined the three representatives of these classes

$$D, D', D'',$$

By calling as reduced the quaternary positive quadratic form which belong to the domains D, D', D'', I have arrived at a modification of the methods of reduction of quaternary positive quadratic forms due to Mr. Charve. †

By virtue of this theorem, the problem of uniform partition of the space in n dimensions by congruent primitive parallelohedra always comes down to the study of parallelohedra corresponding to the positive quadratic forms.

I am inclined to think, without being able to demonstrate, that the theorem introduced is also true for the nonprimitive parallelohedra.

The parallelohedra of the space in 2 and in 3 dimensions have been studied by Mr. Fedorow ¶ which has discovered with the help of purely geometrical considerations, the exitence of two spaces of parallelohedra in the space in 2 dimensions and the existence of five spaces of parallelohedra in the space in 3 dimensions. Mr. Fedorow has demonstrated that there is no other parallelohedra in the space of 2 and of 3 dimensions.

The parallelohedra in 3 dimensions of Mr. Fedorow play an important role in the theory of the structure of crystals. §

First part

Uniform partition of the analytical space in n dimensions with the aid of translations of the same convex polyhedron

---

‡ *Selling,* Mémoire cited

† *Charve,* De la réduction des formes quadratiques quaternaires positives [Of the reduction of quaternary quadratic forms] (Comptes-Rendus des séances de l'Academie du Paris), V. 92, p.782 and Annales de l'École Normale supérieure, 2^{nd} serie, V. XI, p.119

¶ *Fedorow,* Basic principles in the theory of diagrams. St. Petersbourg, 1885 (in Russian)

Fedorow, Reguläre Plan- und Raumteilung. [Regular planar and space partition] (Abhandlungen der K. bayer. Akademie der Wiss. II Cl., XX Bd. II Abt. München, 1899)

See also: *Minkowski,* Allgemeine Lehrsätze über die convexen Polyeder. [General theorems on the onvex polyhedron] (Nachrichten von der Köigl. Gesellschaft der Wissenschaften zu Göttingen, Matheem. -Physikalische Klasses, 1897, p.198)

§ See: *Fedorow,* Courses in Crystallography. St. Petersbourg, 1901 (in Russian)

Soret, Cristallographie physique. [Physical crystallography] Genève, 1894.

Schönflies, Kristallsysteme und Kristallstruktur. [Crystal systems and crystal structure] Leipzig, 1891

Sommerfeldt, Physikalische Kristallographie.[Physical crystallography] Leipzig, 1907.

Section I
General properties of parallelohedra

On the convex polyhedra in n dimensions

1

One will call point of the analytical space in n dimensions any systems $(x_1, x_2, \ldots, x_n)$, or simply (x_i), of real values of variables $x_1, x_2, \ldots, x_n$.

Consider a system of linear inequalities

$$a_{0k} + \sum_{i=1}^{n} a_{ik} x_i \geq 0 \quad (k = 1, 2, \ldots, \sigma) \qquad (1)$$

of any real coefficients.

One will say that the set R of points verifying the inequalities (1) is of n dimensions, if there exist points satisfying the conditions

$$a_{0k} + \sum a_{ik} x_i > 0. \quad (k = 1, 2, \ldots, \sigma)$$

One will call them point, interior to the set R.

Fundamental principle. † *For the set R of points verifying the inequalities (1) to be of n dimensions, it is necessary and sufficient that the equation*

$$\rho_0 + \sum_{k=1}^{\sigma} \rho_k (a_{0k} + \sum a_{ik} x_i) = 0$$

does not reduce into an identity so long as all the parameters $\rho_0, \rho_1, \ldots, \rho_\sigma$ *are positive or zero.*

Definition I. One will call convex polyhedron any set of points verifying a system of linear inequalities, on condition that this set be bounded and of n dimensions.

2

Let us suppose that the inequalities (1) define a convex polyhedron R and suppose that all the inequalities (1) be independent. In such case, the polyhedron R possesses σ faces in $n-1$ dimensions which are defined by the corresponding equations

$$a_{0k} + \sum a_{ik} x_i = 0. \quad (k = 1, 2, \ldots, \sigma)$$

Definition II. Suppose that a point (α_i) *belonging to R verifies the equations*

$$a_{0r} + \sum a_{ir} x_i = 0, \quad (r = 1, 2, \ldots, \mu) \qquad (2)$$

and that one had the inequalities

$$a_{0k} + \sum a_{ik} x_i > 0. \quad (k = \mu + 1, \ldots, \sigma)$$

Designate by ν *the number of dimensions of the set* $P(\nu)$ *composed of points belonging to R and verifying the equations (2). One will call* face in ν dimensions of the polyhedron R *the set* $P(\nu)a$, $(\nu = 0, 1, 2, \ldots, n-1)$.

In the case $\nu = 1$, one will call edge of the polyhedron R a face $P(1)$ and in the case $\nu = 0$, one will call vertex of the polyhedron a face $P(0)$.

For more generality in the notations, one will designate by the symbol $P(n)$ the polyhedron R itself.

Under this restriction, one can introduce the following proposition:

Any point belonging to the polyhedron R is interior to a face $P(\nu)$ *of that polyhedron, where* $\nu = 0, 1, 2, \ldots, n$.

3

Let us suppose that the polyhedron R possesses s vertices

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{is}).$$

Designate by

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{im})$$

all the vertices of R which verify the equations (2).

Theorem. † *The face* $P(\nu)$ *in* ν *dimensions* $(\nu = 0, 1, 2, \ldots, n)$ *of the polyhedron R defined by the equations (2) presents a set of points determined by the aid of equalities*

$$x_i = \sum_{r=1}^{m} \vartheta_r \alpha_{ir} \text{ where } \sum \vartheta_r = 1 \text{ and } \vartheta_r \geq 0. \quad (r = 1, 2, \ldots, m)$$

Set of domains in n dimensions corresponding to the different vertices of a convex polyhedron.

4

Let us suppose a vertex (α_i) of the polyhedron R be determined by the equations

$$a_{0k} + \sum a_{ik} x_i = 0. \quad (k = 1, 2, \ldots, \mu) \qquad (1)$$

---

† The principle announced differs only in the formulation from the fundamental principle explained in my first memoir titled: On some properties of perfect positive quadratic forms. (This journal, V. 133, p. 113)

† See my mémoire cited, Number 12

Definition. One will call domain corresponding to the vertex (x_i) the set A of points determined with the help of equalities

$$x_i = \sum_{k=1}^{\mu} \rho_k a_{ik} \text{ where } \rho_k \geq 0. \ (k = 1, 2, \ldots, \mu) \tag{2}$$

Designate by

$$A_1, A_2, \ldots, A_s \tag{3}$$

the domains corresponding to the different vertices

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{is})$$

of the polyhedron R. By virtue of the definition established, the set (3) of domains enjoys the following properties:

I. All the domains of the set (3) are in n dimensions.

Let us suppose the domain A determined by the equalities (2) be not in n dimensions.

All the points (x_i) belonging to the domain A verify at least one linear equation

$$\sum p_i x_i = 0.$$

By virtue of (2), one will have

$$\sum p_i a_{ik} = 0. \ (k = 1, 2, \ldots, \mu) \tag{4}$$

As the equations (1) define a vertex (α_i) of the polyhedron R, one will find among the systems

$$(a_{11}, \ldots, a_{n1}), (a_{12}, \ldots, a_{n2}), \ldots, (a_{1\mu}, \ldots, a_{n\mu})$$

n systems the determinant of which is not zero; it follow that the equalities (4) are impossible.

II. Any point of the space in n dimensions belong to at least one domain of the set (3).

Let (α_i) be an arbitrary point. Examine the sum

$$\sum a_i \alpha_{ik}, \quad (k = 1, 2, \ldots, s)$$

and suppose that the smallest sum $\sum a_i x_i$ correspond to the vertex (α_i) defined by the equations (1). One will have the inequalities

$$\sum a_i \alpha_{ik} \geq \sum a_i \alpha_i. \quad (k = 1, 2, \ldots, s)$$

By virtue of the theorem of Number 3, one obtains

$$\sum a_i x_i \geq \sum a_i \alpha_i,$$

for any point (x_i) belonging to the polyhedron R.

One concludes that the inequalities

$$\sum a_i \alpha_i - \sum a_i x \geq 0 \text{ and } a_{0k} + \sum a_{ik} x_i \geq 0 \ (k = 1, 2, \ldots, \sigma)$$

can not define a polyhedra in n dimensions and, by virtue of the fundamental principle of Number 1, one will have an identity

$$\rho_0 + \rho(\sum a_i \alpha_i - \sum a_i x_i) + \sum_{k=1}^{\sigma} \rho_k (a_{0k} + \sum a_{ik} x_i) = 0,$$

where

$$\rho_0 \geq 0, \rho \geq 0, \rho_k \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

By making $x_i = \alpha_i$ in this identity, it will become

$$\rho_0 + \sum_{k=1}^{\sigma} \rho_k (a_{0k} + \sum a_{ik} \alpha_i) = 0,$$

and as according to the supposition made

$$a_{0k} + \sum a_{ik} \alpha_i > 0$$

as long as $k = \mu + 1, \ldots, \sigma$, it is necessary that

$$\rho_0 = 0, \rho_{\mu+1} = 0, \ldots, \rho_\sigma = 0,$$

therefore

$$\rho(\sum a_i \alpha_i - \sum a_i x_i) + \sum_{k=1}^{\mu} \rho_k (a_{0k} + \sum a_{ik} x_i) = 0.$$

One draws

$$a_i = \sum_{k=1}^{\mu} \frac{\rho_k}{\rho} a_{ik} \text{ where } \frac{\rho_k}{\rho} \geq 0, \ (k = 1, 2, \ldots, \mu)$$

therefore the points (α_i) belongs to the domain A.

III. A point which is interior to a face $A(\nu)\,(\nu = 0, 1, 2, \ldots, n)$ of any domain of the set (3) belongs only to the domains of the set (3) which are contiguous by the face $A(\nu)$.

Suppose the point (a_i) be interior to a face $A(\nu)$ of the domain A.

By designating with

$$(a_{i1}), (a_{i2}), \ldots, (a_{i\tau}), \tau \leq \mu$$

the points which characterise the face $A(\nu)$, one can put †

$$a_i = \sum_{k=1}^{\tau} \rho_k a_{ik} \text{ where } \rho_k > 0. \quad (k = 1, 2, \ldots, \tau)$$

Suppose that the point (a_i) be interior to another face $A'(\nu')$ of a domain A' which corresponds to a vertex (α'_i). One can put

$$a_i = \sum_{h=1}^{\tau'} \rho'_h a'_{ih} \text{ where } \rho'_h > 0. \ (h = 1, 2, \ldots, \tau')$$

By virtue of these equalities, one will have an identity

$$\rho_0 + \sum_{h=1}^{\tau} \rho_k (a_{0k} + \sum a_{ik} x_i) = \sum_{h=1}^{\tau'} (a'_{0h} + \sum a'_{ih} x_i) \tag{5}$$

By making within this identity $x_i = \alpha_i$, one obtains

$$\rho_0 = \sum_{h=1}^{\tau'} (a'_{0h} + \sum a'_{ih} \alpha_i),$$

and it results that

$$\rho_0 \geq 0.$$

By making within the identity (5) $x_i = x'_i$, one obtains

$$\rho_0 + \sum_{k=1}^{\tau} \rho_k (a_{0k} + \sum a_{ik} \alpha'_i) = 0;$$

consequently $\rho_0 = 0$ and

$$a_{0k} + \sum a_{ik} \alpha'_i = 0. \quad (k = 1, 2, \ldots, \tau)$$

In the same manner, one finds

$$a'_{0h} + \sum a'_{ih} \alpha_i = 0. \quad (h = 1, 2, \ldots, \tau')$$

One concludes that the two faces $A(\nu)$ and $A'(\nu')$ coincide. ‡

By virtue of properties demonstrated of the set (3) of domains, one will say that this set uniformly *partitions* the space in n dimensions.

Definition of the group of vectors

5 *Definition I. One will call vector the set of points determined with the help of equalities*

$$x_i = x_i + u(\alpha'_i - \alpha_i) \text{ where } 9 \leq u \leq 1, \tag{1}$$

(α_i) and (α'_i) being any two different points.

One will designate the vector determined with the help of equalities (1) by the symbol $[\alpha_i, \alpha'_i]$. In this case $\alpha_i = 0\,(i = 1, 2, \ldots, n)$, one will designate the corresponding vector by the symbol $[\alpha'_i]$ and one will call it vector of the point (α'_i).

Definition II. Suppose that

$$[\lambda_{i1}], [\lambda_{i2}], \ldots, [\lambda_{im}] \tag{2}$$

be the vectors of arbitrary points $(\lambda_{i1}), (\lambda_{i2}), \ldots, (\lambda_{im})$. One will call group of vectors *the set G of vectors determined with the help of equalities*

$$\lambda_i = \sum_{k=1}^{m} l_k \lambda_{ik},$$

$l_1, l_2, \ldots, l_m$ being of arbitrary integers.

---

† See my mémoire cited, Number 13

‡ See my mémoire cited, Number 20, p. 133

One will call *basis of the group G of vectors* the vectors (2).

Translation of polyhedra.

6

Definition. Effect a linear transformation of a polyhedron R with the help of a substitution

$$x_i = x_i' - \lambda_i, \ (i = 1, 2, \ldots, n) \qquad (1)$$

the coefficients $\lambda_1, \lambda_2, \ldots, \lambda_n$ *being arbitrary. One will say that one has effected a translation of the polyhedron R the length of the vector* $[\lambda_i]$.

Suppose that the polyhedron R be determined by the inequalities

$$a_{0k} + \sum a_{ik} x_i \geq 0, \quad (k = 1, 2, \ldots, \sigma)$$

The transformed polyhedron R' will be determined, by virtue of (1), by the inequalities

$$a_{0k} + \sum a_{ik}(x_i - \lambda_i) \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

One will call congruent the polyhedra R and R'.

7

Let G be a group of vectors. By effecting the different translations of the poly hedron R the length of vectors belonging to the group G, one will form a set R of congruent polyhedra.

One will say that the set (R) of congruent polyhedra uniformly partition the space in n dimensions in the following conditions.

I. Any point of the space in n dimensions belongs to at least one polyhedron of the set (R).

II. A point which is interior to any one face $P(\nu)$ $(\nu = 0, 1, 2, \ldots, n)$ *of a polyhedron of the set* (R) *belongs to only the polyhedrons of the set* (R) *which are contiguous by the face* $P(\nu)$.

Definition of parallelohedra

8

Definition. One will call parallelohedron any convex polyhedron R possessing a group G of translations with the aid of which one can uniformly fill the space in n dimensions by the polyhedra congruent to the polyhedron R.

By virtue of the definition established, the parallelohedra possess an important property, in knowing:

By effecting a linear transformation of a parallelohedron with the help of a substitution by any real coefficients

$$x_i = \alpha_{i0} + \sum_{k=1}^{n} \alpha_{ik} x_k', \quad (i = 1, 2, \ldots, n)$$

one obtains a convex polyhedron which is also a parallelohedron.

Observe that by virtue of the definition established, any parallelohedron of the space in n dimensions is a parallelohedron.

Properties of the group of vectors of a parallelohedron.

9

Suppose that a parallelohedron R be defined by the inequalities

$$a_{0k} + \sum a_{ik} x_i \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

Designate by G the group of the parallelohedron R and suppose that the group G possesses the basis

$$[\lambda_{i1}], [\lambda_{i2}], \ldots, [\lambda_{im}]. \qquad (2)$$

All the vectors which form the basis of the group G can not verify the same linear equation

$$\sum p_i \lambda_i = 0,$$

because otherwise the set (R) of congruent parallelohedra corresponding to the group G would not fill the space in n dimensions.

One concludes that among the vectors (2) there are n vectors

$$[\lambda_{i1}], [\lambda_{i2}], \ldots, [\lambda_{im}] \qquad (3)$$

the determinant $\pm\Delta$ of which is not zero; one will call them independent.

Theorem I. The numerical value Δ *of the determinant of n independent vectors possesses a limit*

$$\Delta \geq \int_{(R)} dx_1 dx_2 \cdots dx_n \, .$$

Let (α_i) be any point which is interior to the parallelohedron R. Introduce within our researches a parallelepiped K determined with the aid of equalities

$$x_i = \alpha_i + \sum_{k=1}^{n} u_k \lambda_{ik}, \quad (i = 1, 2, \ldots, n) \qquad (4)$$

where

$$-\delta \leq u_k \leq \delta. \quad (k = 1, 2, \ldots, n) \qquad (5)$$

One can choose the positive parameter δ in such manner that all the points of the parallelohedron R defined by the inequalities (1) belong to the parallelepiped K.

Take a positive integer m and determine $(m+1)^n$ systems $(l_1, l_2, \ldots, l_n)$ of integers verifying the inequalities

$$0 \le l_k \le m. \quad (k = 1, 2, \ldots, n) \tag{6}$$

Designate by

$$\lambda_i^{(h)} = \sum_{k=1}^{n} l_k^{(h)} \lambda_{ik}, \quad (h = 1, 2, \ldots, (m+1)^n) \tag{7}$$

$(m+1)^n$ corresponding vectors belonging to the group G.

By applying the translations of the parallelohedra R the length of vectors (7), one obtains $(m+1)^n$ different parallelohedra of the set (R):

$$R^{(h)}. \quad (h = 1, 2, \ldots, (m+1)^n) \tag{8}$$

Designate by H a parallelepiped which is determined by the equalities

$$x_i = \alpha_i + \sum_{k=1}^{n} u_k \lambda_{ik}, \ (i = 1, 2, \ldots, n) \tag{9}$$

where

$$-\delta \le u_k \le m + \delta. \quad (k = 1, 2, \ldots, n) \tag{10}$$

I argue that all the points of parallelohedron (8) belong to the parallelepiped H. In effect, let $(x_i^{(h)})$ be any point of the parallelohedron $R^{(h)}$ $(h = 1, 2, \ldots, (m+1)^n)$. By posing

$$x_i = x_i^{(h)} - \lambda_i^{(h)}, \quad (i = 1, 2, \ldots, n) \tag{11}$$

one obtains a point (x_i) belonging to the parallelohedron R which is congruent to the point given $(x_i^{(h)})$. By virtue of (4), (7) and (11), one obtains

$$x_i^{(h)} = \alpha_i + \sum_{k=1}^{n} (l_k + u_k) \lambda_{ik},$$

and by (5) and (6), it becomes

$$-\delta \le l_k + u_k \le m + \delta, \quad (k = 1, 2, \ldots, n)$$

thus the point $(x_i^{(h)})$ belongs to the parallelepiped H.

It follows that

$$\int_{(H)} dx_1 dx_2 \cdots dx_n \ge \sum_h \int_{(R^h)} dx_1 dx_2 \cdots dx_n. \ (h = 1, 2, \ldots, (m+1)^n)$$

By observing that

$$\int_{(H)} dx_1 dx_2 \cdots dx_n = \Delta (m + 2\delta)^n$$

and that

$$\int_{(R^h)} dx_1 dx_2 \cdots dx_n = \int_{(R)} dx_1 dx_2 \cdots dx_n, \ (h = 1, 2, \ldots, (m+1)^n)$$

one obtains

$$\Delta (m + 2\delta)^n \ge (m+1)^n \int_{(R)} dx_1 dx_2 \cdots dx_n.$$

By making the number m increase indefinitely, one finds

$$\Delta \ge \int_{(R)} dx_1 dx_2 \cdots dx_n.$$

10 *Theorem II. The group G of vectors of a parallelohedron possesses basis formed by n independent vectors.*

Designate by G' a group of vectors having the basis (3). It can be that the two groups G and G' coincide. In that case n vectors (3) present a basis of the group G.

By supposing the contrary, one will have among the vectors (2) at least one vector $[\lambda_i']$ which does not belong to the group G'. By putting

$$\lambda_i' = \sum_{k=1}^{n} l_k' \lambda_{ik} \,,$$

one will have among the numbers $l_1', l_2', \ldots, l_n'$ at least one number which is fractional.

Designate by $l_1, l_2, \ldots, l_n$ the integers verifying the inequalities

$$\left| l_k' - l_k \right| \le \frac{1}{2} \quad (k = 1, 2, \ldots, n)$$

and suppose that $l'_r - l_r \neq 0$.

By designating

$$\lambda'_{ik} = \lambda_{ik}\,,\ (k = 1, 2, \ldots, n, k \neq r) \text{ and } \lambda'_{ir} = \lambda'_i - \sum l_k \lambda_{ik},,$$

one obtains a system of n independent vectors

$$[\lambda'_{i1}], [\lambda'_{i2}], \ldots, [\lambda'_{in}]$$

belonging to the group G, the determinant $\pm\Delta'$ of which verifies the inequality

$$0 < \Delta' \leq \frac{1}{2}\Delta.$$

The procedure explained can not be prolonged indefinitely, by virtue of Theorem I, therefore one will always obtain a system of n vectors forming the basis of the group G.

11 *Theorem III. The numerical value Δ of the determinant of a system of n vectors forming the basis of the group G is expressed by the formula*

$$\Delta = \int_{(R)} dx_1 \cdot dx_2 \cdots dx_n\,.$$

suppose that the system (3) of n vectors presents a basis of the group G.

Introduce in okur studies a parallelepiped H' determined with the help of equalities

$$x_i = \alpha_i + \sum_{k=1}^{n} u_k \lambda_{ik},\ (i = 1, 2, \ldots, n) \tag{12}$$

where

$$\delta \leq u_k \leq m - \delta\,.\ (k = 1, 2, \ldots, n) \tag{13}$$

I argue that any point of the parallelepiped H' belongs to at least one of parallelohedron (8). In effect, let (x'_i) be any point of the parallelepiped H'.

Designate by R^0 a parallelohedron of the set (R) to which belongs the point (x'_i). Let $[\lambda_i]$ be the vector which defines a translation of the parallelohedron R to R^0. By putting

$$x_i = x'_i - \lambda_i\,, \tag{14}$$

one obtains a point (x_i) belonging to the parallelohedron R which is congruent to the point x'_i. By virtue of the supposition made, the vector $[\lambda_i]$ can be determined by the equalities

$$\lambda_i = \sum_{k=1}^{n} l_k \lambda_{ik}\,. \tag{15}$$

As the point (x'_i) belongs to the parallelepiped H', one will present the equalities (14), by (12) and (15), in the following form:

$$x + i = \alpha_i + \sum_{k=1}^{n} (u_k - l_k)\lambda_{ik}\,.$$

The point (x_i) belonging to the parallelohedron R belongs also, by virtue of the supposition made, to the parallelepiped K determined by the equalities (4), by condition of (5). It follows that

$$-\delta \leq u_k - l_k \leq \delta\,, \quad (k = 1, 2, \ldots, n)$$

and as, by (13),

$$\delta \leq u_k \leq m - \delta\,, \quad (k = 1, 2, \ldots, n)$$

it becomes

$$0 \leq l_k \leq m, \quad (k = 1, 2, \ldots, n)$$

therefore the vector $[\lambda_i]$ determined by the equalities (5) is among the vectors (7) and the point examined (x') of the parallelepiped H' belongs to a parallelohedron of the series (8).

It follow that

$$\int_{H'} dx_1 dx_2 \cdots dx_n \leq \sum_h \int_{(R^h)} dx_1 dx_2 \cdots dx_n\,.$$

By making the number m grow indefinitely, one obtains

$$\Delta \leq \int_{(R)} dx_1 dx_2 \cdots dx_n\,.$$

By virtue of Theorem I, it is necessary that

$$\Delta = \int_{(R)} dx_1 dx_2 \cdots dx_n\,.$$

Properties of faces in $n-1$ dimensions of a parallelohedron.

12 Suppose that a parallelohedron R be defined by the independent inequalities

$$a_{0k} + \sum a_{ik}x_i \geq 0 \,. \quad (k = 1, 2, \ldots, \delta)$$

Designate by $P_k (k = 1, 2, \ldots, \sigma)$ the faces in $n-1$ dimensions of the parallelohedron R determined by the corresponding equations

$$a_{0k} + \sum a_{ik}x_i = 0 \,. \qquad (1)$$

Let (α_i) be a point which is interior to the face P_k. Examine a parallelepiped K defined by the equalities

$$x_i = \alpha_i + u_i \text{ where } |u_i| \leq \epsilon \,. (i = 1, 2, \ldots, n) \qquad (2)$$

One can choose a parameter ϵ however small that one will have the inequalities

$$a_{0r} + \sum a_{ir}x_i > 0, \; (r = 1, 2, \ldots, \sigma, r \neq k) \qquad (3)$$

for any point (x_i) of the parallelepiped K. It results in that all the points of the parallelepiped K verifying the inequality

$$a_{0k} + \sum a_{ik}x_i \geq 0 \qquad (4)$$

belong to the parallelohedron R. As the point (α_i) verifies the equation (1), the equation (4) reduces, by reason of (2), to this one here

$$\sum a_{0k}u_i \geq 0 \,.$$

I argue that one can choose a value of the parameter ϵ however sall that all the points of the parallelepiped K verifying the inequality

$$\sum a_{ik}u_i \leq 0$$

will belong to another parallelohedron R_k of the set (R). By relying on the demonstrated properties of the group G of vectors, one will easily demonstrate the proposition stated.

Two parallelohedra R and R_k are contiguous by the face P_k in $n-1$ dimensions. Designate by $[\lambda_{ik}]$ the vector which defines a translation of the parallelohedron R_k to R. The face P_k which is defined in the parallelohedron R by the equation (1) will be defined in the parallelohedron R_k by the equation

$$-a_{0k} - \sum a_{ik}x_i = 0 \,. \qquad (5)$$

By carrying out a translation of the face P_k the length of the vector λ_{ik}, one obtains another face P_k of the parallelohedron R which will be within the parallelohedron determined by the equation

$$-a_{0k} - \sum a_{ik}(x_i - \lambda_{ik}) = 0 \,.$$

One will call parallel the faces P_k and P_k' of the parallelohedron R. We have arrived at the following important result:

All the faces in $n-1$ dimensions of a parallelohedron can be divided into pairs of parallel faces.

13 Designate by

$$R_1, R_2, \ldots, R_\sigma$$

all the parallellohedra which are contiguous to the parallelohedron R by the faces $P_1, P_2, \ldots, P_\sigma$. Designate by

$$[\lambda_{i1}], [\lambda_{i2}], \ldots, [\lambda_{i\sigma}] \qquad (6)$$

the corresponding vectors.

By virtue of the definition of the parallelohedron, the vectors (6) form the basis of the group G. Among the vectors of this group there exist the systems of n vectors which form a basis of the group G.

Congruent faces in different dimensions of a parallelohedron.

14 Suppose that a face $P(\nu)$ in ν dimensions of a parallelohedron R also belongs to the parallelohedra $R_1, R_2, \ldots, R_\tau$ of the set (R). Let (α_i) be a point which is interior to the face $P(\nu)$. One can determine a positive value of the parameter ϵ in such a manner that all the point of the parallelepiped K defined by the equalities

$$x_i = \alpha_i + u_i \text{ where } |u_i| \leq \epsilon \quad (i = 1, 2, \ldots, n)$$

belong to the parallelohedra $R, R_1, R_2, \ldots, R_\tau$.

Designate by $[\lambda_{ik}]$ the vectors the length of which one will carry out the translations of parallelohedra R_k into $R (k = 1, 2, \ldots, \tau)$.

By carrying out the translations of the face $P(\nu)$ the length of vectors $[\lambda_{ik}] \, (k = 1, 2, \ldots, \tau)$, one obtains the new faces

$$P'(\nu), P''(\nu), [\ldots,]P^{(\tau)}(\nu)$$

of the parallelohedron R.

Definition I. One will call congruent the faces of the parallelohedron R

$$P'(\nu), P''(\nu), \ldots, P^{(\tau)}(\nu)$$

in ν dimensions $(\nu = 0, 1, 2, \ldots, n-1)$.

15

Theorem. The number of parallelohedra of the set (R) which are contiguous by the same face in ν dimensions can not be less than $n+1-\nu\,(\nu = 0, 1, 2, \ldots, n-1)$.

Suppose that the face $P(\nu)$ be determined within the parallelohedron R by the equation

$$a_{0r} + \sum a_{ir}x_i = 0. \ (r = 1, 2, \ldots, \mu) \tag{1}$$

Designate by $R_1, R_2, \ldots, R_\mu$ the parallelohedra which are contiguous to R by the faces in $n-1$ dimensions defined by the equations (1). The face $P(\nu)$ will belong to all the parallelohedra $R_1, R_2, \ldots, R_\mu$, therefore

$$\tau \geq \mu.$$

As the face $P(\nu)$ is in ν dimensions, it is necessary that

$$\mu \geq n - \nu,$$

and as a result

$$\tau \geq n - \nu.$$

16

Definition II. One will call simple a face in ν dimensions which belong to only $n+1-\nu$ parallelohedra of the set (R).

Definition II. One will call primitive a parallelohedron, all the faces in different dimensions of which are simple.primitive parallelohedron

The primitive parallelohedra possess many important properties which simplify the study.

In the subsequent studies, one will study only the primitive parallelohedron and all the nonprimitive parallelohedra which can be considered as a boundary of primitive parallelohedra.

I am inclined to think that each primitive parallelohedron can be considered in this point of view, but I have not been successful in demonstrating this.

Section II
Fundamental properties of primitive parallelohedra

Definition of primitive parallelohedra.

17

We have called in Number 16 "primitive parallelohedron" all parallelohedron, all the faces in different dimensions of which are simple.

Theorem I. for a parallelohedron to be primitive it is necessary and sufficient that all the vertices be simple.

The theorem stated is evident by virtue of the definition established.

Theorem II. Two primitive parallelohedra belonging to the set (R) can be contiguous by only one face in $n-1$ dimensions.

Suppose that a face $P(\nu)$ in ν dimensions of a primitive parallelohedron R be determined with the aid of $n-\nu$ equations

$$a_{0r} + \sum a_{ir}x_i = 0. \ (r = 1, 2, \ldots, n-\nu) \tag{1}$$

Designate by $R_1, R_2, \ldots, R_{n-\nu}$ the parallelohedra which are contiguous to the parallelohedron R by the faces in $n-1$ dimensions defined by the aid of equations (1). The face $P(\nu)$ will not belong to the parallelohedra $R_1, R_2, \ldots, R_{n-\nu}$ by virtue of the definition established, thus the theorem introduced is demonstrated.

Edges of primitive parallelohedra of the set (R)

18

Let (α_i) be a vertex of the primitive parallelohedron R determined by n equations

$$a_{0k} + \sum a_{ik}x_i = 0. \ (k = 1, 2, \ldots, n) \tag{1}$$

Designate by $R_1, R_2, \ldots, R_n$ the parallelohedra contiguous to the parallelohedron R by the faces in $n-1$ dimensions determined with the help of equation (1).

By virtue of the definition established, the vertex (α_i) will not belong to the parallelohedra $R_1, R_2, \ldots, R_n$ of the set (R).

Determine n numbers $P_{1k}, P_{2k}, \ldots, P_{nk}$ with the help of equations

$$\sum a_{ir}P_{ik} = 0. \ (r = 1, 2, \ldots, n; r \neq k; k = 1, 2, \ldots, n) \tag{2}$$

The equations (2) do not define the number $P_{1k}, P_{2k}, \ldots, P_{nk}$ to a common factor. Attach to the equations (2) a condition

$$\sum a_{ik}p_{ik} > 0 \ (k = 1, 2, \ldots, n) \tag{3}$$

and consider a vector g_k determined with the help of equalities

$$x_i = \alpha_i + p_{ik}\rho \ \text{ where } \ \rho \geq 0.$$

By attributing to the parameter ρ positive values sufficiently small, one will determine, by (3), the points of the vector g_k belonging to R. By putting

$$\alpha_{ik} = \alpha_i + p_{ik}\rho_k,$$

one will determine a vertex (α_{ik}) of the parallelohedron R adjacent to the vertex (α_i) by an edge $P_k(1)$ of the parallelohedron R $(k = 1, 2, \ldots, n)$. One will characterise the edge $P_k(1)$ by the symbol $[\alpha_i, \alpha_{ik}]$.

Observe that all the points of the edge $P_k(1)$ verifies $n-1$ equations

$$a_{0r} + \sum a_{ir}x_i = 0. \quad (r = 1, 2, \ldots, n; r \neq k)$$

It follows that the edge $P_k(1)$ belongs to the parallelohedra

$$R, R_1, \dots, R_{k-1}, R_{k+1}, \dots, R_n \quad (k = 1, 2, \dots, n)$$

and by virtue of the definition established, does not belong to any other parallelohedron of the set (R).

One concludes that the parallelohedra

$$R_1, R_2, \dots, R_n$$

are contiguous by an edge too. By designating this edge by $P_0(1)$, one will determine it with the symbol $[\alpha_i, \alpha_{i0}]$ by putting

$$\alpha_{i0} = \alpha_i + p_{i0}\rho_0.$$

We have arrived at the following result:

There exist $n + 1$ edges of parallelohedra of the set (R), contiguous by one common vertex of these parallelohedra.

Observe that $n - 1$ edges

$$P_1(1), \dots, P_{k-1}(1), P_{k+1}(1), \dots, P_k(1) \quad (k = 1, 2, \dots, n)$$

define a face in $n - 1$ dimensions which is common to the parallelohedra R and R_k $(k = 1, 2, \dots, n)$. Two parallelohedra R_k and R_h $(k = 1, 2, \dots, n; h = 1, 2, \dots, n)$ are contiguous by a face in $n-1$ dimensions which is defined by $n - 1$ edges

$$P_r(1). \quad (r = 0, 1, 2, \dots, n; r \neq k, r \neq h)$$

Canonical form of equations which define a vertex of a primitive parallelohedron.

19 By conserving the previous notations, one can determine the vertex (α_i) within the parallelohedron R with the help of equations

$$u_k(a_{0k} + \sum a_{ik}x_i) = 0, \ (k = 1, 2, \dots, n) \tag{1}$$

$u_1, u_2, \dots, u_n$ being positive arbitrary parameters. One will say that the equation

$$-u_k(a_{0k} + \sum a_{ik}x_i) = 0 \ \text{ where } \ u_k > 0$$

does not define within the parallelohedron R a face in $n - 1$ dimensions because the inequality

$$-u_k(a_{0k} + \sum a_{ik}x_i) \geq 0$$

will not satisfy all the points of the parallelohedron R.

Theorem. One can determine the positive values of parameters $u_1, u_2, \dots, u_n$ to a common factor, such that by putting

$$a'_{0k} = u_k a_{0k}, a'_{ik} = u_k a_{ik}, \ (i = 1, 2, \dots, n; k = 1, 2, \dots, n)$$

one will define the vertex (α_i) within the parallelohedron R by the equations

$$\sum a'_{ik}(x_i - \alpha_i) = 0, \ (k = 1, 2, \dots, n) \tag{2}$$

and one will define the vertex (α_i) within the parallelohedron R_k $(k = 1, 2, \dots, n)$ by the equations

$$\begin{cases} \sum(a'_{ih} - a'_{ik})(x_i - \alpha_i) = 0, \ (h = 1, 2, \dots, n; h \neq k) \\ -\sum a'_{ik}(x_i - \alpha_i) = 0. \ (k = 1, 2, \dots, n) \end{cases} \tag{3}$$

Take an arbitrary positive parameter δ and determine the parameters $u_1, u_2, \dots, u_n$ after the equations

$$u_k \sum a_{ik}(\alpha_i - \alpha_{i0}) = \delta. \ (k = 1, 2, \dots, n) \tag{4}$$

I argue that the values $u_1, u_2, \dots, u_n$ obtained satisfy the conditions of the theory stated.

To demonstrate this, observe in the first place that the equations (4) define the positive values of $u_1, u_2, \dots, u_n$. In effect, we have seen in Number 18 that the edge $P_0(1)$ defined by the equalities

$$x_i = \alpha + u(\alpha_{i0} - \alpha_i) \ \text{ where } \ 0 \leq u \leq 1 \tag{5}$$

does not belong to the parallelohedra $R_1, R_2, \dots, R_n$. One concludes that by attributing to the parameter u any negative values sufficiently small, one will determine by the equality (5) a point which will be interior to the parallelohedron R. It follows that

$$\sum a_{ik}(\alpha_i - \alpha_{i0}) > 0, \quad (k = 1, 2, \dots, n)$$

and the equations (4) give

$$u_k > 0. \quad (k = 1, 2, \dots, n)$$

This established, designate by

$$\sum a^{(k)}_{ir}(x_i - \alpha_i) = 0 \ (r = 1, 2, \dots, n) \tag{6}$$

the equations which define the vertex (α_i) in the parallelohedron R_k $(k = 1, 2, \dots, n)$

Observe that n edges $P_r(1)$ $(r = 0, 1, 2, \ldots, n, r \neq k)$ are contiguous by the vertex (α_i) in the parallelohedron R_k. Each equation (6) will be verified by $n-1$ edges. One can thus put

$$\begin{cases} \sum a_{ik}^{(k)}(\alpha_{ir} - \alpha_i) = 0, \ (r = 1, 2, \ldots, n; r \neq k) \\ \sum a_{ik}^{(k)}(\alpha_{i0} - \alpha_i) > 0 \end{cases} \tag{7}$$

and

$$\begin{cases} \alpha a_{ih}^{(k)}(\alpha_{ir} - \alpha_i) = 0, \ (r = 0, 1, 2, \ldots, n; r \neq k, r \neq h) \\ \sum a_{ih}^{(k)}(\alpha_{ih} - \alpha_i) > 0. \ (h = 1, 2, \ldots, n; h \neq k) \end{cases} \tag{8}$$

The conditions established define the coefficients of equations (6) to a common positive factor, which can be arbitrarily chosen.

Observe that the coefficients of equations (1), which define the vertex (α_i) in the parallelohedron R are also determined to a common positive factor and satisfy the conditions

$$\begin{cases} \sum a_{ik}(\alpha_{ir} - \alpha_i) = 0, \ (r = 1, 2, \ldots, n; r \neq k; k = 1, 2, \ldots, n) \\ \sum a_{ik}(\alpha_{ik} - \alpha_i) > 0 \end{cases} \tag{9}$$

The equalities (4), (7), (8) and (9), one takes

$$\left.\begin{aligned} a_{ik}^{(k)} &= -\delta_k u_k a_{ik}, \\ a_{ih}^{(k)} &= \delta_h(u_h a_{ih} - u_k a_{ik}), \end{aligned}\right\} (i = 1, 2, \ldots, n; h = 1, 2, \ldots, n; h \neq k)$$

where $\delta_1, \delta_2, \ldots, \delta_n$ are positive factors. One can put

$$\delta_1 = 1, \delta_2 = 1, \ldots, \delta_n = 1,$$

and the equations (6) become

$$\sum(u_h a_{ih} - u_k a_{ik})(x_i - \alpha_i) = 0, \ (h = 1, 2, \ldots, n; h \neq k)$$

$$-\sum u_k a_{ik}(x_i - \alpha_i) = 0.$$

The theorem introduced is thus demonstrated.

One will say that the equations (2) and (3) which define the vertex (α_i) in the contiguous parallelohedron $R, R_1, \ldots, R_n$ are presented in the canonical form.

We have seen in Number 18 that the parallelohedra R_k and R_h $(k = 1, 2, \ldots, n; h = 1, 2, \ldots, n)$ are contiguous by a face in $n-1$ dimensions. As this face is characterised by the edges $P_r(1)$ $(r = 0, 1, 2, \ldots, n; r \neq k; r \neq h)$, one will determine it in the parallelohedron R_h by the canonical equation

$$\sum(a'_{ik} - a'_{ih})(x_i - \alpha_i) = 0.$$

20 *Canonical form of inequalities which define a positive parallelohedron.*

Suppose that a primitive parallelohedron R is determined with the help of independent inequalities

$$a_{0k} + \sum a_{ik} x_i \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

Bu designating with $u_1, u_2, \ldots, u_\sigma$ of arbitrary positive parameters, one will determine the parallelohedron R with the help of independent inequalities

$$u_k(a_{0k} + \sum a_{ik} x_i) \geq 0. \ (k = 1, 2, \ldots, \sigma) \tag{1}$$

We will see how all the problem of the study of primitive parallelohedra comes down to the appropriate choice of parameters $u_1, u_2, \ldots, u_\sigma$.

Fundamental Theorem. One can determine the positive values of parameters $u_1, u_2, \ldots, u_\sigma$ to a common factor, such that by putting

$$a'_{0k} = u_k a_{0k}, a'_{ik} - u_k a_{ik}, \quad (i = 1, 2, \ldots, n; k = 1, 2, \ldots, \sigma)$$

one will determine the parallelohedron R with the help of inequalities

$$a'_{0k} + \sum a'_{ik} x_i \geq 0 \ (k = 1, 2, \ldots, \sigma) \tag{2}$$

which enjoy the following property: all the vertices of the parallelohedron R will be determined by the equations presented in the canonical form.

One will call the inequalities (2) canonical.

By conserving the previous notations, suppose that one had chosen the parameters $u_1, u_2, \ldots, u_n$ in such a manner that the vertex (α_i) is determined by the canonical equations

$$a'_{0k} + \sum a'_{ik} x_i = 0. \quad (k = 1, 2, \ldots, n)$$

Examine the equations which define a vertex (α_{ik}) $(k = 1, 2, \ldots, n)$ of the parallelohedron R adjacent to the vertex (α_i) by the edge $P_k(1)$.

The vertex (α_{ik}) satisfies $n-1$ equations

$$a'_{0h} + \sum a'_{ih}x_i = 0. \ (h = 1, 2, \ldots, n; h \neq k) \tag{3}$$

Designate by

$$b_{0k} + \sum b_{ik}x_i = 0 \tag{4}$$

the n^{th} equation which defines the vertex (α_{ik}).

Determine the positive parameters v_h $(h = 1, 2, \ldots, n, h \neq k)$ and v_k corresponding to the equations (3) and (4), which reduces to these equations in the canonical form:

$$v_h(a'_{0h} + \sum a'_{ih}x_i) = 0 \quad (h = 1, 2, \ldots, n; h \neq k)$$

and

$$v_k(b_{0k} + \sum b_{ik}x_i) = 0.$$

I argue that one can put

$$v_h = 1. \quad (h = 1, 2, \ldots, n; h \neq k)$$

To demonstrate this, examine the canonical equation which defines in the parallelohedron R_h $(h = 1, 2, \ldots, n; h \neq k)$ a face in $n-1$ dimensions common to the parallelohedra R_r and R_h $(r = 1, 2, \ldots, n; r \neq h, r \neq, k)$.

By virtue of the theorem of Number 19, this face will be determined within R_h by the canonical equation

$$\sum (a'_{ir} - a'_{ih})(x_i - \alpha_i) = 0.$$

Besides, this same face will be determined in R_h, by virtue of the supposition made, by the canonical equation

$$\sum (v_r a'_{ir} - v_h a'_{ih})(x_i - \alpha_i) = 0.$$

It results in that

$$v_r a'_{ir} - v_h a'_{ih} = \delta(a'_{ir} - a'_{ih}), \quad (i = 1, 2, \ldots, n)$$

and so

$$v_r = \delta, v_h = \delta,$$

thus

$$v_r = v_h. \quad (r = 1, 2, \ldots, n; r \neq k; r \neq h)$$

As the parameters v_h $(h = 1, 2, \ldots, n)$ are defined to a factor, one can put

$$v_h = 1, \quad (h = 1, 2, \ldots, n; h \neq k)$$

and it only remains to determine the parameter v_k in order to define the vertex (α_{ik}) by the canonical equations.

21

By applying the procedure explained to all the vertices of the parallelohedron R adjacent to the vertex (α_i) and so on, one will successively determine the values of various parameters corresponding to all the inequalities (1).

It can turn out that one determines for one inequality the value of the corresponding parameter in various manners. I argue that all these values of the same parameter coincide.

The problem posed is extremely difficult. It is within this group of studies explained that is manifested their true geometrical characteristic, and one does not manage to master the difficulties which arise as a result with the help of geometrical methods.

Set of simplexes corresponding to the various vertices of a primitive parallelohedron.

22

We have seen in Number 4 that the various vertices of a parallelohedron R $(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{is})$ correspond to the domains

$$A_1, A_2, \ldots, A_s \tag{1}$$

which uniformly fill the space in n dimensions.

By conserving the previous notations, examine a domain A which corresponds to the vertex (α_i) of the parallelohedron R.

The domain A is composed of points determined by the equalities

$$x_i = \sum_{k=1}^{n} \rho_k a_{ik} \text{ where } \rho_k \geq 0. \ (k = 1, 2, \ldots, n) \tag{2}$$

The domain A possesses n faces in $n-1$ dimensions which correspond to n edges $P_k(1), (k = 1, 2, \ldots, n)$ contiguous by the vertex (α_i).

One will call the domain A simple.

Extract from the domain A a simplex L by the solution with the help of equalities

$$x_i = \sum_{k=1}^{n} \vartheta_k u_k a_{ik} \text{ where } \sum \vartheta_k \leq 1 \text{ and } \vartheta_k \geq 0. \ (k = 1, 2, \ldots, n)$$

The simplex L possesses $n+1$ faces in $n-1$ dimensions which are opposite to $n+1$ vertices

$$(0), (u_1 a_{i1}), (u_2 a_{i2}), \ldots, (u_n a_{in}),$$

Examine the face of the simplex L which is opposite to the vertex (0). One can present the equation which defines this face in the form

$$1 - \sum p_i x_i = 0. \tag{3}$$

It follows that one will have an inequality

$$1 - \sum p_i x_i > 0$$

for any point of L which does not belong to the face examined.

As the vertices of L: $(u_k a_{ik})(k = 1, 2, \ldots, n)$ satisfy the equation (3), one has

$$\sum p_i a_{ik} = \frac{1}{u_k}, \quad (k = 1, 2, \ldots, n) \tag{4}$$

thus

$$\sum p_i a_{ik} > 0. \quad (k = 1, 2, \ldots, n).$$

By virtue of (2), one obtains the inequality

$$\sum p_i x_i > 0$$

which holds for any point (x_i) of the domain A, the vertex (0) being excluded.

23 Examine in the same manner n domains $A_1, A_2, \ldots, A_n$ which are contiguous to the domain A by faces in $n-1$ dimensions.

One will take from the simple domain $A_k a$, $(k = 1, 2, \ldots, n)$ defined by the equalities

$$x_i = \sum \rho_h a_{ih} + \rho_k b_{ik} \text{ where } \rho_k \geq 0 \text{ and } \rho_h \geq 0, \ (h = 1, 2, \ldots, n; h \neq k)$$

a simplex L_k composed of points

$$x_i = \sum \vartheta_h u_h a_{ih} + \vartheta_k v_k b_{ik} \text{ where } \sum \vartheta_h + \vartheta_k \leq 1, \vartheta_k \geq 0, \vartheta_h \geq 0 \quad (h = 1, 2, \ldots, n; h \neq k)$$

Designate by

$$1 - \sum p_{ik} x_i = 0$$

the equation of the face of the simplex L_k which is opposite to the vertex (0).

One will have the equalities

$$\sum p_{ik} a_{ih} = \frac{1}{u_b} \quad (h = 1, 2, \ldots, n; h \neq k)$$

and

$$\sum p_{ik} b_{ih} = \frac{1}{u_k}.$$

By virtue of equalities (4), one obtains

$$\sum p_{ik} a_{ih} = \sum p_i a_{ih}. \quad (h = 1, 2, \ldots, n; h \neq k)$$

It follows that

$$\sum p_{ik} x_i = \sum p_i x_i, \tag{5}$$

for any point (x_i) belonging to the face common to domains A and A_k.

By applying the procedure explained to the domains which are contiguous to the domains $A_1, A_2, \ldots, A_n$ and so on, one will extract from any domain of the set (1) a corresponding simplex.

It can turn out that one extracts from the same domain the corresponding simplex by various manners. I argue that all these simplexes coincide.

It is clear that the problem stated does not differ from a formulation of the problem put forward in Number 21.

We shall show a new formulation of this problem.

On a function defined by the set of simplexes corresponding to the various vertices of a primitive parallelohedron.

24 Introduce within our study a function $P(x_1, x_2, \ldots, x_n)$ of variables $x_1, x_2, \ldots, x_n$ by defining as follows.

1. One will determine the function $P(x_1, x_2, \ldots, x_n)$ in the domain A by the formula

$$P_{(A)}(x_1, x_2, \ldots, x_n) = \sum_{i=1}^{n} p_i x_i.$$

2. In the domains A_k $(k = 1, 2, \dots, n)$ contiguous to the domain A by the faces in $n-1$ dimensions, one will determine the function $P(x_1, x_2, \dots, x_n)$ by the formula

$$P_{(A_k)}(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} P_{ik} x_i. \quad (k = 1, 2, \dots, n)$$

Let

$$A, A', A'', \dots, A^{(m)} \tag{1}$$

be a series of domains which are successively contiguous by the faces in $n-1$ dimensions. One will successively take from these domains the following simplexes.

$$L, L', L'', \dots, L^{(m)}$$

and one will determine the corresponding function.

$$\sum_{i=1}^{n} p_i x_i, \sum_{i=1}^{n} p_i' x_i, \sum_{i=1}^{n} p_i'' x_i, \dots, \sum_{i=1}^{n} p_i^{(m)} x_i.$$

One will define the function $P(x_1, x_2, \dots, x_n)$ in the domains (1) by the formula

$$P_{(A^{(k)})}(x_1, x_2, \dots, x_n) = \sum_{i=1}^{n} p_i^k x_i. \quad (k = 1, 2, \dots, m)$$

Fundamental Theorem. The function $P(x_1, x_2, \dots, x_n)$ defined by the conditions 1, 2, and 3 is continuous and uniform in all the space in n dimensions.

Observe that the fundamental introduced only give as a new formulation of the fundamental theorem of Number 20.

Take an arbitrary closed contour C. By traversing the contour C, one can determine a series of domains successively contiguous by faces in $n-1$ dimensions in which belong the points of the contour C:

$$A^{(0)}, A', A'', \dots, A^{(m)}, A^{(0)}, A', \dots$$

To demonstrate this, take a point (ξ_{i0}) of the contour C and designate by C_0 a curve which is being traversed within a domain $A^{(0)}$ leaving from the initial point (ξ_{i0}). Suppose that the curve C_0 does not coincide with the contour C and designate by (ξ_{i1}) the final point of the curve C_0.

Suppose that on leaving the point (ξ_i) one got out of the domain $A^{(0)}$ and that one entered inside the domain A'. Designate by C_1 a group of contour C which one has traversed in the domain A' when leaving the point (ξ_{i1}) and so on and so forth. Suppose that one had divided with the help of the procedure described the contour C into groups

$$C_0, C_1, \dots, C_m, C_0$$

which belong to the domains

$$A^{(0)}, A', A'', \dots, A^{(m)}, A^{(0)}. \tag{2}$$

It can turn out that two adjacent domains of this series $A^{(k)}$ and $A^{(k+1)}$ are not contiguous by a face in $n-1$ dimensions. One inserts in this case between the domains $A^{(k)}$ and $A^{(k+1)}$ new domains of the solutions as follows:

A point $(\xi_{i,k+1})$ which is the final point of the curve C_k and which gives the initial point of the curve C_{k+1} belongs, by virtue of the supposition made, to the domains $A^{(k)}$ and $A^{(k+1)}$. One concludes that the point $(\xi_{i,k+1})$ is interior to a face $A^{(k)}(\nu)$ in ν dimensions which is common to the domains $A^{(k)}$ and $A^{(k+1)}$.

One can determine a parallelepiped K with the help of equalities

$$x_i = \xi_{i,k+1} + u_i \text{ where } |u_i| \le \epsilon, \ (i = 1, 2, \dots, n)$$

of manner such that the points of the parallelepiped D do not belong to the domains of the series (1) (Number 22) which are contiguous by the face $A^{(k)}(\nu)$.

Take with the parallelepiped K two points (x_{ik}) and $x_{i,k+1}$ which ae interior to the domain $A^{(k)}$ and $A^{(k+1)}$ and take within the parallelepiped K a curve $C^{(k)}$ which joins the points (x_{ik}) and $(x_{i,k+1})$. One can choose this curve in such a manner that it does not pass beyond any face of domains (1) (Number 22) of which the number of dimension s is less than $n-1$.

Suppose that the curve $C^{(k)}$ traverse the domain

$$A^{(k)}, A_1^{(k)}, \dots, A_\mu^{(k)}, A^{(k+1)}.$$

By virtue of the supposition made, the domains obtained are successively contiguous by the faces in $n-1$ dimensions. All these domains are contiguous pairwisely by the face $A^{(k)}(\nu)$.

In the same manner, one will examine all the pairs of adjacent domains of the series (2) and one will form the series

$$A^{(0)}, A', \dots, A^{(m)}, A^{(0)}$$

of domains successively contiguous by the faces in $n-1$ dimensions to which belong all the points of the closed contour C given.

25 This established, observe that the fundamental theorem introduced is true in the case where all the domains (2) are contiguous in at least one edge.

In effect, suppose that one had successively taken away from the domains (2) the simplexes

$$L^{(0)}L', L'', \ldots, L^{(m)}, L^{(m+1)} \tag{3}$$

I argue that the simplex $L^{(m+1)}$ taken from the domain $A^{(0)}$ coincide with the simplex $L^{(0)}$. To demonstrate this, designate by

$$\sum p_i^{(m+1)} x_i = \delta \sum p_i^{(0)} x_i \,. \tag{4}$$

By virtue of the supposition made, the domains (2) are contiguous by at least one edge. Let (a_i) be a point of this edge.

As the domains $A^{(0)}$ and A' are contiguous by a face in $n-1$ dimensions, one will have, as we have seen this in Number 23, an equality

$$\sum p_i^{(0)} x_i = \sum p'_i x_i \tag{5}$$

which holds for any point (x_i) of thee face common to the domains $A^{(0)}$ and A'.

By making $x_i = a_i$, one obtains

$$\sum p^{(0)} a_i = \sum p'_i a_i$$

In the same manner, one will obtain

$$\sum p_i^{(0)} a_i = \sum p'_i a_i = \ldots \sum p_i^{(m)} a_i = \sum p_i^{(m+1)} a_i \,.$$

On the other hand, the identity (4) gives

$$\sum p_i^{(0)} a_i = \delta \sum p_i^{(m+1)} a_i \,,$$

and as $\sum p_i^{(0)} a_i > 0$, then $\delta = 1$, therefore

$$\sum p_i^{(m+1)} x_i = \sum p_i^{(0)} x_i$$

and the two simplexes $L^{(0)}$ and $L^{(m+1)}$ coincide.

By virtue of the definition established, one will determine the function $P(x_1, \ldots, x_n)$ in the domain $A^{(0)}$ by the formula

$$p_{(A^0)}(x_1, x_2, \ldots, x_n) = \sum p_i^{(}0)x_i$$

by leaving the domain $A^{(0)}$ and by returning to within that domain after having traversed the path C.

26 We will see that the general case can be brought back to the case examined. To this effect, suppose the projection of any one contour C evaluated in relation to surface S is determined by the equation

$$\sum x_i^2 = 1.$$

By putting

$$x'_i = \frac{x_i}{\sqrt{\sum x_i^2}}, \quad (i = 1, 2, \ldots, n) \tag{6}$$

one will call the point (x'_i) the projection of the point (x_i) within the surface S.

Designate by C' a projection of this contour C.

Suppose that by traversing the contour C', one returns to the initial point (ξ'_i) with the same solution of the function $P(x_1, \ldots, x_n)$ of which when leaving that point. I argue that one will return to the corresponding point ξ_i of the contour C with the same solution of the function $P(x_1, \ldots, x_n)$.

To demonstrate this, it suffices to observe that the points (ξ_i) and (ξ'_i), by virtue of equalities (6), belong to the same domains of the series (2).

One concludes that it suffices to examine the different closed contours belonging to the surface S.

27 Introduce in our study a function $d(x_i, x'_i)$ being defined by the formula

$$d(x_i, x'_i) = \sqrt{\sum (x'_i - x_i)^2}.$$

One will call distance between two points (x_i) and (x'_i) the corresponding value of the function $d(x_i, x'_i)$.

Lemma. One can determine a positive parameter δ satisfying the following condition: every closed contour C belonging to the surface S will be situated in the domains which are contiguous by at least one edge, if the distance of all the point of the contour C, each of all to the rest, do not exceed the limit δ.

Let (ξ_i) be a point of the contour C belonging to the domain A. Put

$$\xi_i = \sum_{k=1}^{n} \rho_k a_{ik} \text{ where } \rho_k \geq 0. \quad (k = 1, 2, \ldots, n)$$

By virtue of the equation

$$\sum \xi_i^2 = 1,$$

the sum $\sum_{k=1}^{n} \rho_k$ is not less than a positive fixed limit.

$$\sum_{k=1}^{n} \rho_k \geq \tau. \tag{7}$$

Suppose that the contour C is not situated entirely within the domain A.
Let (ξ_i') be a point of C which does not belong to the domain A. By putting

$$\xi_i' = \sum_{k=1}^{n} \rho_k' a_{ik}, \tag{8}$$

one will have among the numbers $\rho_1', \rho_2', \ldots, \rho_n'$ at least one negtive number.
Suppose, to fix an ideas, that

$$\rho_1' \geq 0, \rho_2' \geq 0, \ldots, \rho_\mu' \geq 0 \tag{9}$$

and that

$$\rho_{\mu+1}' < 0, \rho_{\mu+2}' < 0, \ldots, \rho_n' < 0. \tag{10}$$

After the supposition made, one has the inequality

$$d(\xi_i, \xi_i') \geq \delta.$$

One can choose the parameter δ, of such a manner that one had the inequalities

$$|\rho_k' - \rho_k| < \epsilon, \ (k = 1, 2, \ldots, n) \tag{11}$$

ϵ being a positive parameter also small as one would wish.
By (10), one obtains

$$0 \leq \rho_k < \epsilon, -\epsilon < \rho_k' < 0. \ (k = \mu + 1; \mu + 2, \ldots, n) \tag{12}$$

Choose among the numbers $\rho_1, \rho_2, \ldots, \rho_n$ the one which is the largest. By virtue of the inequality (7), this number can not be less than $\frac{\tau}{n}$. By supposing that

$$\epsilon < \frac{\tau}{n},$$

one will find the number looked for among the numbers $\rho_1, \rho_2, \ldots, \rho_\mu$. Suppose, to fix the ideas, that

$$\rho_1 > \frac{\tau}{n}.$$

the inequality (11) gives

$$\rho_1' > \frac{\tau}{n} - \epsilon. \tag{13}$$

This posed, suppose that the point (ξ_i') belonged to the domain A' and put

$$\xi_i' = \sum_{k=1}^{n} u_k a_{ik}' \text{ where } u_k \geq 0. \ (k = 1, 2, \ldots, n) \tag{14}$$

Designate by (α_i) and α_i' two vertices of the parallelohedron R corresponding to domains A and A' by defining them by the equations

$$a_{0k} + \sum a_{ik} x_i = 0, \quad (k = 1, 2, \ldots, n)$$

and by the equations

$$a_{0k}' + \sum a_{ik}' x_i = 0. \ (k = 1, 2, \ldots, n) \tag{15}$$

By virtue of equality (8) and (14), one obtains an identity

$$\rho_0' + \sum_{k=1}^{n} \rho_k' (a_{0k} + \sum a_{ik} x_i) = \sum_{k=1}^{n} u_k (a_{0k}' + \sum a_{ik}' x_i). \tag{16}$$

By making in this identity $x_i' = \alpha_i$, one finds

$$\rho_0' = \sum u_k (a_{0k}' + \sum a_{0k}' \alpha_i) \geq 0. \tag{17}$$

By making in the identity (16) $x_i = \alpha_i'$, it will become

$$\rho_0' + \sum_{k=1}^{n} \rho_k' (a_{0k} + \sum a_{ik} \alpha_i') = 0. \tag{18}$$

Suppose that

$$a_{01} + \sum a_{i1} \alpha_i' > 0.$$

By virtue of (7), (12), (13) and (17), one will have

$$\rho_0' + \sum_{k=1}^{\mu} \rho_k'(a_{0k} + \sum a_{ik}\alpha_i') > (\frac{\tau}{n} - \epsilon)(a_{01} + \sum a_{i1}\alpha_i'),$$

$$\sum_{k=\mu+1}^{n} \rho_k'(a_{0k} + \sum a_{ik}\alpha_i') \geq -\epsilon \sum_{k=\mu+1}^{n} (a_{0k} + \sum a_{ik}\alpha_i'),$$

and the equality (18) gives

$$\frac{\tau}{n}(a_{01} + \sum a_{i1}\alpha_i') < \epsilon \left[a_{01} + \sum a_{i1}\alpha_i' + \sum_{k=\mu+1}^{n} (a_{0k} + \sum a_{ik}\alpha_i') \right]. \tag{19}$$

Designate

$$A = \frac{\tau}{n}(a_{01} + \sum a_{i1}\alpha_i') \text{ and } B = a_{01} + \sum a_{i1}\alpha_i' + \sum_{k=\mu+1}^{n} (a_{0k} + \sum a_{ik}\alpha_i'),$$

one will have

$$A > 0 \text{ and } B > 0,$$

and as a result

$$\epsilon > \frac{A}{B}. \tag{20}$$

One could determine the ratio $\frac{A}{B}$ correspondent to the different vertices of the parallelohedron R. Designate by ω the smallest of these ratios which is not zero. The parameter ϵ being arbitrary, one can suppose that

$$\epsilon \leq \omega.$$

The inequality (20) becomes impossible, it is therefore necessary that $A = 0$ or [to put it] differently

$$a_{01} + \sum a_{i1}\alpha_i' = 0.$$

By virtue of the equality obtained, the coefficients of the equation

$$a_{01} + \sum a_{i1}x_i = 0$$

are proportional to those of an equation which is among the equations (15).

By putting

$$a_{01} + \sum a_{i1}x_i = u(a_{0h}' + \sum a_{ih}'x_i) \text{ where } u > 0,$$

one will have

$$a_{i1} = ua_{ih}'. \quad (i = 1, 2, \ldots, n)$$

We have arrived at the following result: all the domain traversed by the contour C examined are contiguous by at least one edge which is characterised by the point (a_{i1}).

28 We are now in the state of reaching the demonstration of the fundamental theorem announced.

Let C be any contour belonging to the surface S. Suppose that on leaving the point (ξ_i) one passes via the points $(\xi_i^{(0)})$, (ξ_i'), (ξ_i'') and one returns to the point (ξ_i).

The path around the contour C can be replaced by the paths $C^{(0)}$ and C'.

The contour $C^{(0)}$ will be composed of a group $(\xi_i) - (\xi_i^{(0)})$ of C, of the vector $[\xi_i^{(0)}, \xi_i'']$ and of a group $(\xi_i^{(0)}) - (\xi_i') - (\xi_i'')$ of the contour C and of the vector $[\xi_i'', \xi_i^{(0)}]$.

Suppose that by traversing the paths $C^{(0)}$ and C' one uniformly defined the function $P(x_1, x_2, \ldots, x_n)$. In this case the trajectory by the group $(\xi_i^{(0)}) - (\xi_i')$ of the contour C can be replace by the path the length of the vector $[\xi_i^{(0)}, \xi_i'']$.

By replacing the group $(\xi_i^{(0)}) - (\xi_i') - (\xi_i'')$ of the contour C by the vector $[\xi_i^{(0)}, \xi_i'']$, one will transform the contour C to $C^{(0)}$, thus, by traversing the contour C, one will return to the point (ξ_i), by virtue of suppositions made, with the same solution of the function $P(x_1, x_2, \ldots, x_n)$.

Two contours $C^{(0)}$ and C' can be examined in the same manner and so on.

Suppose that one had determined the contours

$$C_1, C_2, \ldots, C_m \tag{21}$$

which replace the path C. By supposing that the function $P(x_1, x_2, \ldots, x_n)$ be uniform the length of contour (21), one will demonstrate that it will be uniform the length of the contour C given.

This established, observe that we can always choose the contours (21), of such a manner that their contours satisfy the conditions of the lemma of the previous Number. In this case, any contour (21) will be situated within domains which are contiguous by at least one edge. We have seen in Number 25 that by traversing the same contours one will always return to the point of departure by the same solution of the function $P(x_1, x_2, \ldots, x_n)$ as while leaving this point. It is thus demonstrated that any closed contour C possesses the same property.

We have demonstrated that the function $P(x_1, x_2, \ldots, x_n)$ is uniformly defined in any domain of the set (1) (Number 22). It remains to demonstrate that the function $P(x_1, x_2, \ldots, x_n)$ is well defined in any point of the space in n dimensions.

Suppose that a point ξ_i belongs to two domains A and $A^{(0)}$.

I argue that the function $P(x_1, x_2, \ldots, x_n)$ for the point ξ_i will have one same value in the domain A and in the domain $A^{(0)}$.

To demonstrate this, one will form a series of domains

$$A, A', \ldots, A^{(m)}, A^{(0)}$$

which are successively contiguous by faces in $n-1$ dimensions and in which belongs the point ξ_i.

As the point ξ_i belongs to the face common to domains A and A', one will have ,by virtue of the formula (5) of Number 23,

$$P_{(A)}(\xi_1, \xi_2, \ldots, \xi_n) = P_{(A')}(\xi_1, \xi_2, \ldots, \xi_n).$$

In the same manner, one obtains

$$P_{(A')}(\xi_1, \xi_2, \ldots, \xi_n) = P_{(A'')}(\xi_1, \xi_2, \ldots, \xi_n),$$
$$\ldots$$
$$P_{(A^{(m)})}(\xi_1, \xi_2, \ldots, \xi_n) = P_{(A^{(0)})}(\xi_1, \xi_2, \ldots, \xi_n).$$

It results in that

$$P_{(A)}(\xi_1, \xi_2, \ldots, \xi_n) = P_{(A^{(0)})}(\xi_1, \xi_2, \ldots, \xi_n).$$

The fundamental theorem announced is thus demonstrated.

Canonical form of inequalities which define the set (R) of primitive parallelohedron S.

29

Choose within the set (R) of primitive parallelohedra any parallelohedron R_0 . Suppose that the parallelohedron R_0 is determined with the help of canonical inequalities

$$a_{0k} + \sum a_{ik}x_i \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

Observe that we can replace these inequalities by the following canonical inequalities:

$$u(a_{0k} + \sum a_{ik}x_i) \geq 0, \quad (k = 1, 2, \ldots, \sigma)$$

u being a positive arbitrary parameter.

Designate by R_k $(k = 1, 2, \ldots, \sigma)$ the parallelohedron which is contiguous to the parallelohedron R_0 by the face determined within R_0 by the equation

$$a_{0k} + \sum a_{ik}x_i = 0 \tag{1}$$

and suppose that the vector $[\lambda_{ik}]$ defined a translation of the parallelohedron R_k to R_0.

It follows that the parallelohedron $R_k$will be determined by the canonical inequalities

$$a_{0h} + \sum a_{ih}(x_i + \lambda_{ik}) \geq 0, \quad (h = 1, 2, \ldots, \sigma)$$

or by the canonical inequalities

$$u_k[a_{0h} + \sum a_{ih}(x_i + \lambda_{ik})] \geq 0, \;\; (h = 1, 2, \ldots, \sigma) \tag{2}$$

u_k being an arbitrary positive parameter.

The face P_k in $n-1$ dimensions common to the parallelohedra R and R_k is defined in the parallelohedron R_0 by the equation (1). Within the parallelohedron R_k, the face P_k will be determined by an equation in which the coefficients are proportional to those of the equation

$$-a_{0k} - \sum a_{ik}x_i = 0.$$

One can choose the positive parameter u_k, of a manner such that one had the identity

$$-a_{0k} - \sum a_{ik}x_i = u_k(a_{0h} + \sum a_{ih}(x_i + \lambda_{ik})).$$

In this case, the inequality

$$-a_{0k} - \sum a_{ik}x_i \geq 0$$

is found among the inequalities (2) which define the parallelohedron R_k.

One will say that these inequalities are represented in the canonical form.

30

Observe an important property of canonical inequalities which define the parallelohedra $R_0, R_1, R_2, \ldots, R_\sigma$.

Let (α_i) be a vertex of the parallelohedron R_0 determined by the canonical equations

$$a_{0k} + \sum a_{ik}x_i = 0. \; (k = 1, 2, \ldots, n) \tag{3}$$

Examine the canonical equations which define the vertex (α_i) in the parallelohedron R_k $(k = 1, 2, \ldots, n)$.

The equations (3) being canonical, one will determine the vertex (α_i) within the parallelohedron R_k, by virtue of the theorem of Number 19, by the equation

$$\sum(a_{ih} - a_{ik})(x_i - \alpha_i) = 0, \quad (h = 1, 2, \dots, n, h \neq k)$$
$$-\sum a_{ik}(x_i - \alpha_i) = 0 \quad (k = 1, 2, \dots, n)$$

By virtue of the supposition made, the inequality

$$-\sum a_{ik}(x_i - \alpha_i) \geq 0$$

exists among the canonical inequalities (2) which define the parallelohedron R_k, which results in that the inequalities

$$\sum(a_{ih} - a_{ik})(x_i - \alpha_i) \geq 0, \quad (h = 1, 2, \dots, n, h \neq k)$$

also exist among the canonical inequalities (2).

One concludes that the canonical equation

$$\sum(a_{ih} - a_{ik})(x_i - \alpha_i) = 0$$

define in the parallelohedron R_k a face in $n-1$ dimensions which is common to the parallelohedra R_k and R_h. the same face will be determined in the parallelohedron R_h by a canonical equation

$$\sum(a_{ik} - a_{ih})(x_i - \alpha_i) = 0.$$

31

by applying the procedure explained, one can determine the canonical inequalities which define the parallelohedra contiguous to the parallelohedra $R_1, R_2, \dots, R_\sigma$ and so on.

For every parallelohedron R of the set (R), one can form a series of parallelohedra

$$R_0, R', R'', \dots, R^{(m)}, R$$

which are successively contiguous. One will determine successively the canonical inequalities that define the parallelohedra of this series.

One could arrive at the parallelohedron R by other ways and determine the canonical inequalities which define the parallelohedron R in various manners.

We shall se that the canonical inequalities which define a parallelohedron of the set (R) do not depend on the path by which one arrives at the parallelohedron (R) leaving from the principal parallelohedron R_0.

Generatrix function of the set (R) of primitive parallelohedra.

32

Consider a set (R) of primitive parallelohedra. Suppose any parallelohedron R of the set (R) be characterised by a vector $[\lambda_i]$ which defines a translation of parallelohedra R to a principal parallelohedron R_0.

Designate by G the group of vectors $[\lambda_i]$ which correspond to the different parallelohedra of the set (R).

Introduce in our study a function

$$V(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_n)$$

of variables $x_1, x_2, \dots, x_n$ and parameters $\lambda_1, \lambda_2, \dots, \lambda_n$ by defining it within the space in n dimensions and for the group G such that:

1. Within the principal parallelohedron R_0, one will write

$$V(x_1, x_2, \dots, x_n, 0, 0, \dots, 0,) = 0.$$

2. Within the parallelohedron R_k which is contiguous to R_0, one will write

$$V(x_1, x_2, \dots, x_n, \lambda_{1k}, \lambda_{2k}, \dots, \lambda_{nk} = a_{0k} + \sum a_{ik}x_i, \ (k = 1, 2, \dots, \sigma)$$

providing that in the parallelohedron R_0 the canonical equation

$$a_{0k} + \sum a_{ik}x_i = 0$$

had the face in $n-1$ dimensions common to the parallelohedra R_0 and R_k.

3. By supposing that the parallelohedra R and R' characterised by the vectors $[\lambda_i]$ and λ'_i are contiguous by a face in $n-1$ dimensions which is defined within R by a canonical equation

$$a_0 + \sum a_i x_i = 0,$$

one will write

$$V(x_1, x_2, \dots, x_n, \lambda_1,', \lambda'_2, \dots, \lambda'_n) =$$
$$V(x_1, x_2, \dots, x_n, \lambda_1, \lambda_2, \dots, \lambda_n) + a_0 + \sum a_i x_i.$$

33

Let R be any parallelohedron of the set (R) characterised by a vector $[\lambda_i]$. One will form a series of parallelohedra.

$$R_0, R', \dots, R^{(m)}, R$$

which are successively contiguous by faces in $n-1$ dimensions. Designate by

$$a_0^{(0)} + \sum a_i^{(0)} x_i = 0$$

the equation of the faces common to the parallelohedra R_0 and R' and defined in R_0; designate by

$$a_0' + \sum a_i' x_i = 0$$

the equaion of the face common to the parallelohedra R' and R'' defined in R' and so on.

By applying the definition established, one will determine the function

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$$

by the formula

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{k=0}^{n} (a_0^{(k)} + \sum a_i^{(k)} x_i).$$

34 *Fundamental theorem. The function $V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$ is well defined for any vector $[\lambda_i]$ of the group G.*

Suppose that one had formed a series of parallelohedra

$$R, R', R'', \ldots, R^{(m)}, R \qquad (1)$$

which are successively contiguous. On leaving the parallelohedron R with any solution of the function $V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$, one will return inside the parallelohedron R after having traversed the parallelohedra (1) with a solution of the function $V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$ which, by virtue of the definition established, is expressed by the vertex

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) + \sum_{k=0}^{n} (a_0^{(k)} + \sum a_i^{(k)} x_i).$$

We shall demonstrate that one will always have

$$\sum_{k=0}^{n} (a_0^{(k)} + \sum a_i^{(k)} x_i) = 0.$$

Examine, in the first place, the case where all the parallelohedra (1) are contiguous by at least one vertex (α_i). By virtue of Theorem II of Number 17, all the primitive parallelohedra (1) will be in this case contiguous one to one through faces in $n-1$ dimensions.

Designate by

$$a_{0k} + \sum a_{ik}(x_i - \alpha_i) = 0, \quad (k = 1, 2, \ldots, m)$$

the canonical equation of the face common to the parallelohedra $R^{(k)}$ and R $(k = 1, 2, \ldots, m)$ defined within the parallelohedron R.

We have seen in Number 30 that the canonical equation of the face common to the parallelohedra R' and R'' and defined within R' will be

$$\sum (a_{i2} - a_{i1})(x_i - \alpha_i) = 0$$

and so on and so forth. One obtains the formulae

$$\begin{aligned} a_0^{(0)} + \sum a_i^{(0)} x_i &= \sum a_{i1}(x_i - \alpha_i), \\ a_0' + \sum a_i' x_i &= \sum (a_{i2} - a_{i1})(x_i - \alpha_i), \\ &\ldots \\ a_0^{(m-1)} + \sum a_i^{(m-1)} x_i &= \sum (a_{im} - a_{i,m-1})(x_i - \alpha_i), \\ a_0^{(m)} + \sum a_i^{(m)} x_i &= -\sum a_{im}(x_i - \alpha_i), \end{aligned}$$

and it follows that

$$\sum_{k=0}^{m} (a_0^{(k)} + \sum a_i^{(k)} x_i) = 0.$$

35 We shall see that the general case can be brought back to the case examined.

Theorem. One can determine a positive parameter δ, in a manner that every closed contour C is found within the parallelohedra which are contiguous by at least one vertex, providing that the distance between any two points of the contour C does not exceed the limit δ.

Observe, in the first place, that the distance between the points (ξ_i) and (ξ_i') belonging to the two parallelohedra which are not contiguous can not be less than a fixed limit. To demonstrate this, suppose that the point (ξ_i) belong to the parallelohedron R defined with the help of inequalities

$$a_{0k} + \sum a_{ik}x_i \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

Designate by $R_1, R_2, \ldots, R_\sigma$ the parallelohedra which are contiguous to R and examine the set K of points belonging to the parallelohedra $R_1, R_2, \ldots, R_\sigma$.

Designate by

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{is})$$

the vertices of parallelohedron R and designate by

$$(\alpha_{i1}^{(h)}), (\alpha_{i2}^{(h)}), \ldots, (\alpha_{is}^{(h)}), \quad (h = 1, 2, \ldots, \sigma)$$

the vertices of parallelohedron R_h $(h = 1, 2, \ldots, \sigma)$.

By virtue of the supposition made, one will have the inequalities

$$a_{0h} + \sum a_{ih}\alpha_{ik}^{(h)} \leq 0, \ (k = 1, 2, \ldots, s; h = 1, 2, \ldots, \sigma).$$

Designate by ρ the smallest numerical value of vertices

$$a_{0h} + \sum a_{ih}\alpha_{ik}^{(h)} \quad (k = 1, 2, \ldots, s; h = 1, 2, \ldots, \sigma)$$

which does not become zero. By virtue of supposition made, one will have the inequality

$$\rho + a_{0h} + \sum a_{ih}\alpha_{ik}^{(h)} \leq 0,$$

on condition that

$$a_{0h} + \sum a_{ih}\alpha_{ik}^{(h)} < 0,$$

where $(k = 1, 2, \ldots, s, \ h = 1, 2, \ldots, \sigma)$.

This established, take any point (ξ_i') which does not belong to the set K. Examine the points of a vector $[\xi_i, \xi_i']$. By putting

$$x_i = \xi_i + u(\xi_i' - \xi_i) \text{ where } 0 \leq u \leq 1,$$

let us think the parameter u of a continuous manner within the interval $0 < u < 1$. One will determine a point

$$\xi_i^{(0)} = \xi_i + u_0(\xi_i' - \xi_i) \text{ where } 0 < u_0 < 1 \tag{2}$$

which belongs to the boundary of the set K, that is to say to a face in $n-1$ dimensions of parallelohedra $R_1, R_2, \ldots, R_\sigma$ and which also belongs to another parallelohedron R'.

Suppose that the point $(\xi_i^{(0)})$ belongs to the parallelohedron R_h. The parallelohedra R_h and R' will be contiguous by a face in $n-1$ dimensions.

Designate by

$$(\alpha_{i1}^{(h)}), (\alpha_{i2}^{(h)}), \ldots, (\alpha_{it}^{(h)}) \tag{3}$$

the vertices of parallelohedron R_h which belong to this face.

None of these vertices verifies the equation

$$a_{0h} + \sum a_{ih}x_i = 0$$

because otherwise the face examined would belong to two parallelohedron of the series $R, R_1, \ldots, R_\sigma$, which is contrary to the hypothesis.

Therefore one will have the inequalities

$$\rho + a_{0h} + \sum a_{ih}\alpha_{ik}^{(h)} \leq 0. \quad (k = 1, 2, \ldots, t)$$

The point $(\xi_i^{(0)})$ belonging to the face of R_h, which is characterised by the vertices (3), probably determined by the equations

$$\xi_i^{(0)} = \sum_{k=1}^{k=t} \vartheta_k\alpha_{ik}^{(h)} \text{ where } \sum_{k=1}^{k=t} \vartheta_k = 1 \text{ and } \vartheta_k \geq 0. \, (k = 1, 2, \ldots, t)$$

Of the previous inequalities, one draws

$$\rho + a_{0h} + \sum a_{ih}\xi_i^{(0)} \leq 0.$$

By observing that on the other hand one has

$$a_{0h} + \sum a_{ih}\xi_i \geq 0, \tag{4}$$

one finds, by (2),

$$\rho + a_{0h} + \sum a_{ih}\xi_i' < 0. \tag{5}$$

By virtue of inequalities (4) and (5), the distance $d(\xi_i, \xi_i')$ can not be smaller than a fixed limit d.

Solution of the centre of the primitive parallelohedra

36 This established, examine a contour C formed which the points had the mutual distance that does not surpass δ . By supposing that

$$\delta < d,$$

one will have a contour C which is situated within the contiguous parallelohedra two to two. [one to one]

Let ξ_i be any point of the contour C belonging to the parallelohedron R. Suppose that not all the points of contour C belong to R and designate by ξ_i' a point of contour C which does not belong to R. Put

$$\xi_i = \sum_{k=1}^{s} \vartheta_k \alpha_{ik} \text{ where } \sum \vartheta_k = 1 \text{ and } \vartheta_k \geq 0,\ (k = 1, 2, \ldots, s) \tag{6}$$

$$\xi_i' = \sum_{k=1}^{s} \vartheta_k' \alpha_{ik} \text{ where } \sum \vartheta_k' = 1 \tag{7}$$

As the point (ξ_i') does not belong to R, one will have among the numbers $\vartheta_1', \vartheta_2', \ldots, \vartheta_s'$ at least one number which will be negative. Suppose, to fix the ideas, that

$$\vartheta_1' \geq 0, \ldots, \vartheta_\mu' \geq 0 \text{ and } \vartheta_{\mu+1}' < 0, \ldots, \vartheta_s' < 0 \tag{8}$$

One can choose the parameter δ as small that one would have the inequalities

$$|\vartheta_k' - \vartheta_k| < \epsilon, \quad (k = 1, 2, \ldots, s) \tag{9}$$

ϵ being a positive parameter also as small as one would like. By virtue of (8), it will become

$$0 \leq \vartheta_k < \epsilon,\ -\epsilon < \vartheta_k' < 0. \quad (k = \mu + 1, \ldots, s) \tag{10}$$

Observe how the largest one among the numbers $\vartheta_1, \vartheta_2, \ldots, \vartheta_s$ can not be smaller than $\frac{1}{s}$ by (6) by supposing that

$$\epsilon < \frac{1}{s},$$

one wil find the required number among the number $\vartheta_1, \vartheta_2, \ldots, \vartheta_\mu$. Suppose, for fixing ideas, that

$$\vartheta_1 > \frac{1}{s}.$$

By virtue of (9), it will become

$$\vartheta_1' > \frac{1}{s} - \epsilon \tag{11}$$

We have demonstrated that the point (ξ_i') can not belong to these parallelohedra $R_1, R_2, \ldots, R_\sigma$ which are contiguous to R. By supposing that the point (ξ_i') belong to the parallelohedron R_h, one will have an equality

$$a_{0h} + \sum a_{ih} \xi_i' < 0 \tag{12}$$

Observing that by virtue of equations (7)

$$a_{0h} + \sum a_{ih} \xi_i' = \sum_{k=1}^{s} \vartheta_k' (a_{0h} + \sum a_{ih} \alpha_{ik}),$$

one obtains, because of (12),

$$\sum_{k=1}^{s} \vartheta_k' (a_{0h} + \sum a_{ih} \alpha_{ik}) < 0$$

Of this inequality one draws, by (10) and (11),

$$\frac{1}{2}(a_{0h} \sum a_{ih} \alpha_{i1}) - \epsilon \left[a_{0h} \sum a_{ih} \alpha_{i1} + \sum_{k=\mu+1}^{s} (a_{0h} + \sum a_{ih} \alpha_{ik}) \right] < 0\,.$$

By putting

$$A = \frac{1}{2}(a_{0h} \sum a_{ih} \alpha_{i1}) \text{ and } B = a_{0h} + \sum a_{ih} \alpha_{i1} + \sum_{k=\mu+1}^{s} (a_{0h} + \sum a_{ih} \alpha_{ik}),$$

suppose that $A > 0$; the previous inequality gives $B > 0$, thus

$$\epsilon > \frac{A}{B} \tag{13}$$

Observe that the numbers A and B do not change when one replace the parallelohedron by any parallelohedron of the set (R). One concludes that the ratio $\frac{A}{B}$ which does not vanish possess a positive minimum ω.

By supposing that

$$\epsilon < \omega,$$

the inequality (13) becomes impossible and it is necessary that $A = 0$ or otherwise

$$a_{0h} + \sum a_{ih}\alpha_{i1} = 0$$

We have arrived at the following result: all the parallelohedra within which is situated the contour examined C are contiguous by the vertex (α_{i1}).

With the help of the lemma of Number 35, one will easily demonstrate the fundamental theorem stated by repeating the reasoning explained in Number 28.

Fundamental properties of the generatrix function $V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$

37 *Theorem I. Suppose that two vectors* $[\lambda_i]$ *and* $[\lambda_i^{(0)}]$ *characterise two parallelohedra* R *and* $R^{(0)}$ *of the set* (R)*. One will have an inequality*

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) > V\left(x_1, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}\right),$$

on condition that the point (x_i) *be interior to the parallelohedron* $R^{(0)}$.

Let$(\xi_i^{(0)})$ be any point which is interior to the parallelohedron $R^{(0)}$.

Take a point (ξ_i) which is interior to the parallelohedron R and examine a vector $[\xi_i^{(0)}, \xi_i]$ determined by the equations

$$x_i = \xi_i^{(0)} + u(\xi_i - \xi_i^{(0)}) \quad \text{where } 0 \leq u \leq 1\,.$$

A group of the vector $[\xi_i^{(0)}, \xi_i]$ belong to the parallelohedron R^0. Designate,

$$\xi_i' = \xi_i^{(0)} + u_1(\xi_i - \xi_i^{(0)}) \quad \text{where } 0 < u < 1$$

and suppose that the vector $[\xi_i^{(0)}, \xi_i']$ represents the group of the vector $[\xi_i^{(0)}, \xi_i]$ which belong to $R^{(0)}$.

The second group $[\xi_i', \xi_i]$ of the vector $[\xi_i^{(0)}, \xi_i]$ does not possess any point (ξ_i') common to the parallelohedron $R^{(0)}$. The point (ξ_i') belongs to a face $p^{(0)}(\nu)$ of the parallelohedron $R^{(0)}$. One will choose among the parallelohedra which are contiguous by the face $p^{(0)}(\nu)$ a parallelohedron R' which contains a group of the vector $[\xi_i', \xi_i]$.

Designate

$$\xi_i'' = \xi_i + u_2(\xi_i - \xi_i^{(0)}) \quad \text{where } u_1 < u_2 \leq 1$$

and suppose that the vector $[\xi_i', \xi_i'']$ represents a group of the vector $[\xi_i', \xi_i]$ which belongs to the parallelohedron R' and so on.

Let us suppose that one has determined m points of the vector $[\xi_i^{(0)}, \xi_i]$

$$\xi_i^{(k)} = \xi_i^{(0)} + u_k(\xi_i - \xi_i^{(0)})\,, \; (k = 1, 2, \ldots, m) \tag{1}$$

where

$$0 < u_1 < u_2 < \cdots < u_m < 1 \tag{2}$$

which correspond to the vectors $[\xi_i^{(0)}, \xi_i'], [\xi_i', \xi_i''], \ldots, [\xi_i^{(m)}, \xi_i]$ belonging to the parallelohedra

$$R^{(0)}, R', \ldots, R^{(m-1)}, R$$

successively contiguous.

Designate by

$$a_0^{(k)} + \sum a_i^{(k)} x_i = 0, \quad (k = 0, 1, 2, \ldots, m-1)$$

the canonical equation of the face common to the parallelohedra $R^{(k)}$ and $R^{(k+1)}$ which is defined within the parallelohedron $R^{(k)}$.

By virtue of the established definition in Number 32, one will have a formula

$$\begin{aligned} V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = V\left(x_1, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}\right) \\ + \sum_{k=0}^{m-1}\left(a_0^{(k)} + \sum a_i^{(k)} x_i\right) \end{aligned} \tag{3}$$

Examine the sum

$$a_0^{(k)} + \sum a_i^{(k)} \xi_i^{(0)} \quad \text{and} \quad a_0^{(k)} + \sum a_i^{(k)} \xi_i. \quad (k = 0, 1, 2, \ldots, m-1)$$

By virtue of the supposition made, the point $(\xi_i^{(k+1)})$ verifies the equation

$$a_0^{(k)} + \sum a_i^{(k)} \xi_i^{(k+1)} = 0$$

As the point $(\xi^{(k)})$ belongs to the parallelohedron $R^{(k)}$, one will have an inequality

$$a_0^{(k)} + \sum a_i^{(k)} \xi_i^{(k)} \geq 0\,.$$

By virtue of (1) and (2), one obtains

$$a_0^{(k)} + \sum a_i^{(k)}\xi_i^{(0)} \geq 0 \text{ and } a_0^{(k)} + \sum a_i^{(k)}\xi_i \leq 0\,.\ (k = 0, 1, 2, \ldots, m-1)$$

As the point $(\xi_i^{(0)})$ is interior to the parallelohedron $R^{(0)}$, we will have

$$a_0^{(0)} + \sum a_i^{(0)}\xi_i^{(0)} > 0 \text{ and } a_0^{(0)} + \sum a_i^{(0)}\xi_i < 0\,,$$

It results in that

$$\sum_{k=0}^{m-1}(a_0^{(0)} + \sum a_i^{(0)}\xi_i^{(0)}) > 0 \text{ and } \sum_{k=0}^{m-1}(a_0^{(k)} + \sum a_i^{(k)}\xi_i) < 0\,.$$

By substituting in the formula (3), one obtains

$$V(\xi_1^{(0)}, \xi_2^{(0)}, \ldots, \xi_n^{(0)}, \lambda_1, \lambda_2, \ldots, \lambda_n) > V(\xi_1^{(0)}, \xi_2^{(0)}, \ldots, \xi_n^{(0)}, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)})$$

and

$$V(\xi_1, \xi_2, \ldots, \xi_n, \lambda_1, \lambda_2, \ldots, \lambda_n) < V(\xi_1, \xi_2, \ldots, \xi_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)})\,.$$

Theorem II. Suppose that the parallelohedra $R^{(0)}, R', \ldots, R^{(n-\nu)}$ *be contiguous by a face* $p(\nu)$ *in* ν *dimensions. By designating by* $[\lambda_i^{(k)}]$, $(k = 0, 1, 2, \ldots, n-\nu)$ *the vectors which characterise these parallelohedra, one will have an inequality*

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) > V(x_1, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}),$$

on condition that the point (x_i) *be interior to a face* $p(\nu)$ *and that the vector* $[\lambda_i]$ *is not among the vectors* $[\lambda_i^{(k)}]$, $(k = 1, 2, \ldots, (n-\nu))$.

By supposing that $\lambda_i = \lambda_i^{(k)}$, one will have the equation

$$V(x_1, x_2, \ldots, x_n, \lambda_1^{(k)}, \lambda_2^{(k)}, \ldots, \lambda_n^{(k)}) = V(x_1, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}).$$
$$(k = 1, 2, \ldots, (n-\nu))$$

One will easily demonstrate the announced Theorem II by repeating the reasonings which have been established previously.

38 The results obtained open a new way for the researches concerning the primitive parallelohedra. One can consider the set (R) of primitive parallelohedra under a new point of view, in knowing:

Each parallelohedron $R^{(0)}$ *of the set* (R) *characterised by the vector* $[\lambda_i^{(0)}]$ *presents a set of points* (x_i) *verifying the inequality* $V(x_i, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) \geq V(x_i, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)})$, *for any vector* $[\lambda_i]$ *belonging to group* G.

We have seen in Number 32 that for the principal parallelohedron R_0 of the set (R) one has

$$V(x_i, x_2, \ldots, x_n, 0, 0, \ldots, 0) = 0.$$

It follows that the principal parallelohedron R_0 is defined by the inequality

$$V(x_i, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) \geq 0$$

which holds for any vector $[\lambda_i]$ of group G.

Solution of the quadratic function $V(x_i, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$

39 Suppose that the principal parallelohedron R_0 is determined with the help of canonical inequalities

$$a_{0k} + \sum a_{ik}x_i \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

Designate by $[\lambda_{ik}]$ the vector which defines a translation of the parallelohedron R_k to R_0 $(k = 1, 2, \ldots, \sigma)$.

Take two parallelohedra R_k and R_0 contiguous to the parallelohedron R_0 through the faces P_k and P_h which are not parallel. Put

$$\lambda = \lambda_k + \lambda_{ih}$$

and designate by R the parallelohedron of the set (R) characterised by the vector $[\lambda_i]$.

The parallelohedron R is contiguous to the parallelohedra R_k and R_h through the faces which are congruent to the faces P_h and P_k.

One can thus form the series

$$R_0, R_k, R \text{ and } R_0, R_h, R$$

of parallelohedra which are successively contiguous.

Let us suppose that the parallelohedron R_k is determined with the help of canonical equations

$$u_k[a_{0r} + \sum a_{ir}(x_i + \lambda_{ik})] \geq 0. \quad (r = 1, 2, \ldots, \sigma)$$

The face of the parallelohedron R_k which is congruent to the face P_h will be determined by the equation

$$u_k[a_{0h} + \sum a_{ih}(x_i + \lambda_{ik})] = 0.$$

It results in that the function $V(x_i, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$ is expressed by the sum

$$V(x_i, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = a_{0k} + \sum a_{ik} x_i + u_k \left[a_{0h} + \sum a_{ih}(x_i + \lambda_{ik}) \right].$$

In the same manner, one obtains

$$V(x_i, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = a_{0h} + \sum a_{ih} x_i + u_h \left[a_{0k} + \sum a_{ik}(x_i + \lambda_{ih}) \right].$$

By virtue of the fundamental theorem of Number 34, one will have an identity

$$a_{0k} + \sum a_{ik} x_i + u_k \left[a_{0h} + \sum a_{ih}(x_i + \lambda_{ik}) \right] = a_{0h} + \sum a_{ih} x_i + u_h \left[a_{0k} + \sum a_{ik}(x_i + \lambda_{ih}) \right].$$

It follows that

$$a_{0k} + u_k(a_{0h} + \sum a_{ih}\lambda_{ik}) = a_{0h} + u_h(a_{0k} + \sum a_{ik}\lambda_{ih}) \tag{1}$$

and

$$a_{ik} + u_k a_{ih} = a_{ih} + u_h a_{ik}. \quad (i = 1, 2, \ldots, n)$$

We have supposed that the coefficients a_{ik} and a_{ih}, $(i = 1, 2, \ldots, n)$ would not be proportional , thus it is necessary that

$$u_k = 1 \text{ and } u_h = 1$$

We have arrived at the following important result:
it Any parallelohedron R characterised by a vector $[\lambda_i]$ will be determined by the canonical inequalities

$$a_{0k} + \sum a_{ik}(x_i + \lambda_i) \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

Observe that by virtue of (1), one will have the equation

$$\sum a_{ik}\lambda_{ih} = \sum a_{ih}\lambda_{ik}.$$

In this equation, one can attribute to the indices k and h the values $k = 1, 2, \ldots, \sigma; h = 1, 2, \ldots, \sigma$.

40 *Theorem. The vectors*

$$[\lambda_{i1}], [\lambda_{i2}], \ldots, [\lambda_{i\sigma}]$$

form the basis of the group G. By posing

$$\lambda_i = \sum_{k=1}^{\sigma} l_k \lambda_{ik} \tag{2}$$

where $l_1, l_2, \ldots, l_\sigma$ are arbitrary integers, one will determine each vector $[\lambda_i]$ of the group G. By indicating

$$a_i = \sum_{k=1}^{\sigma} l_k a_{ik}, \tag{3}$$

one will define the function $V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$ by the formula

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = \sum_{k=1}^{\sigma} l_k (a_{0k} - \frac{1}{2}\sum_{i=1}^{n} a_{ik}\lambda_{ik} + \sum_{i=1}^{n} a_{ik} x_i) + \frac{1}{2}\sum_{i=1}^{n} a_i \lambda_i \tag{4}$$

Let us suppose that the formula (4) is verified by the vectors $[\lambda_i^{(0)}]$ and $[\lambda_i']$ which are defined by the equations

$$\lambda_i^{(0)} = \sum_{k=1}^{n} l_k^{(0)} \lambda_{ik} \text{ and } \lambda_i' = \sum_{k=1}^{n} l_k' \lambda_{ik}. \ (i = 1, 2, \ldots, n) \tag{5}$$

We will see that the formula (1) will also be true for the vector $[\lambda_i]$ determined by the equations

$$\lambda_i = \lambda_i^{(0)} + \lambda_i'.$$

Let us indicate by $R, R^{(0)}$ and R' the parallelohedra characterised by the vectors $[\lambda_i], [\lambda_i^{(0)}]$ and $[\lambda_i']$.
The parallelohedron $R^{(0)}$ will be determined by the canonical inequalities

$$a_{0k} + \sum a_{ik}(x_i + \lambda_i^{(0)}) \geq 0. \quad (k = 1, 2, \ldots, \sigma)$$

One concludes that the function $V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n)$ is expressed by the formula

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = V(x_1, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}) + U(x_1, x_2, \ldots, x_n, \lambda', \lambda_2', \ldots, \lambda_n'), \tag{6}$$

Where the function $U(x_1, x_2, \ldots, x_n, \lambda', \lambda'_2, \ldots, \lambda'_n)$ represents the generatrix function determined with the condition that the parallelohedron $R^{(0)}$ have been chosen for the principal parallelohedron.

By designating

$$a_i^{(0)} = \sum_{k=1}^{\sigma} l_k^{(0)} a_{ik} \text{ and } a'_i = \sum_{k=1}^{\sigma} l'_k a_{ik}, \ (i = 1, 2, \ldots, n) \tag{7}$$

one will have, by virtue of the supposition made,

$$V(x_1, x_2, \ldots, x_n, \lambda_1^{(0)}, \lambda_2^{(0)}, \ldots, \lambda_n^{(0)}) =$$
$$\textstyle\sum_{k=1}^{\sigma} l_k^{(0)} \left(a_{0k} - \frac{1}{2}\sum a_{ik}\lambda_{ik} + \sum a_{ik}x_i\right) + \frac{1}{2}\sum a_i^{(0)}\lambda_i^{(0)},$$

$$U(x_1, x_2, \ldots, x_n, \lambda', \lambda'_2, \ldots, \lambda'_n) =$$
$$\textstyle\sum_{k=1}^{\sigma} l'_k \left(a_{0k} + \sum a_{ik}\lambda_i^{(0)} - \frac{1}{2}\sum a_{ik}\lambda_{ik} - \sum a_{ik}x_i\right) + \frac{1}{2}\sum a'_i\lambda'_i.$$

Let us put

$$l_k = l_k^{(0)} + l'_k. \quad (k = 1, 2, \ldots, \sigma)$$

By virtue of (6) one obtains

$$\begin{aligned} V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = {} & \textstyle\sum l_k \left(a_{0k} - \frac{1}{2}\sum a_{ik}\lambda_{ik} + \sum a_{ik}x_i\right) \\ & \textstyle + \frac{1}{2}\sum a_i^{(0)}\lambda_i^{(0)} + \frac{1}{2}\sum a'_i\lambda'_i + \sum_{k=1}^{\sigma}\sum_{i=1}^{\sigma} a_{ik}l'_k\lambda_i^{(0)}. \end{aligned} \tag{8}$$

Let us examine the sum

$$\frac{1}{2}\sum a_i^{(0)}\lambda_i^{(0)} + \frac{1}{2}\sum a'_i\lambda'_i + \sum_{k=1}^{\sigma}\sum_{i=1}^{n} a_{ik}l'_k\lambda_i^{(0)}. \tag{9}$$

By virtue of (5), one will hve

$$\sum_{i=1}^{n} a_{ik}\lambda_i^{(0)} = \sum_{h=1}^{\sigma}\sum_{i=1}^{n} a_{ik}l_h^{(0)}\lambda_{ih}.$$

We have seen in Number 39 that

$$\sum_{i=1}^{n} a_{ik}\lambda_{ih} = \sum_{i=1}^{n} a_{ih}\lambda_{ik},$$

therefore

$$\sum_{i=1}^{n} a_{ik}\lambda_i^{(0)} = \sum_{h=1}^{\sigma}\sum_{i=1}^{n} a_{ih}l_h^{(0)}\lambda_{ik}$$

and, because of (7), this becomes

$$\sum_{i=1}^{n} a_{ik}\lambda_i^{(0)} = \sum_{i=1}^{n} a_i^{(0)}\lambda_{ik}.$$

It follows that

$$\sum_{k=1}^{\sigma}\sum_{i=1}^{n} a_{ik}l'_k\lambda_i^{(0)} = \sum a_i^{(0)}\lambda'_i.$$

By virtue of (7), one will also have

$$\sum_{k=1}^{\sigma}\sum_{i=1}^{n} a_{ik}l'_k\lambda_i^{(0)} = \sum_{i=1}^{n} a'_i\lambda_i^{(0)}.$$

One can therefore present the sum (9) under the form

$$\begin{aligned} & \frac{1}{2}\sum a_i^{(0)}\lambda_i^{(0)} + \frac{1}{2}\sum a'_i\lambda'_i + \sum_{k=1}^{n}\sum_{i=1}^{n} a_{ik}l'_k\lambda_i^{(0)} \\ & = \frac{1}{2}\left(\sum a_i^{(0)}\lambda_i^{(0)} + \sum a_i^{(0)}\lambda'_i + \sum a'_i\lambda_i^{(0)} + \sum a'_i\lambda'_i\right) \\ & = \frac{1}{2}\sum\left(a_i^{(0)} + a_i^{(0)})(\lambda_i^{(0)} + \lambda'_i\right). \end{aligned}$$

As

$$a_i^{(0)} + a'_i = a_i \text{ and } \lambda_i^{(0)} + \lambda'_i = \lambda_i,$$

the formula (8) can be written

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) =$$
$$\sum_{k=1}^{\sigma} l_k \left(a_{0k} - \frac{1}{2}\sum a_{ik}\lambda_{ik} + \sum a_{ik}x_i\right) + \frac{1}{2}\sum a_i\lambda_i.$$

It is easy to verify the formula (4) in the case

$$\lambda_i = \pm\lambda_{ik}. \quad (k = 1, 2, \ldots, \sigma)$$

This results in that the formula (4) holds for any vector $[\lambda_i]$ belonging to the group G.
Theorem II. The group G possesses a basis formed of n vector

$$[\pi_{i1}], [\pi_{i2}], \ldots, [\pi_{in}].$$

By putting

$$\lambda_i = \sum_{k=1}^{n} l_k \pi_{ik} \tag{11}$$

where $l_1, l_2, \ldots, l_n$ are arbitrary integers, one will determine each vector $[\lambda_i]$ of the group G. By indicating

$$V(x_1, x_2, \ldots, x_n, \pi_1, \pi_2, \ldots, \pi_n) = p_{0k} + \sum_{i=1}^{n} p_i x_i, \quad (k = 1, 2, \ldots, n)$$

and

$$a_i = \sum_{k=1}^{n} l_k p_{ik}, \tag{12}$$

one will have the formula

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) =$$
$$\sum_{k=1}^{n} l_k \left(p_{0k} - \frac{1}{2} \sum_{i=1}^{n} p_{ik}\pi_{ik} + \sum_{i=1}^{n} p_{ik} x_i \right) + \frac{1}{2} \sum_{i=1}^{n} a_i \lambda_i.$$

One will easily demonstrate Theorem II introduced with the help of the formula (4).

Let us notice that the sum $\sum a_i\lambda_i$ presents, by virtue of equation (11) and (12), a quadratic form of integer variables $l_1, l_2, \ldots, l_n$

$$\sum a_i \lambda_i = \sum_{k=1}^{n} \sum_{h=1}^{n} A_{kh} l_k l_h,$$

where one has put

$$A_{kh} = \frac{1}{2} \sum_{i=1}^{n} p_{ik}\pi_{ih} + \frac{1}{2} \sum_{i=1}^{n} p_{ih}\pi_{ik}, \quad (k = 1, 2, \ldots, n; h = 1, 2, \ldots, n)$$

We will see that the quadratic form $\sum\sum A_{kh} l_k l_h$ obtained is positive.

41 *Theorem I. Any primitive parallelohedron possesses a centre.*

Designate by (ζ_i) the point satisfying the equations

$$p_{0k} - \frac{1}{2} \sum_{i=1}^{n} p_{ik}\pi_{ik} + \sum_{i=1}^{n} p_{ik}\zeta_i = 0 \quad (k = 1, 2, \ldots, n) \tag{1}$$

I say that the point (ζ_i) represents the centre of the principal parallelohedron R_0
To demonstrate this, put

$$\lambda_{ih} = \sum_{k=1}^{n} l_k^{(h)} \pi_{ik} \quad (h = 1, 2, \ldots, \sigma)$$

By virtue of Theorem II of Number 40, one obtains

$$V(x_1, x_2, \ldots, x_n, \lambda_{1h}, \lambda_{2h}, \ldots, \lambda_{nh}) =$$
$$\sum l_k^{(h)} \left(p_{0k} - \frac{1}{2} \sum p_{ik}\pi_{ik} + \sum p_{ik} x_i \right) + \frac{1}{2} \sum_{i=1}^{n} \left(p_{i1} l_1^{(h)} + \ldots + p_{in} l_n^{(h)} \right) \lambda_{ih}$$

On the other hand, by virtue of the definition established in Number 32, one has

$$V(x_1, x_2, \ldots, x_n, \lambda_{1h}, \lambda_{2h}, \ldots, \lambda_{nh}) = a_{0h} + \sum a_{ih} x_i$$

It follows that

$$a_{ih} = \sum_{k=1}^{n} l_k^{(h)} p_{ik}, \quad (h = 1, 2, \ldots, \sigma) \tag{2}$$

and

$$a_{0h} = \sum_{k=1}^{n} l_k^{(h)} \left(p_0 k - \frac{1}{2} \sum p_{ik}\pi_{ik} \right) + \frac{1}{2} \sum a_{ih}\lambda_{ih} \; (h = 1, 2, \ldots, \sigma) \tag{3}$$

Multiply the equation (1) by ($l^{(h)}$) and by attributing to the index k the values $1, 2, \ldots, n$, add the equations obtained, it becomes, by (2) and (3),

$$a_{0h} - \frac{1}{2}\sum a_{ih}\lambda_{ih} + \sum a_{ih}\zeta_i = 0\,(h = 1, 2, \ldots, \sigma) \tag{4}$$

That posed, take any one point (x_i) belonging to the parallelohedron R_0 .

For the point (ζ_i) to be the centre of the parallelohedron R_0, it is necessary and sufficient that the point (x'_i) determined by the equations

$$x'_i = 2\zeta_i - \lambda_i,\ (i = 1, 2, \ldots, n) \tag{5}$$

also belongs to the parallelohedron R_0 .

By virtue of the supposition made, one will have the inequalities

$$a_{0h} + \sum a_{ih}x_i \geq 0.\ (h = 1, 2, \ldots, \sigma) \tag{6}$$

By noticing that by (4) and (5)

$$a_{0h} + \sum a_{ih}x'_i = -a_{0h} - \sum a_{ih}(x_i - \lambda_{ih})$$

and that the inequality

$$-a_{0h} - \sum a_{ih}(x_i - \lambda_{ih}) \geq 0$$

is found among the inequalities (6), one obtains

$$a_{0h} + \sum a_{ih}x'_i \geq 0. \quad (h = 1, 2, \ldots, \sigma)$$

It is therefore demonstrated that the point (ξ_i) represents the centre of the parallelohedron R_0.

Let us notice that the centre (ξ_i) is interior to the parallelohedron R_0.

To demonstrate this, let us suppose that a point (x_i) is interior to the parallelohedron R_0.

One will have the inequalities

$$a_{0h} + \sum a_{ih}x_i > 0. \quad (h = 1, 2, \ldots, \sigma)$$

Among these inequalities can be found the inequalities

$$-a_{0h} - \sum a_{ih}(x_i - \lambda_{ih}) > 0. \quad (h = 1, 2, \ldots, \sigma)$$

By taking the summation of these inequalities, one obtains

$$\sum a_{ih}\lambda_{ih} > 0, \quad (h = 1, 2, \ldots, \sigma)$$

and, because of the equation (4), it becomes

$$a_{0h} + \sum a_{ih}\xi_i > 0. \quad (h = 1, 2, \ldots, \sigma)$$

42

Theorem II. The quadratic form

$$\sum_{i=1}^{n}(p_{i1}l_1 + p_{i2}l_2 + \ldots + p_{in}l_n)(\pi_{i1}l_1 + \pi_{i2}l_2 + \ldots + \pi_{in}l_n)$$

Apply Theorem I of Number 37 to the centre (ζ_i) of the principal parallelohedron R_0, one will have the inequality

$$V(\zeta_1, \zeta_2, \ldots, \zeta_n, \lambda_1, \lambda_2, \ldots, \lambda_n) > 0, \tag{7}$$

whatever the vector [λ_i] of the group G may be, the vector [0] being excluded.

By virtue of Theorem II of Number 40 and, [by virtue] of the equation (1), it becomes

$$p(\zeta_1, \zeta_2, \ldots, \zeta_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = \frac{1}{2}\sum(p_{i1}l_1 + \ldots + p_{in}l_n)(\pi_{i1}l_1 + \ldots + \pi_{in}l_n)$$

and, from (7), one finds

$$\sum (p_{i1}l_1 + p_{i2}l_2 + \ldots + p_{in}l_n)\,(\pi_{i1}l_1 + \pi_{i2}l_2 + \ldots + \pi_{in}l_n) > 0$$

The inequality obtained holds, whatever the integer values of the variable $l_1, l_2, \ldots, l_n$ may be, the system $l_1 = 0, l_2 = 0, \ldots, l_n = 0$ being excluded.

Continuous group of the linear transformations of the primitive parallelohedra

43

Applying a linear transformation of the principal primitive parallelohedron R_0 with the help of a substitution

$$x_i = \alpha_{i0} + \sum_{k=1}^{n}\alpha_{ik}x'_k, \quad (i = 1, 2, \ldots, n)$$

with any real coefficients and of the determinant which does not vanish.

One obtains a new primitive parallelohedron R' which will be determined with the help of the canonical inequalities

$$a'_{0h} + \sum_{k=1}^{n} a'_{kh} x'_k \geq 0, \quad (h = 1, 2, \ldots, \sigma)$$

where one has put

$$a'_{0h} = a_{0h} + \sum_{i=1}^{n} a_{ih}\alpha_{i0}, \; a'_{kh} = \sum_{i=1}^{n} a_{ih}\alpha_{ik} \qquad (1)$$

$$(k = 1, 2, \ldots, h; h = 1, 2, \ldots, \sigma)$$

The group G' of vectors corresponding to the parallelohedron R' obtained will be determined by the equations

$$\lambda_i = \sum_{k=1}^{n} \alpha_{ik}\lambda_k, \qquad (2)$$

on condition that the vector $[\lambda_i]$ of the group G corresponds to the vector $[\lambda'_i]$ in the group G' .

Designate

$$V(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_n) = p_0 + \sum_{i=1}^{n} p_i x_i$$

and

$$V(x'_1, x'_2, \ldots x'_n, \lambda'_1, \lambda'_2, \ldots, \lambda'_n) = p'_0 + \sum_{i=1}^{n} p'_i x_i$$

By virtue of the formula (4) of Number 40 and [by virtue] of the equation (1) and (2) one obtains

$$p'_0 = p_0 + \sum_{i=1}^{n} p + i\alpha_{i0}, \; p'_k = \sum_{i=1}^{n} p_i\alpha_{ik} \quad (k = 1, 2, \ldots, n)$$

Of which result $[\pi_i]$ and $[\pi'_i]$ being any two corresponding vectors, one will have

$$\sum_{i=1}^{n} p_i\pi_i = \sum_{i=1}^{n} p'_i\pi'_i \qquad (3).$$

Theorem. The quadratic form

$$\sum_{k=1}^{n}\sum_{h=1}^{n} A_{kh} l_k l_h = \sum_{i=1}^{n} (p_{i1}l_1 + p_{i2}l_2 + \ldots + p_{in}l_n)(\pi_{i1}l_1 + \pi_{i2}l_2 + \ldots + \pi_{in}l_n)$$

44 Carry out a transformation of the primitive parallelohedra of the set (R) with the help of a substitution

$$p_0 k - \frac{1}{2}\sum_{i=1}^{n} p_{ik}\pi_{ik} + \sum_{i=1}^{n} p_{ik}x_i = x'_k \quad (k = 1, 2, \ldots, n)$$

One obtains a set of the primitive parallelohedra (R').

The corresponding value of the function $V(x'_1, x'_2, \ldots, x'_n, \lambda'_1, \lambda'_2, \ldots, \lambda'_n)$ for the set (R') will be expressed by the formular

$$V(x'_1, x'_2, \ldots, x'_n, \lambda'_1, \lambda'_2, \ldots, \lambda'_n) = \sum_{i=1}^{n} l_i x'_i + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} l_i l_j .$$

By virtue of the theorem Number 38, the principal parallelohedron of the set (R) will be determined by the inequalities

$$\frac{1}{2}\sum\sum A_{ij} l_i h_j + \sum_{i=1}^{n} l_i x_i \geq 0$$

which hold, whatever the integer values of $l_1, l_2, \ldots, l_n$ may be.

The various parallelohedra of the set (R') will be determined by the inequalities

$$\frac{1}{2}\sum\sum A_{ij} l_i l_j + \sum l_i x_i \geq \frac{1}{2}\sum\sum A_{ij} l_i^{(0)} l_j^{(0)} + \sum l_i^{(0)} x_i \qquad (4)$$

Each parallelohedron of the set (R') will be characterised by a corresponding system $(l_i^{(0)})$ of integers $l_1^{(0)}, l_2^{(0)}, \ldots, l_n^{(0)}$.

Observe how one could replace the base of the group G formed of n vectors by another base also formed of n vectors, these two bases will be equivalent, by virtue of Theorem III of Number 11; the corresponded positive quadratic form $\sum\sum A_{ij} l_i l_j$ will be replaced by an equivalent form; the inequalities (4) define within this case the set of the parallelohedra which can be transformed as the set (R') with the help of a corresponding linear substitution on integer coefficients and of the determinant ± 1 .

The following remarkable theorem is thus demonstrated.

Theorem. By applying the linear transformation of a primitive parallelohedron with the help of the substitutions in some real coefficients which form a group continuous for linear substitutions, one obtains a set of primitive parallelohedra which is perfectly determined by a class of equivalent positive quadratic form, on condition that one does not consider as being different the quadratic forms with proportional coefficients.

We have seen how any positive quadratic form defines, by the help of the inequalities (4), a set of congruent parallelohedra which can be primitives or not.

Section III

Solution of the parallelohedra with the aid of positive quadratic form

Definition of the convex polyhedron corresponding to a positive quadratic form

45 Let $\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j$ be an arbitrary positive quadratic form in n variables $x_1, x_2, \ldots, x_n$. Imagine a set R of points (α_i) satisfying the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j + 2\sum_{i=1}^{n} \alpha_i x_i > \geq 0,$$

whatever may be the integer values of $x_1, x_2, \ldots, x_n$.

By virtue of the definition established, the set R enjoys the following properties:

1. The set R is in n dimensions
2. The point (0) represents the centre of the set R
3. The set R is convex.

Take we a system of arbitrary parameters $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$ and examine a vector g composed of points (α_i) which are determined by the equation

$$\alpha_i = \rho\epsilon_i \quad \text{where } \rho \geq 0$$

It is easy to demonstrate that there exists an interval

$$0 \leq \rho \leq \rho_0 \quad \text{where } \rho_0 > 0$$

which correspond with the points of vector g belonging to the set R .

By posing

$$\alpha_{i0} = \rho_0\epsilon_i,$$

one obtain a vector $[\alpha_{i0}]$ the points of which belong to the set R. The point (α_{i0}) belongs to the boundary of the set R, that is to say: the point (α_{i0}) satisfies the inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_{i0}x_i \geq 0, \tag{1}$$

whatever may be the integer values of $x_1, x_2, \ldots, x_n$ and satisfy at least one equation

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_{i0}l_i = 0, \tag{2}$$

$l_1, l_2, \ldots, l_n$ being the integers which do not vanish.

Designate

$$\alpha_{i1} = -\alpha_{i0} - \sum_{j=1}^{n} a_{ij}l_j, \quad (i = 1, 2, \ldots, n) \tag{3}$$

one will have, by (2), the equation

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_{i1}x_i = \sum\sum a_{ij}(l_i - x_i)(l_j - x_j) + 2\sum \alpha_{i0}(l_i - x_i)$$

and, by virtue of (1), one obtains

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_{i1}x_i \geq 0, \tag{4}$$

therefore the point (α_{i1}) also belongs to the set R .

By adding the inequalities (1) and (4), one finds, from (3),

$$\sum\sum a_{ij}x_ix_j - \sum\sum a_{ij}x_il_j \geq 0\,.$$

The inequality obtained holds, whatever the integer values of $x_1, x_2, \ldots, x_n$; this inequality can be written

$$\sum\sum a_{ij}l_il_j \leq \sum\sum a_{ij}(l_i - 2x_i)(l_j - 2x_j)$$

One concludes that the system (l_i) is nothing but a representation of the minimum of the positive quadratic form $\sum\sum a_{ij}x_ix_j$ determined in the set composed of all the systems of integers which are contiguous to the system l_i with respect to the modulo 2.

The number of such systems is finite. Suppose that all these systems form a series

$$(l_{i1}), (l_{i2}), \ldots, (l_{i\sigma}) \tag{5}$$

46 *Theorem. The set R presents a convex polyhedron determined with the aid of the inequalities*

$$\sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_i l_{ik} \geq 0\,(k = 1, 2, \ldots, \sigma) \tag{6}$$

By virtue of the definition established, each point (α_i) of the set R satisfies these inequalities. Suppose that a point (α_i) satisfying these inequality does not belong to the set R. One will determine in this case a positive value of the parameter ρ in the interval $0 < \rho < 1$, such that

$$\alpha_i^{(0)} = \rho\alpha_i \quad \text{where } 0 < \rho < 1, \tag{7}$$

one obtains a point $(\alpha_i^{(0)})$ belonging to the boundary of the set R. The point $(\alpha_i^{(0)})$ will satisfy, as we have seen, an equation

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_i^{(0)} l_i = 0 \tag{8}$$

characterised by a system (l_i) belonging to the series (6) .

By virtue of the equation obtained, one has

$$\sum \alpha_i^{(0)} l_i < 0$$

and, by (7), it becomes

$$\sum \alpha_i l_i < 0\,.$$

By presenting the equation (8) in the form

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i = 2(1-\rho)\sum \alpha_i l_i,$$

one will have the inequality

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i < 0,$$

which is contrary to the hypothesis.

Independent inequalities which define the convex polyhedron corresponding to a positive quadratic form

47 It may be the case that among the inequalities (6) of the previous number there are independent inequalities. Suppose, for example, that the inequality

$$\sum\sum a_ijl_il_j + 2\sum \alpha_i l_i \geq 0 \tag{1}$$

ve dependent. One will have in this case an identity

$$\sum\sum a_ijl_il_j + 2\sum \alpha_i l_i = \rho_0 + \sum_{k=1}^{\sigma}\rho_k \left(\sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_i l_ik\right) \tag{2}$$

where

$$\rho_0 \geq 0, \quad \rho_k \geq 0, \quad (k = 1, 2, \ldots, \sigma)\,.$$

We have seen in Numer 45 that the inequality

$$\sum\sum a_{ij}x_ix_j = \sum\sum a_{ij}x_il_j \geq 0$$

holds whatever the integer values of $x_1, x_2, \ldots, x_m$ may be.

By making in the identity (R)

$$\alpha_i = -\frac{1}{2}\sum a_{ij}l_j,$$

one obtains

$$\rho_0 + \sum \rho_k \left(\sum\sum a_{ij}l_{ik}l_{jk} - \sum\sum a_{ij}l_{ik}l_j\right) = 0$$

and consequently

$$\rho_0 = 0,\ \rho_k \left(\sum\sum a_{ij}l_{ik}l_{jk} - \sum\sum a_{ij}l_{ik}l_j\right) = 0\ (k = 1, 2, \ldots, \sigma)$$

By supposing that $\rho_k \neq 0$, one will have

$$\sum\sum a_{ij}l_{ik}l_{jk} - \sum\sum a_{ij}l_{ik}l_j = 0$$

thus

$$\sum\sum a_{ij}l_il_j = \sum\sum a_{ij}(l_i - 2l_ik)(l_j - 2l_{jk})\,.$$

By virtue of the equation obtained, the system $(l_i - 2s_{ik})$ is in the series (5) of Number 45. This is a condition necessary for the inequality (1) to be dependent.

48 *Theorem. For an inequality*

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i \geq 0 \tag{3}$$

to be independent, it is necessary and sufficient that the quadratic form $\sum\sum a_{ij}x_ix_j$ *does not possess as two minimum representations* (l_i) *and* $(-l_i)$ *in the set composed of all the systems of integers which are contiguous to the system* (l_i) *with regard to the modulus 2.*

We have demonstrated that the condition studied is sufficient. It remains to be demonstrated that this condition is necessar.

Let us suppose that the inequality (3) is independent. In this case

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i = 0$$

defines a face P in $n-1$ dimensions of the polyhedron R.

Let (α_i) be a point which is interior to the face P. One has the inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i > 0, \tag{4}$$

whatever the integer values of $x_1, x_2, \ldots, x_n$ may be, the two systems (0) and (l_i) being excluded. By putting, as we have done in Number 45,

$$\alpha_i' = -\alpha_i - \sum a_{ij}l_j, \tag{5}$$

one will also have an inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i' x_i > 0 \tag{6}$$

which holds for any integer values of $x_1, x_2, \ldots, x_n$, the two systems (0) and (l_i) being excluded. By adding the inequality (4) and (6) one finds, by (5),

$$\sum\sum a_{ij}x_ix_j - \sum a_{ij}x_il_j > 0$$

in other words

$$\sum\sum a_{ij}l_il_j < \sum\sum a_{ij}(l_i - 2x_i)(l_j - 2x_j).$$

The inequality obtained holds for any integer values of $x_1, x_2, \ldots, x_n$, the two systems (0) and (l_i) being excluded.

The theorem introduced is thus demonstrated.

Corollary. The number of the independent inequalities which define the polyhedron R corresponding to a positive quadratic form can not exceed the limit $2(2^n - 1)$.

Set (R) of parallelohedra defined by a positive quadratic form.

49 *Theorem. Let us suppose that the convex polyhedron R corresponding to a positive quadratic form* $\sum\sum a_{ij}x_ix_j$ *is determined with the help of the inequalities*

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i \geq 0.$$

By applying the translations of polyhedron R the length of the vector determined by the equations

$$\lambda_i = -\sum a_{ij}l_j,$$

$l_1, l_2, \ldots, l_n$ *being the arbitrary integers, one will make up a set (R) of congruent polyhedra which uniformly partition space in n dimensions.*

Let us indicate with R' the polyhedron which are obtained with the help of a translation of the polyhedron R the length of the vector $[\lambda_i]$. The polyhedron R' will be determined by the inequalities

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j + 2\sum_{i=1}^{n}(\alpha_i + \sum_{j=1}^{n} a_{ij}l_j)x_i \geq 0.$$

This inequality can be written

$$\sum\sum a_{ij}(x_i + l_i)(x_j + l_j) + 2\sum \alpha_i(x_i + l_i) \geq \sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i.$$

One concludes that the polyhedron R' will be determined by the inequalities

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i \geq \sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i \tag{1}$$

which hold, whatever the integer values of variables $x_1, x_2, \ldots, x_n$ may be.

One will say that the polyhedron R' congruent with the polyhedron R is characterised by the system (l_i).

Let us indicate by (R) the set of all the polyhedra congruent to polyhedron R and which are characterised by the various systems (l_i) of integers.

I argue that the set (R) uniformly fills the space in n dimensions.

Let us take an arbitrary point (α_i) in the space in n dimensions and find the polyhedron of the set (R) of which belongs the point (α_i). In this effect, determine a minimum representation (l_i) of the form

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i$$

in the set E composed of all the systems (x_i) of integer values of the variables $x_1, x_2, \ldots, x_n$.

One will have the inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i \geq \sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i$$

which holds in the set E. As a result, the point (α_i) belongs to the polyhedron of the set (R) characterised by the system (l_i).

Let us suppose that the point (α_i) belongs to the various polyhedra of the set (R): $R, R', \ldots, R^{(\mu)}$ characterised by the systems

$$(l_i), (l_{i1}), \ldots, (l_{i\mu}). \tag{2}$$

By virtue of (1), one obtains the inequalities

$$\sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_i l_{ik} = \sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i. \tag{3}$$

$$(k = 1, 2, \ldots, \mu)$$

It follows that one will have the inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i > \sum\sum a_{ij}l_il_j + 2\sum \alpha_i l_i,$$

for any integer values of $x_1, x_2, \ldots, x_n$, the systems (2) being excluded.

One concludes that the point (α_i) is interior to a face common to the polyhedra $R, R', \ldots, R^{(\mu)}$ and defined by the equations (3).

We have arrived at the following result: *Every positive quadratic form defines a set (R) of congruent parallelohedra which can be primitive or not.*

Algorithm for the search for the minimum of the form $\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i$ in the set E.

50 Let us suppose that one had determined the independent inequalities

$$\sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_i l_{jk} \geq 0 \quad (k = 1, 2, \ldots, \sigma)$$

which define the parallelohedron R corresponding to a positive quadratic form $\sum\sum a_{ij}x_ix_j$

With the help of the systems

$$(l_{i1}), (l_{i2}), \ldots, (l_{i\sigma})$$

of integers, one can resolve many problems of the arithmetic theory of positive quadratic form.

We seek, for example, the minimum of the form

$$\sum\sum a_{ij}x_ix_j = 2\sum a_i x_i \tag{1}$$

in the set E composed of all the systems (x_i) with integers, $\alpha_1, \alpha_2, \ldots, \alpha_n$ being arbitrary parameters given.

The values of $x_1, x_2, \ldots, x_n$ which correspond to the absolute minimum of the function (1) verify the equations

$$\sum_{i=1}^{n} a_{ik}x_i + a_k = 0 \quad (k = 1, 2, \ldots, n)$$

We designate by (ξ_i) the point verifying these equations. By posing

$$\xi_i = l_i + r_i$$

we determine the integers $l_1, l_2, \ldots, l_n$ under the conditions

$$|r_i| \leq \frac{1}{2} \qquad (i = 1, 2, \ldots, n)$$

In the case $r_i = 0\,(i = 1, 2, \ldots, n)$ the system (l_i) is the one we have sought. We suppose that all the numbers r_i $(i = 1, 2, \ldots, n)$ do not vanish. We pose

$$\alpha_i^{(0)} = \alpha_i + \sum_{j=1}^{n} a_{ij}l_j$$

and examine the point $(\alpha_i^{(0)})$.

Let us suppose that the point (α_i) belongs to the parallelohedron of the set (R) which is characterised by the system (l_i), therefore the system represents the minimum of the form (1) .

In the case where the point $(\alpha_i^{(0)})$ does not belong to the parallelohedron R, we determine a value ρ_0 in the interval $0 < \rho_0 < 1$ of parameter ρ, in the manner such that the point $(\rho_0\alpha_i^{(0)})$ belongs to a face of the parallelohedron R . Suppose that this face be determined by the equation

$$\sum\sum a_{ij}l_{ih}l_{jh} + 2\sum \alpha_i l_{ih} = 0$$

One will have an equation

$$\sum\sum a_{ij}l_{ih}l_{jh} + 2\rho_0\sum \alpha_i^0 l_{ih} = 0 \text{ where } 0 < \rho_0 < 1$$

Let

$$\alpha_i' = \alpha_i^0 + \sum_{j=1}^{n} a_{ij} l_{jh}$$

and examine anew the point (α_i') and so on. I say that one will always determine a representation of the minimum of the form (1) by repeating many times the procedure explained. To demonstrate, suppose that one had determined with the help of the algorithm shown a series of points

$$\left(\alpha_i^{(0)}\right), \left(\alpha_i'\right), \ldots, \left(\alpha_i^{(k)}\right), \ldots \tag{2}$$

and a series of systems

$$\left(l_i^{(0)}\right), \left(l_i'\right), \ldots, \left(l_i^{(k)}\right), \ldots,$$

verifying the equations

$$\alpha_i^{(k)} = \alpha_i^{(k-1)} + \sum_{j=1}^{n} a_{ij} l_j^{(k-1)} \quad (k = 1, 2, \ldots) \tag{3}$$

and the equations

$$\sum\sum a_{ij} l_i^{(k)} l_j^{(k)} + 2\sum \rho_k \alpha_i^{(k)} l_i^{(k)} = 0 \quad \text{where } 0 < \rho_k < 1$$

$$(k = 0, 1, 2, \ldots)$$

By virtue of these equations, one finds

$$\sum\sum a_{ij} l_i^{(k)} l_j^{(k)} + 2\sum \alpha_i^{(k)} l_i^{(k)} < 0 \quad (k = 0, 1, 2, \ldots) \tag{4}$$

By designating

$$m_i^{(k)} = l_i + l_i^{(0)} + \cdots + l_i^{(k-1)} \quad (k = 1, 2, \ldots) \tag{5}$$

and

$$m_i^{(0)} = l_i\,,$$

one obtains, from (3),

$$\alpha_i^{(k)} = \alpha_i + \sum_{j=1}^{n} a_{ij} m_j^{(k)} \quad (k = 0, 1, 2, \ldots) \tag{6}$$

By substituting in the inequality (4), one gets

$$\sum\sum a_{ij}\left(l_i^{(k)} + m_i^{(k)}\right)\left(l_j^{(k)} + m_j^{(k)}\right) + 2\sum d_i\left(l_i^{(k)} + m_i^{(k)}\right) <$$
$$\sum\sum a_{ij} m_i^{(k)} m_j^{(k)} + 2\sum \alpha_i m_i^{(k)}$$

This inequality, by (5), can be written

$$\sum\sum a_{ij} m_i^{k+1} m_j^{(} k + 1) + 2\sum d_i m_i^{k+1} < \sum\sum a_{ij} m_i^{(k)} m_j^{(k)} + 2\sum d_i m_i^{(k)}$$

$$(k = 0, 1, 2, \ldots)$$

The number of the systems $(m_i^{(k)})$ of integers verifying these inequalities is limited. One concludes that the series of points (2) will always end by a point $(\alpha_i^{(k)})$ belonging to parallelohedron R . By virtue of the equation (6), the system $(m_i^{(k)})$ represents the minimum of the form $\sum\sum a_{ij}x_i x_j + 2\sum\sum \alpha_i x_i$ in the set E. The problem described comes down to the search for all the parallelohedra of the set (R) which are contiguous by a face in the interior of which the point $(\alpha_i^{(k)})$ is to be found. One will determine all these parallelohedra by successively determining the parallelohedra which are contiguous to R through the faces in $n-1$ dimensions and so on and so forth.

Properties of the systems of integers which characterise the faces in $n-1$ dimensions of the parallelohedron corresponding to a positive quadratic form

51 Suppose that the systems

$$\pm(l_{i1}), \pm(l_{i2}), \ldots, \pm(l_i\tau) \tag{1}$$

characterises the faces in $n-1$ dimensions of parallelohedron R corresponding to a positive quadratic form

$$\sum\sum a_{ij} x_i x_j$$

Theorem I. The elements $l_1k, l_2k, \ldots, l_mk$ of any system (l_{ik}) belonging to the series (1) have no common divisor.

We have seen in Number 45 that the numbers $l_1k, l_2k, \ldots, l_nk$ verify the inequality

$$\sum\sum a_{ij} x_i x_j - \sum\sum a_{ij} x_i l_{jk} \geq 0$$

in the set E. By letting

$$l_{ik} = \delta t_i \quad \text{where } \delta \geq 1$$

and by putting $x_i = t_i$ in the previous inequality, one gets

$$\sum\sum a_{ij}t_it_j - \delta\sum\sum a_{ij}t_it_j \geq 0$$

and it is necessary that $\delta = 1$.

52 *Theorem II. Suppose that n systems*

$$(p_{i1}), (p_{i2}), \ldots, (p_{in}) \tag{2}$$

represent n consecutive minima

$$M_1 \leq M_2 \leq \ldots M_n$$

of the positive quadratic form $\sum\sum a_{ij}x_ix_j$. *All the systems (2) are in the series (1).*

By virtue of the definition for the system of n consecutive minima, one will have an inequality

$$M_k = \sum\sum a_{ij}p_{ik}p_{jk} \leq \sum\sum a_{ij}x_ix_j \quad (k = 1, 2, \ldots, n)$$

as long as all the numbers $x_1, x_2, \ldots, x_n$ can not be presented in the form

$$M_k = \sum_{r=1}^{k-1} u_r p_i r\,,$$

the system (0) being excluded.

Suppose that the system (p_{ik}) does not belong to the series (1). In this case there exists a system (t_i) of all the numbers verifying the inequality

$$\sum\sum a_{ij}p_{ik}p_{jk} \geq \sum\sum a_{ij}(p_{ik} - 2t_i)(p_{jk} - 2t_j)$$

On letting

$$q_i = p_{ik} - t_i \tag{3}$$

one presents the previous inequality in the form

$$\sum\sum a_{ij}t_it_j + \sum\sum a_{ij}q_iq_j \qquad (4)\,.$$

By supposing that the two systems (t_i) and (q_i) are different from the system (0), one will have, by virtue of the inequality obtained, the equation

$$t_i = \sum_{r=1}^{k-1} u_r p_i r\,, \quad q_i = \sum_{r=1}^{k-1} v_r p_i r$$

and, from (3), it follows that

$$p_{ik} = \sum_{r=1}^{k-1} (u_r + v_r) p_i r\,.$$

The equations obtained are impossible, since otherwise the determinant of n systems (2) would vanish, which is contrary to the hypothesis.

As a result, the inequality (4) does not hold at condition where either

$$t_i = 0 \quad \text{or} \quad t_i = p_{ik} \quad (i = 1, 2, \ldots, n)$$

It is therefore demonstrated that the system (p_{ik}), $(k = 1, 2, \ldots, n)$ belongs to the series (1).

Corollary. All the representations for the arithmetic minimum of the positive quadratic form $\sum\sum a_{ij}x_ix_j$ are within the series (1).

53 *Theorem III. The numerical value of determinant of any n systems which belong to the series (1) is less than n*!

Choose any n systems in the series (1)

$$(l_{i1}), (l_{i2}), \ldots, (l_{in})$$

which the determinant $\pm\omega$ does not vanish. Let us indicate

$$\alpha_{ik}^{(0)} = \frac{1}{2}\sum_{j=1}^{n} a_{ij}l_{jk} \text{ and } \alpha'_{ik} = -\frac{1}{2}\sum_{j=1}^{n} a_{ij}l_{jk} \; (k = 1, 2, \ldots, n) \tag{5}$$

By virtue of the inequalities

$$\sum\sum a_{ij}x_ix_j \pm \sum\sum a_{ij}x_il_{jh} \geq 0 \quad (h = 1, 2, \ldots, \tau)$$

which holds in the set E, $2n$ points (5) that belong to the parallelohedron R corresponding to the quadratic form $\sum\sum a_{ij}x_ix_j$.

Let us choose any n points among $2n$ points (5), making sure that two points corresponding to the same index k value are not among the ones chosen. One forms in this manner $2n$ systems composed of n points

$$\left(\alpha^{(0)}_{ih_1}\right), \left(\alpha^{(0)}_{ih_2}\right), \ldots, \left(\alpha^{(0)}_{ih_\mu}\right), (\alpha'_{ih_{\mu+1}}), \ldots, \left(\alpha'_{ih_n}\right)$$

$h_1, h_2, \ldots, h_n$ being any permutation of the indices $1, 2, \ldots, n$ and $\mu = 0, 1, 2, \ldots, n$.

We designate, to summarise,

$$\alpha^{(h)}_{ik} = \alpha^{(0)}_{ih_k}, \ (k = 1, 2, \ldots, \mu) \ \alpha^{(h)}_{ik} = \alpha'_{ih_k}, \qquad (6)$$
$$(k = \mu + 1, \ldots, n; h = 1, 2, \ldots, 2^n)$$

and examine a simplex K_h determined by the equation

$$x_i = \sum_{k=1}^{n} \vartheta_k \alpha^{(h)}_{ik} \ \text{ where } \sum_{k=1}^{n} \vartheta_k \leq 1 \ \text{ and } \ \vartheta_k \geq 0 \ (k = 1, 2, \ldots, n)$$

All the simplexes K_h, $(h = 1, 2, \ldots, 2^n)$ belong to the parallelohedron R . Any point (α_i), which is interior to a simplex K_h, does not belong to any other simplex of the series formed. This results in an inequality

$$\sum_h \int_{(K_h)} dx_1 dx_2 \cdots dx_n < \int_{(R)} dx_1 dx_2 \cdots dx_n \quad (h = 1, 2, \ldots, 2^n) \qquad (7)$$

54 On designating by D the determinant

$$D = \begin{vmatrix} a_1 1 & a_1 2 & \cdots & a_1 n \\ \vdots & \vdots & \ddots & \vdots \\ a_n 1 & a_n 2 & \cdots & a_n n \end{vmatrix}$$

of the quadratic form $\sum\sum a_{ij} x_i x_j$, one has by virtue of (5) and (6)

$$\int_{(K_h)} dx_1 dx_2 \cdots dx_n = \frac{\omega}{n!} \cdot \frac{D}{2^n}$$

and the inequality (7) gives

$$\frac{\omega}{n!} D < \int_{(R)} dx_1 dx_2 \cdots dx_n \qquad (8)$$

This established, we observe how the group G of vectors corresponding to the parallelohedron R possesses a basis formed by n vectors

$$[a_{i1}], [a_{i2}], \ldots, [a_{in}] .$$

By virtue of Theorem III of Number 11, it follows that

$$\int_{(R)} dx_1 dx_2 \cdots dx_n = D . \qquad (9)$$

By substituting in the inequality (8), one would obtain

$$\omega < n!$$

§ D.4 G. F. Voronoi, 1909

New application of continuous parameters to the theory of quadratic form.
Second Memoir
Studies on the primitive parallelohedra by Mr. Georges Voronoi in Warsaw
Second Part
Domains of quadratic forms corresponding to the various types of primitive parallelohedra
Section IV
Various types of primitive parallelohedra
[*Journal für die reine und angewandte Mathematik*, V. 136, p. 67–181, 1909]
[translated by K N Tiyapan]

On the number of faces in $n-1$ dimensions of primitive parallelohedron.

55 *Theorem. The number of faces in $n-1$ dimensions of a primitive parallelohedron is equal to* $2(2^n - 1)$.

Let us suppose that a primitive parallelohedron R corresponding to a positive quadratic form $\sum\sum a_{ij} x_i x_j$ is defined by the independent inequalities

$$\sum\sum a_{ij} l_{ik} l_{jk} \pm 2 \sum \alpha_i l_{ik} \geq 0. \ (k = 1, 2, \ldots, \tau) \qquad (1)$$

We have seen in Number 48 that any system

$$\pm(l_{1k}, l_{2k}, \ldots, l_{nk}), \ (k = 1, 2, \ldots, \tau) \qquad (2)$$

represents the minimum of the quadratic form $\sum\sum a_{ij}x_ix_j$ in the set composed of all the systems of integers which are congruent to the system $\pm(l_{ik})$ with respect to the modulus 2. The form $\sum\sum a_{ij}x_ix_j$ possesses in this set only two minimum representations $\pm(l_{ik})$.

Let us divide the set E, composed of all the systems (x_i) of integers $x_1, x_2, \dots, x_n$, into 2^n classes

$$E_0, E_1, \dots, E_m \text{ where } m = 2^n - 1$$

with regard to the modulus 2 and suppose that the set E_0 is composed of systems the elements of which have the common divisor 2.

All the systems (2) do not belong to the different sets

$$E_1, E_2, \dots, E_m \text{ where } m = 2^n - 1.$$

It follows that

$$\tau \leq 2^n - 1.$$

I argue that $\tau = 2^n - 1$. Let us suppose that among the systems (2) there are not found the systems belonging to a set E and we determine the minimum of the form $\sum\sum a_{ij}x_ix_j$ in the set E_h. Let (l_i) be a representation of this minimum.

Let us indicate by

$$(\alpha_{i1}), (\alpha_{i2}), \dots, (\alpha_{is}) \tag{3}$$

the vertices of the parallellohedron R defined by the inequalities (1) and examine the values of the function $\sum\sum a_{ij}l_il_j + 2\sum \alpha_il_i$ which correspond to the different vertices (3). Let us suppose that the sum $\sum\sum a_{ij}l_il_j + 2\sum \alpha_{ik}l_i$ be the smallest one.

By virtue of the supposition made, one will have the inequalities

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_{ir}l_i \geq \sum\sum a_{ij}l_il_j + 2\sum \alpha_{ik}l_i. \, (r = 1, 2, \dots, s) \tag{4}$$

By noticing that each point (α_i) belonging to the parallelohedron R can be determined by the equalities

$$\alpha_i = \sum_{r=1}^{s} \vartheta_r\alpha_{ir} \text{ where } \sum \vartheta_r = 1 \text{ and } \vartheta_r \geq 0, (r = 1, 2, \dots, s)$$

one will deduce the inequalities (4) an inequality

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_il_i \geq \sum\sum a_{ij}l_il_j + 2\sum \alpha_{ik}l_i$$

which holds for any poit (α_i) belonging to the parallelohedron R. The system (l_i) which represents the minimum of the form $\sum\sum a_{ij}x_ix_j$ in the set E_h verifies the inequality

$$\sum\sum a_{ij}x_ix_j - \sum\sum a_{ij}x_il_j \geq 0$$

in the set E. It results that the point

$$\xi_i = -\frac{1}{2}\sum_{j=1}^{n} \alpha_{ij}l_j$$

belong to the parallelohedron R. By making in the inequality (5) $\alpha_i = \xi_i$, one notices

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_{ik}l_i \leq 0.$$

The vertex (α_{ik}) of the parallelohedron R verifies the inequality

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_{ik}l_i \geq 0,$$

therefore it is necessary that

$$\sum\sum a_{ij}l_il_j + 2\sum \alpha_{ik}l_i = 0. \tag{6}$$

This stated, let us notice that the vertex (α_ik) of the primitive parallelohedron is simple.

Let us indicate by

$$\sum\sum a_{ij}l_{ir}^{(k)}l_{jr}^{(k)} + 2\sum \alpha_{ik}l_{ir}^{(k)} = 0, \quad (r = 1, 2, \dots, n)$$

n equations which define the vertex in the parallelohedron R. As the vertex (α_{ik}) is simple, one will have an inequality

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_{ik}x_i > 0,$$

whatever the integer values of $x_1, x_2, \dots, x_n$ may be, the system (0) and the systems

$$(l_{i1}^{(k)}), (l_{i2}^{(k)}), \dots, (l_{in}^{(k)}) \tag{7}$$

being excluded. By virtue of the equality (6), the system (l_i) is found among the systems (7) which all belong to the series (R).

It is therefore demonstrated that

$$\tau = 2^n - 1$$

and that the number of faces in $n-1$ dimensions of the parallelohedron R is equal to

$$2\tau = 2(2^n - 1).$$

Definition of the type of primitive parallelohedra.

56 Let us examine a primitive parallelohedron R determined with the help of independent inequalities

$$\sum\sum a_{ij} l_{ik} l_{jk} + 2\sum \alpha_i l_{ik} \geq 0. \ (k = 1, 2, \ldots, \sigma \text{ where } \sigma = 2(2^n - 1))$$

Let us indicate with

$$(\alpha_{i1}), (\alpha_{i2}), \ldots, (\alpha_{is})$$

the vertices of the parallelohedron R. One will determine with the help of equations

$$\sum\sum a_{ij} l_{ir}^{(k)} l_{jr}^{(k)} + 2\sum \alpha_{ik} l_{ir}^{(k)} = 0. \quad (r = 1, 2, \ldots, n, k = 1, 2, \ldots, s)$$

Each vertex (α_{ik}) $(k = 1, 2, \ldots, s)$ is characterised by n systems of integers

$$(l_{i1}^{(k)}), (l_{i2}^{(k)}), \ldots, (l_{in}^{(k)}), \quad (k = 1, 2, \ldots, s) \tag{1}$$

the determinant $\pm\omega_k$ of which does not cancel each other out.

Let us indicate, to make short, n systems (1) by a symbol

$$\{l_{ir}^{(k)}\}.$$

All the vertices of the primitive parallelohedron R will be characterised by a set of symbols

$$\{l_{ir}^{(1)}\}, \{l_{ir}^{(2)}\}, \ldots, \{l_{ir}^{(s)}\}. \tag{2}$$

This declared, let us examine another primitive parallelohedron R' corresponding to another positive quadratic form $\sum\sum a'_{ij} x_i x_j$. It can turn out that all the vertices of the parallelohedron R' will alsobe characterised by the symbols (2). One will say in this case that the two parallelohedra R and R' belong to the same type.

Definition. One will call the various parallelohedra all the vertices of which are characterised by the set of symbols (2), "belonging to the same type."

57 One can characterise a type of primitive parallelohedra in many ways.

Let us consider a set (R) of congruent primitive parallelohedra which corresponds to a positive quadratic form $\sum\sum a_{ij} x_i x_j$.

All the vertices of parallelohedra belonging to the set (R) can be divided into classes of congruent vertices. Let us indicate by τ the number of incongruent vertices belonging to the various classes.

Any vertex of a primitive parallelohedron is congruent to n vertices of parallelohedron, this results in that

$$S = (n+1)\tau.$$

Let (α_i) be any one vertex of parallelohedra of the set (R). One will define it with the help of $n+1$ equations

$$\sum\sum a_{ij} l_{ik} l_{jk} + 2\sum \alpha_i l_{ik} = A. \ (k = 0, 1, 2, \ldots, n) \tag*{3}$$

The $n+1$ systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{in})$$

characterise $n+1$ parallelohedra of the set (R) which are contiguous by the vertex (α_i). By indicating with (l_i) a system of arbitrary integers, one will characterise by the systems

$$(l_{i0} + l_i), (l_{i1} + l_i), \ldots, (l_{in} + l_i) \tag{4}$$

all the congruent vertices of parallelohedra of the set (R).

By attributing to the variables $l_1, l_2, \ldots, l_n$ any arbitrary values, one will characterise by $n+1$ systems (4) a class of congruent vertices.

One concludes this that a type of primitive parallelohedra can be characterised by τ systems (4).

58 To have more convenience in the notations, let us introduce in our studies the linear functions

$$u = \sum_{i=1}^{n} l_i x_i, \text{ and } u_k = \sum_{i=1}^{n} l_{ik} x_i. \quad (k = 0, 1, 2, \ldots, n)$$

One will say that the symbol $(u_0, u_1, \ldots, u_n)$ characterise the vertex (α_i) determined by the equations (3); the symbol $(u_0 + u, u_1 + u, \ldots, u_n + u)$, u being a linear function in arbitrary integer coefficients, characterise a vertex congruent to the vertex (α_i).

Let us suppose that one had characterised by the symbols

$$(u_0^{(k)}, u_1^{(k)}, \ldots, u_n^{(k)}), \quad (k = 1, 2, \ldots, \tau) \tag{5}$$

τ congruent vertices of primitive parallelohedra belonging to the set (R). One will say that the set of symbols (5) characterise a type of primitive parallelohedra.

59 Let us examine the faces in various dimensions of primitive parallelohedra belonging to the same type.

Let $P(\nu)$ be a face in ν dimensions ($\nu = 0, 1, 2, \ldots, n-1$) of parallelohedra of the set (R) defined by the equations

$$\sum\sum a_{ij} l_{ik} l_{jk} + 2\sum \alpha_i l_{ik} = \sum\sum a_{ij} l_{i0} + 2\sum \alpha_i l_{i0}.(k = 1, 2, \ldots, n-\nu)$$

One will characterise this face by $n+1-\nu$ systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{i,n-\nu})$$

or by $n+1-\nu$ corresponding linear functions.

$$u_0, u_1, \ldots, u_{n-\nu}.$$

All the faces in ν dimensions of parallelohedra of the set (R) which are congruent to the face $P(\nu)$ will be characterised by the systems

$$(l_{i0} l_i), (l_{i1} l_i), \ldots, (l_{i,n-\nu} l_i)$$

or by the corresponding linear functions

$$u_0 + u, u_1 + u, \ldots, u_{n-\nu} + u.$$

By making, for example, $l_i = -l_{i0}$ one obtains $n-\nu$ systems

$$(l_{i1} - l_{i0}), (l_{i2} - l_{i0}), \ldots, (l_{i,n-\nu} - l_{i0}) \tag{6}$$

which enjoy the following property: all the determinants of the order $(n-\nu)^2$ which one can form from $n-\nu$ systems (6) do not cancel one another at the same time. Let us indicate by $\omega^{(n-\nu)}$ the greatest common divisor of these determinants. By declaring

$$x_i = \sum_{k=1}^{n-\nu} (l_{ik} - l_{i0}) \xi_k, \tag{7}$$

one will present a system (x_i) of integers by the linear forms where $\xi_1, \xi_2, \ldots, \xi_{n-\nu}$ are integer or rational numbers which belong to $\omega^{(n-\nu)}$ sets

$$\xi_k = \vartheta_{kr} + y_k \quad (k = 1, 2, \ldots, n-\nu; r = 1, 2, \ldots, \omega^{(n-\nu)}) \tag{8}$$

where $y_1, y_2, \ldots, y_{n-\nu}$ are arbitrary integers. Among the sets (8) is found a set where $\vartheta_{kr} = 0$, $k = 1, 2, \ldots, n-\nu$ and which is composed of integer values of $\xi_1, \xi_2, \ldots, \xi_{n-\nu}$.

In the case $\omega^{(n-\nu)} = 1$, the equalities (7) are possible only on condition that the number $\xi_1, \xi_2, \ldots, \xi_{n-\nu}$ be integer.

The set (8) play an important role in the subsequent studies.

Let us indicate by the symbol $\sigma_{n-\nu}$ the number of incongruent faces in ν dimensions of primitive parallelohedra belonging to the type examined. By indicating with the symbol S_ν the number of faces in ν dimensions of corresponding primitive parallelohedron, one will have a formula

$$S_\nu = (n+1-\nu)\sigma_{n-\nu}. \quad (\nu = 0, 1, 2, \ldots, n-1) \tag{9}$$

Definition of the set (L) of simplexes characterising a type of primitive parallelohedra

60 Let us suppose that $n+1$ systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}) \tag{1}$$

characterise a vertex of primitive parallelohedra belonging to the type examined.

Definition I. One will call correlative to the vertex of primitive parallelohedra characterised by the systems (1) *a simplex L having $n+1$ vertices*

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}).$$

The simplex L presents a set of points determined by the equalities

$$x_i = \sum_{k=0}^{n} \vartheta_k l_{ik} \text{ where } \sum_{k=0}^{n} \vartheta_k = 1 \text{ and } \vartheta_k \geq 0. \quad (k = 0, 1, 2, \ldots, n)$$

Let us indicate by (L) the set of simplexes correlative to the various vertices of a set (R) of primitive parallelohedra belonging to the type examined.

Definition II. One will say that a type of primitive parallelohedra is characterised by the set (L) of simplexes.

One will call congruent two simplexes characterised by the vertices

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}) \text{ and } (l_{i0} + l_i), (l_{i1} + l_i), \ldots, (l_{in} + l_i),$$

$l_1, l_2, \ldots, l_n$ being arbitrary integers.

All the simplexes of the set (L) can be divided into classes of congruent simplexes; the number of classes is expressed by the symbol σ_n defined by the formula (9) of the previous number.

With the help of equations

$$x_i = \sum_{k=0}^{n-\nu} \vartheta_k l_{ik} \text{ where } \sum_{k=0}^{n-\nu} \text{ and } \vartheta_k \geq 0, \quad (k = 0, 1, 2, \ldots, n-\nu)$$

one will determine a face in $n-\nu$ dimensions of the simplex L which is correlative to the face in ν dimensions of parallelohedra characterised by the systems

$$(l_{i0}), (l_{i1}), \dots, (l_{i[,]n-\nu}).$$

One concludes that the number of incongruent faces in $n-\nu$ dimensions of the set (L) of simplexes is expressed by the symbol $\sigma_{n-\nu}$ $(\nu = 0, 1, 2, \dots, n-1)$.

As all the vertices of simplexes of the set (L) are congruent, one will declare $\sigma_0 = 1$, and the formula (9) of Number 59

$$S_\nu = (n+1-\nu)\sigma_{n-\nu}$$

will hold for the values of $\nu = 0, 1, 2, \dots, n$, provided that one would admit $S_n = 1$.

61 *Theorem I. The set (L) of simplexes uniformly fills the space in n dimensions.*

Let us suppose that a point (x_i) be interior to a face of the simplex L characterised by the systems

$$(l_{i0}), (l_{i1}), \dots, (l_{i\nu}). \tag{2}$$

One will have

$$x_i = \sum_{k=0}^{\nu} \vartheta_k l_{ik} \text{ where } \sum_{k=0}^{\nu} \vartheta_k = 1 \text{ and } \vartheta_k > 0. \; (k = 0, 1, 2, \dots, \nu) \tag{3}$$

Let us suppose that the point (x_i) be interior to a face in ν' dimensions of another simplex L' characterised by the vertices

$$(l'_{i0}), (l'_{i1}), \dots, (l'_{i\nu}).$$

One can write

$$x_i = \sum_{k=0}^{\nu'} \vartheta'_k l'_{ik} \text{ where } \sum_{k=0}^{\nu'} \vartheta'_k = 1 \text{ and } \vartheta'_k > 0. a; (k = 0, 1, 2, \dots, \nu') \tag{4}$$

Let $\sum\sum a_{ij}x_ix_j$ be a positive quadratic form which defines a set (R) of primitive parallelohedra belonging to the type examined.

Let us indicate by (α_i) and (α'_i) two vertices of parallelohedra of the set (R) which are correlative to the simplexes L and L'. One will have the equalities

$$\left\{ \begin{aligned} \sum\sum a_{ij} l_{ik} l_{jk} + 2\sum \alpha_i l_{ik} &= A, \\ \sum\sum a_{ij} l'_{ik} l'_{jk} + 2\sum \alpha'_i l'_{ik} &= A'. \end{aligned} \right. \quad (k = 0, 1, 2, \dots, n) \tag{5}$$

By putting down

$$\left\{ \begin{aligned} \sum\sum a_{ij} l'_{ik} l'_{jk} + 2\sum \alpha_i l'_{ik} &= A + \rho_k, \\ \sum\sum a_{ij} l_{ik} l_{jk} + 2\sum \alpha'_i l_{ik} &= A' + \rho'_k, \end{aligned} \right. \quad (k = 0, 1, 2, \dots, n) \tag{6}$$

one will have the inequalities

$$\rho_k \geq 0 \text{ and } \rho'_k \geq 0. \quad (k = 0, 1, 2, \dots, n)$$

From equalities (5) and (6) one derives

$$\left\{ \begin{aligned} A' - A + 2\sum(\alpha_i - \alpha'_i) l'_{ik} &= \rho_k, \\ A - A' + 2\sum(\alpha'_i - \alpha_i) l_{ik} &= \rho'_k. \end{aligned} \right. \quad (k = 0, 1, 2, \dots, n)$$

By virtue of equalities (3) and (4), one obtains,

$$A' - A + 2\sum(\alpha_i - \alpha'_i)x_i = \sum_{k=0}^{\nu'} \rho_k \vartheta'_k,$$

$$A - A' + 2\sum(\alpha'_i - \alpha_i)x_i = \sum_{i=0}^{\nu} \rho'_k \vartheta_k.$$

By making the sum of these equalities, one finds

$$\sum_{k=0}^{\nu'} \rho_k \vartheta'_k + \sum_{k=0}^{\nu} \rho'_k \vartheta_k = 0.$$

It follows, because of (3) and (4), that

$$\rho_k = 0, \;\; (k = 0, 1, 2, \dots, \nu') \;\; \text{and } \rho'_k = 0. \;\; (k = 0, 1, 2, \dots, nu)$$

let us notice that the equality $\rho_k = 0$ is possible only on the condition that the system (l'_{ik}) is found among the vertices of the simplex L, similarly, the equality $\rho'_k = 0$ is possible only on condition that the system (l_{ik}) is found among the vertices of the simplex L'.

One concludes that the systems

$$(l'_{i0}), (l'_{i1}), \dots, (l'_{i\nu'}) \tag{7}$$

characterise a face of the simplex L and that the systems (2) characterise a face of the simplex L'. As a point (x_i) can not be interior to two different faces of the same simplex, it results in that the systems (2) and (7) coincide; therefore the two simplexes L and L' are contiguous through the faces in ν dimensions characterised by the systems (2).

It remains to demonstrate that any point (x_i) of the space in n dimensions belongs to at least one simplex of the set (L).

To demonstrate this, let us take any one point (ξ_i) which is interior to the simplex L and draw any one curve C which joins the points (ξ_i) and (x_i). I say that all the points of the curve C will be situated in the simplexes

$$L, L', \dots, L^{(m)}$$

belonging to the set (L). In effect, let us suppose that the point (x_i) not belong to the simplex L. The curve C will go beyond in one point (ξ'_i) the boundary of the simplex L and will pass through a simplex L' which is contiguous to the simplex L through a face in any one number of dimensions and so on and so forth.

62

Theorem II. A point (x_i) the elements $x_1, x_2, \dots, x_n$ of which are integers can be only one vertex of simplexes of the set (L).

Let us notice that there exist simplexes of the set (L) which passes the vertex (0); the number of these simplexes is expressed by the symbol S_0.

By effecting the translations of these simplexes the length of the vector $[x_i]$, one will obtain S_0 simplexes which possess the vertex (x_i). By virtue of Theorem I, the point (x_i) can not belong to other simplexes of the set (L).

Corollary. Suppose that a point (x_i) the elements $x_1, x_2, \dots, x_n$ of which are integers, is not found among the vertices

$$(l_{i0}), (l_{i1}), \dots, (l_{in})$$

of a simplex L. By writing

$$x_i = \sum_{i=0}^{n} \vartheta_k l_{ik} \text{ where } \sum_{k=0}^{n} \vartheta_k = 1,$$

one will have among the numbers $\vartheta_0, \vartheta_1, \dots, \vartheta_n$ at least one negative number.

Properties of symbols S_ν and σ_ν $(\nu = 0, 1, 2, \dots, n)$.

63

Let us take any one positive integer m and consider a set K of points which are congruent to m^n points

$$\frac{g_1}{m}, \frac{g_2}{m}, \dots, \frac{g_n}{m}$$

which one obtains by attributing to the numbers $g_1, g_2, \dots, g_n$ the integer values verifying the inequalities

$$0 \leq g_k < m. \quad (k = 1, 2, \dots, n)$$

Let us take any one point $(\frac{x_i}{m})$ of the set K and suppose that the point $(\frac{x_i}{m})$ be interior to any one face $P(\nu)$ of simplexes of the set (L) $(\nu = 0, 1, 2, \dots, n)$. By virtue of Theorem I of number 61, all points of the set K which are interior to the face $P(\nu)$ can not be congruent.

Let us indicate by

$$P_k^{(\nu)}, \quad (k = 1, 2, \dots, \sigma_\nu; \nu = 0, 1, 2, \dots, n)$$

the various incongruent faces of simplexes of the set (L) and by the symbol

$$m_k^{(\nu)} \quad (k = 1, 2, \dots, \sigma_\nu; \nu = 0, 1, 2, \dots, n)$$

let us indicate the number of points of the set K which are interior to the face $P_k(\nu)$. ONe will have a formula

$$\sum_{\nu=0}^{n} \sum_{k=1}^{\sigma_\nu} m_k^{(\nu)} = m^n. \tag{1}$$

64

It is easy to determine the value of the symbol $m_k^{(\nu)}$. Let us indicate by

$$l_{i0}^{(k)}, l_{i1}^{(k)}, \dots, l_{i\nu}^{(k)}$$

the vertices of the face $P_k(\nu)$ and let us write

$$\frac{x_i}{m} = \sum_{r=0}^{\nu} \vartheta_r l_{ir}^{(k)} \text{ where } \sum \vartheta_r = 1 \text{ and } \vartheta_r > 0. \quad (r = 0, 1, 2, \dots, \nu)$$

These equalities can be written

$$\frac{x_i}{m} - l_{i0}^{(k)} = \sum_{r=1}^{\nu} \vartheta_r (l_{ir}^{(k)} - l_{i0}^{(k)}).$$

By indicating, to make short,

$$x_i - m l_{i0}^{(k)} = t_i, \;\; l_{ir}^{(k)} - l_{i0}^{(k)} = p_{ir}, \quad (r = 1, 2, \ldots, \nu)$$

and

$$m\vartheta_r = \tau_r, \quad (r = 1, 2, \ldots, \nu)$$

one will have

$$t_i = \sum_{r=1}^{\nu} \tau_r p_{ir} \text{ where } \sum_{r=1}^{\nu} \tau_r < m \text{ and } \tau_r > 0.\ (r = 1, 2, \ldots, \nu) \tag{2}$$

Let us indicate by $\omega_k^{(\nu)}$ the greatest common divisor of determinants of the order ν^2 which one can form from ν systems

$$(p_{i1}), (p_{i2}), \ldots, (p_{i\nu})$$

and suppose that the forms (2) represent the integers $t_1, t_2, \ldots, t_n$, provided that the numbers $\tau_1, \tau_2, \ldots, \tau_\nu$ belong to one of $\omega_k^{(\nu)}$ sets

$$\tau_r = \xi_{rh} + y_r \text{ where } r = 1, 2, \ldots, \nu, h = 1, 2, \ldots, \omega_k^{(\nu)}, \tag{3}$$

$y_1, y_2, \ldots, y_\nu$ being arbitrary integers.

One can suppose that

$$0 < \xi_{rh} \leq 1. \quad (r = 1, 2, \ldots, \nu, h = 1, 2, \ldots, \omega_k^{(\nu)})$$

By substituting the expressions of τ_r $(r = 1, 2, \ldots, \nu)$ derived from equalities (3) in the inequalities (2), one obtains

$$\xi_{rh} + y_r > 0, (r = 1, 2, \ldots, \nu) \quad \sum_{r=1}^{\nu} (\xi_{rh} + y_r) < m. \tag{4}$$

Let us indicate

$$\sum_{r=1}^{\nu} \xi_{rh} = a_h + \xi_h \tag{5}$$

where the integer a_h is determined after the conditions

$$0 \leq \xi_p < 1. \tag{6}$$

The inequalities (4) can be replaced by the following ones:

$$y_r \geq 0, \;\; (r = 1, 2, \ldots, \nu) \text{ and } \sum_{r=1}^{\nu} y_r \leq m - a_h - 1.$$

The number of systems $(y_1, y_2, \ldots, y_\nu)$ of integers $y_1, y_2, \ldots, y_\nu$ verifying these inequalities is equal to

$$\frac{(m - a_h)(m + 1 - a_h) \cdots (m + \nu - 1 - a_h)}{1 \cdot 2 \cdots \nu.}$$

By replacing with $a_{hk}^{(\nu)}$ the number a_h corresponding to the various sets (3), one obtains the formula

$$m_k^{(\nu)} = \sum_{h=1}^{\omega_k^{(\nu)}} \frac{(m - a_{hk}^{(\nu)})(m + 1 - a_{hk}^{(\nu)}) \cdots (m + \nu - 1 - a_{hk}^{(\nu)})}{1 \cdot 2 \cdots \nu}. \tag{7}$$

By substituting in the equality (1), one finds

$$\sum_{\nu=0}^{n} \sum_{k=1}^{\sigma_\nu} \sum_{h=1}^{\omega_k^{(\nu)}} \frac{(m - a_{hk}^{(\nu)}) \cdots (m + \nu - 1 - a_{hk}^{(\nu)})}{1 \cdot 2 \cdots \nu} = m^n. \tag{8}$$

The formula obtained holds, whatever may be the positive integer value of m. One concludes this that this formula presents an identity.

65 By comparing the coefficients of m^n in the formula (8), one finds

$$\sum_{k=1}^{\sigma_n} \omega_k^n = n!.$$

It follows that

$$\sigma_n \leq n!.$$

Let us introduce in our studies the finite difference of different orders by defining them by the formula

$$\Delta^{(\mu)} f(m) = \sum_{k=0}^{\mu} (-1)^{\mu-k} \frac{\mu!}{k!(\mu-k)!} f(m+\mu).$$

The formula (8) gives

$$\sum_{\nu=\mu}^{n} \sum_{k=1}^{\sigma_\nu} \sum_{h=1}^{\omega_k^{(\nu)}} \frac{(m+\mu-a_{hk}^{(\nu)})\cdots(m+\nu-1-a_{hk}^{(\nu)})}{1\cdot 2\cdots(\nu-\mu)} = \Delta^{(\mu)}(m^n). \qquad (\mu = 0,1,2,\ldots,n)$$

By making $m = 1$ in this formula and by noticing that

$$\frac{(\mu+1-a_{hk}^{(\nu)})\cdots(\nu-a_{hk}^{(\nu)})}{1\cdot 2\cdots(\nu-\mu)} \geq 0$$

since, because of (5) and (6)

$$a_{hk}^{(\nu)} \leq \nu,$$

one finds

$$\sum_{k=1}^{\sigma_\mu} \omega_k^{(\mu)} \leq \Delta^{(\mu)}(m^n)_{m=1}. \quad (\mu = 0,1,2,\ldots,n) \tag{9}$$

It follows that

$$\sigma_\mu \leq \Delta^{(\mu)}(m^n)_{m=1}. \quad (\mu = 0,1,2,\ldots,n)$$

We have seen in Number 60 that

$$S_\nu = (n+1-\nu)\sigma_{n-\nu}, \quad (\nu = 0,1,2,\ldots,n) \tag{10}$$

therefore

$$S_\nu \leq (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}. \quad (\nu = 0,1,2,\ldots,n)$$

66

Let us examine the conditions which have to be fulfilled for the symbols S_ν $(\nu = 0,1,2,\ldots,n)$ to be expressed by the formula

$$S_\nu = (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}. \quad (\nu = 0,1,2,\ldots,n) \tag{11}$$

By virtue of inequalities (9), it is necessary that

$$\omega_k^{(\mu)} = 1. \quad (k = 1,2,\ldots,\sigma_\mu; \mu = 0,1,2,\ldots,n) \tag{12}$$

These are the conditions necessary and sufficient for the formula (11) to hold. In effect, in the case $\omega_k^{(\nu)} = 1$, the formula (7) becomes

$$m_k^{(\nu)} = \frac{(m-\nu)(m+1-\nu)\cdots(m-1)}{1\cdot 2\cdots \nu},$$

and the equality (8) takes the form

$$\sum_{\nu=0}^{n} \sigma_\nu \frac{(m-\nu)\cdots(m-1)}{1\cdot 2\cdots\nu} = m^n.$$

It follows that

$$\sum_{\nu=\mu}^{n} \sigma_\nu \frac{(m+\mu-\nu)\cdots(m-1)}{1\cdot 2\cdots(\nu-\mu)} = \Delta^{(\mu)}(m^n),$$

and by making $m = 1$, one obtains

$$\sigma_\mu = \Delta^{(\mu)}(m^n)_{m=1}. \quad (\mu = 0,1,2,\ldots,n)$$

It results in, because of (10), the formula (11).

Let us notice that the conditions (12) come down to a single condition

$$\sigma_n = n!.$$

We will see that there exists primitive parallelohedra which satisfy this condition.

67

Theorem. The faces in 1, 2, 3 and 4 dimensions of simplexes of the set (L) enjoy the property that

$$\omega_k^(\nu) = 1. \quad (k = 1,2,\ldots,\sigma_\nu; \nu = 1,2,3,4)$$

The demonstration of the theorem introduced does not present difficulties.

Corollary. The number of faces in different dimensions of primitive parallelohedra in the space of 2, 3 and 4 dimensions is expressed by the formula (11).

1. By making in the formula (11) $n = 2$, one obtains

$$S_0 = 6 \text{ and } S_1 = 6.$$

2. By making in the formula (11) $n = 3$, one obtains

$$S_0 = 24, \quad S_1 = 36, \quad S_2 = 14.$$

3. By making in the formula (11) $n = 4$, one obtains

$$S_0 = 120, \quad S_1 = 240, \quad S_2 = 150, \quad S_3 = 30.$$

By studying the primitive parallelohedra in the space of 5 dimensions, I have come across parallelohedra the number of faces of which is not expressed by the formula (11).

68 We have seen that, in the case $\omega_k^{(\nu)} = 1$, one has

$$a_{kh}^{(\nu)} = \nu.$$

It is easy to demonstrate that, in the case $\omega_k^{(\nu)} > 1$, one will have this equality for a single set (3) which is composed of integers; for all the sets which remain, one will have the inequalities

$$2 \leq a_{hk}^{(\nu)} \leq \nu - 2. \quad (\nu \geq 5) \tag{13}$$

Let us make in the formula (8) $m = 0$. By noticing that

$$\frac{(-a_{hk}^{(\nu)})(1 - a_{hk}^{(\nu)}) \cdots (\nu - 1 - a_{hk}^{(\nu)})}{1 \cdot 2 \cdots \nu} = 0$$

so long as $a_{hk}^{(\nu)} \neq \nu$, one finds

$$\sum_{\nu=0}^{n} (-1)^\nu \sigma_\nu = 0.$$

By making in the formula (8) $m = -1$, one obtains, because of (13),

$$\sum_{\nu=0}^{n} (-1)^\nu (\nu + 1) \sigma_\nu = (-1)^n.$$

By substituting in this formula the expression of σ_ν derived from the formula (10), one will have

$$\sum_{\nu=0}^{n} (-1)^\nu S_\nu = 1. \tag{14}$$

Let us notice that the equality obtained expresses a property of faces in different dimensions of primitive parallelohedra which is common to all the convex polyhedra of the space in n dimensions. † By making in the formula (14) $n = 3$, one will have

$$S_0 - S_1 + S_2 - S_3 = 1,$$

and as $S_3 = 1$, this becomes

$$S_0 + S_2 = 2 + S_1.$$

This is the well known formula of Euler. ‡

Regulators and characteristics of edges of primitive parallelohedra.

69 Let us examine the set (R) of primitive parallelohedra belonging to a type of parallelohedra characterised by a set (L) of simplexes.

Let (α_i) be a vertex of parallelohedra of the set (R) determined by the equations

$$\sum \sum a_{ij} l_{ik} l_{jk} + 2 \sum \alpha_i l_{ik} = A. \quad (k = 1, 2, \ldots, n) \tag{1}$$

The system L correlative to the vertex (α_i) is characterised by the systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}). \tag{2}$$

Let us indicate by $(\alpha_{ik}$ $(k = 0, 1, 2, \ldots, n))$ the vertices adjacent to the vertex (α_i) (Number 18). The simplex L_k $(k = 0, 1, 2, \ldots, n)$ correlative to the vertex (α_{ik}) will be characterised by the systems which one obtains from system (2) by replacing the vertex (l_{ik}) of the simplex L by a corresponding vertex (l'_{ik}) of the simplex L_k. The two simplexes L and L_k are contiguous by a face in $n-1$ dimensions P_k $(k = 0, 1, 2, \ldots, n)$ which is characterised by the systems

$$(l_{ih}). \quad (k = 0, 1, 2, \ldots, n; h \neq k)$$

---

† See: *Poincaré*, Sur la généralisation d'un théorème d'Euler relatif aux polyèdres. [On the generalisation of the theorem of Euler relative to the polyhedra] (Comptes Rendus des Séances de l'Académie de Paris, V. 117, p. 144)

‡ Euler, Elementa doctrinae Solidorum. (Novi Comment. Petrop. 1758.)

The face P_k of simplexes L and L_k is correlative to an edge $[\alpha_i, \alpha_{ik}]$ of parallelohedra of the set (R).

Let us indicate by

$$\sum p_{ik}x_i = \delta_k, \quad (k = 0, 1, 2, \ldots, n)$$

the equation of the face P_k. As one has

$$\sum p_{ik}l_{ik} = \delta_k, \quad (h = 0, 1, 2, \ldots, n; h \neq k)$$

it becomes

$$\sum p_{ik}(l_{ik} - l_{ir}) = 0. \ \ (h = 0, 1, 2, \ldots, n; r = 0, 1, 2, \ldots, n; h \neq k; r \neq k) \tag{3}$$

The equalities obtained define the number $p_{1k}, p_{2k}, \ldots, p_{nk}$ to a common factor close by. By supposing that $p_{1k}, p_{2k}, \ldots, p_{nk}$ be integer not having common divisor, one will determine by the equality (3) two systems (p_{ik}) and $(-p_{ik})$. One will call characteristic of the edge $[\alpha, \alpha_{ik}]$ or of the correlative face P_k one of the two systems $\pm(p_{ik})$ likewise.

By noticing that

$$\sum p_{ik}l_{ik} \neq \delta_k,$$

one will attach, for more precision, a supplementary condition

$$\sum p_{ik}l_{ik} > \delta_k.$$

Definition. One will call characteristic of the face P_k with regard to the simplex L the system (p_{ik}) which is well defined by the conditions

$$\sum p_{ik}l_{ik} > \delta_k, \ \sum p_{ik}l_{ih} = \delta_k. \ \ (h = 0, 1, 2, \ldots, n, h \neq k) \tag{4}$$

Let us notice that the characteristic of the face P_k with regard to the simplex L_k will be the system $(-p_{ik})$. In effect, one will have

$$\sum p_{ik}l'_{ik} \neq \delta_k.$$

Let us suppose that

$$\sum p_{ik}l'_{ik} > \delta_k.$$

In this case the two simplexes L and L_k would be situated on the same side of the face P_k, and one could find a point interior to the simplex L which would be interior to the simplex L_k too, this is contrary to Theorem I demonstrated in Number (6). It is therefore necessary that

$$\sum p_{ik}l'_{ik} < \delta_k,$$

and the system $(-p_{ik})$ presents the characteristic of the face P_k with regard to the simplex L_k.

70 One will determine the vertex (α_{ik}) $(k = 0, 1, 2, \ldots, n)$ correlative to the simplex L_k by the equations

$$\sum\sum a_{ij}l_{ih}l_{jh} + 2\sum \alpha_{ik}l_{ih} = A_k \ \ (h = 0, 1, 2, \ldots, n; h \neq k) \tag{5}$$

and

$$\sum\sum a_{ij}l'_{ik}l'_{jk} + 2\sum \alpha_{ik}l'_{ik} = A_k. \tag{6}$$

From the equalities (1) and (5), one derive

$$2\sum(\alpha_{ik} - \alpha_i)l_{ih} = A_k - A. \ \ (h = 0, 1, 2, \ldots, n; h \neq k) \tag{7}$$

As a result because of (3), one will have

$$\alpha_{ik} - \alpha_i = p_{ik}\rho_k. \ \ (i = 1, 2, \ldots, n; k = 0, 1, 2, \ldots, n) \tag{8}$$

On the ground of the supposition made, the vertices (α_i) and (α_{ik}) $(k = 0, 1, 2, \ldots, n)$ of primitive parallelohedra of the set (R) are simple.

It follows that,

$$\sum\sum a_{ij}l'_{ik}l'_{jk} + 2\sum \alpha_{ik}l'_{ik} > A$$

and

$$\sum\sum a_{ij}l_{ik}l_{i[\text{sic}]k} + 2\sum \alpha_{ik}l_{ik} > A_k.$$

By virtue of (1) and (6), one obtains

$$2\sum(\alpha_{ik} - \alpha_i)l_{ik} > A_k - A \ \text{ and } \ 2\sum(\alpha_{ik} - \alpha_i)l'_{ik} < A_k - A$$

and, because of (8), it becomes

$$2\rho_k\sum p_{ik}l_{ik} > A_k - A \ \text{ and } \ 2\rho_k\sum p_{ik}l'_{ik} < A_k - A. \tag{9}$$

As by virtue of (7) and (8), one has

$$2\rho_k \sum p_{ik} l_{ih} = A_k - A, \quad (h = 0, 1, 2, \ldots, n; h \neq k) \tag{10}$$

the inequalities (9) can be written

$$2\rho_k \sum p_{ik}(l_{ik} - l_{ih}) > 0,\ 2\rho_k \sum p_{ik}(l'_{ik} - l_{ih}) < 0. \qquad (h = 0, 1, 2, \ldots, n; h \neq k) \tag{11}$$

By noticing that because of (4)

$$\sum p_{ik}(l_i - l_{ih}) > 0, \quad (h = 0, 1, 2, \ldots, n; h \neq k) \tag{12}$$

one finds

$$\rho_k > 0, \quad (k = 0, 1, 2, \ldots, n) \tag{13}$$

and the second inequality (11) gives

$$\sum p_{ik}(l'_{ik} - l_{ih}) < 0, \quad (h = 0, 1, 2, \ldots, n; h \neq k) \tag{14}$$

or differently, because of (4),

$$\sum p_{ik} l'_{ik} < \delta_k, \quad (k = 0, 1, 2, \ldots, n) \tag{15}$$

that which we have demonstrated by another method.

71 By substituting in (6) the expression of α_{ik} derived from the equality (8), one obtains

$$\sum\sum a_{ij} l'_{ik} l'_{jk} + 2 \sum \alpha_i l'_{ik} + 2\rho_k \sum p_{ik} l'_{ik} = A_k.$$

One will present this equality, because of (1), under the form

$$\sum\sum a_{ij} l'_{ik} l'_{jk} + 2 \sum \alpha_i l'_{0k} - \sum\sum a_{ij} l_{ik} l_{jk} - 2 \sum \alpha_i l_{ik} = A_k - A - 2\rho_k \sum p_{ik} l'_{ik},$$

and lastly, by virtue of (10),

$$2\rho_k \sum p_{ik}(l_{ih} - l'_{ik}) = \sum\sum a_{ij} l'_{ik} l_{jk} + 2 \sum \alpha_i l'_{ik} - \sum\sum \alpha_{ij} l_{ik} l_{jk} - 2 \sum \alpha_i l_{ik} \tag{16}$$

where $h = 0, 1, 2, \ldots, n, h \neq k, k = 0, 1, 2, \ldots, n$.

Definition II. One will call regulator of the edge $[\alpha_i, \alpha_{ik}]$ or of the correlative face P_k the positive parameter ρ_k defined by the formulae (8) and (16).

Let us notice that on the ground of equalities (3) and (8) the congruent edges and the congruent correlative faces possess the same regulator and the same characteristic.

One can determine the regulator ρ_k by other formulae.

Let us write

$$l'_{ik} = \sum_{r=0}^{n} \vartheta_r^{(k)} l_{ir} \text{ where } \sum_{r=0}^{n} \vartheta_r^{(k)} = 1. \quad (k = 1, 2, \ldots, n) \tag{17}$$

On the grounds of equations (1) and (17), one obtains

$$\sum\sum a_{ij} l'_{ik} l'_{jk} + 2 \sum \alpha_i l'_{ik} - \sum\sum a_{ij} l_{ik} l_{jk} - 2 \sum \alpha_i l_{ik} = \sum\sum a_{ij} l'_{ik} l'_{jk} - \sum_{r=0}^{n} \vartheta_r^{(k)} \sum\sum a_{ij} l_{ir} l_{jr}.$$

By substituting in the formula (16), one finds

$$2\rho_k \sum p_{ik}(l_{ih} - l'_{ik}) = \sum\sum a_{ij} l'_{ik} l'_{jk} - \sum_{r=0}^{n} \vartheta_r^{(k)} \sum\sum a_{ij} l_{ir} l_{jr} \tag{18}$$

where $h = 0, 1, 2, \ldots, n; h \neq k; k = 0, 1, 2, \ldots, n$.

The formula obtained makes visible an important property of the regulator ρ_k: *the regulator ρ_k is expressed by a linear function of coefficients of the quadratic form $\sum\sum a_{ij} x_i x_j$.* By writing

$$\rho_k = \sum\sum p_{ij}^{(k)} a_{ij}, \tag{19}$$

one will have the rational coefficients $p_{ij}^{(k)} = p_{ji}^{(k)}$, $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$.

72 By virtue of the formula (19), the regulator ρ_k will be perfectly determined if one knows the corresponding coefficients $p_{ij}^{(k)}$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$.

As the coefficients a_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$ of the quadratic form $\sum\sum a_{ij}x_ix_j$ do not play any role in the determination of coefficients $p_{ij}^{(k)}$ $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$ which depend only on the simplexes L and L_k, one can replace in the previous formula the coefficients a_{ij} by the coefficients x_ix_j $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$.

By introducing the linear functions, as we have done in Number 58,

$$u_r = \sum l_{ir}x_i, \; v_r = \sum l'_{ir}x_i, \quad (r = 0, 1, 2, \ldots, n)$$

let us indicate by

$$u_r^{(k)} \text{ and } v_r^{(k)} \quad (r = 0, 1, 2, \ldots, n)$$

the values of these functions which correspond to the values of variables $x_1, x_2, \ldots, x_n$

$$x_i = p_{ik}. \quad (k = 0, 1, 2, \ldots, n)$$

By virtue of (4), one will have

$$u_h^{(k)} = \delta_k. \quad (h = 0, 1, 2, \ldots, n; h \neq k, u_k^{(k)} > \delta_k)$$

By virtue of (15), one will have

$$v_k^{(k)} < \delta_k.$$

Let us notice that the numbers $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$ defined by the equality (7) will be determined by the equalities

$$v_k = \sum_{r=0}^{n} \vartheta_r^{(k)} u_r \text{ where } \sum \vartheta_r^{(k)} = 1.$$

By replacing in the formula (18) the coefficients a_{ij} by the coefficients x_ix_j, $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$, one obtaines

$$2\rho_k(\delta_k - v_k^{(k)}) = (v_k)^2 - \sum_{r=0}^{n} \vartheta_r^{(k)}(u_r)^2. \quad (k = 0, 1, 2, \ldots, n) \qquad 20$$

To return the formula obtained to the formula (19), it suffices to replace in the equality

$$\rho_k = \sum\sum p_{ij}^{(k)} x_ix_j$$

the coefficients x_ix_j by the coefficients a_{ij}, $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$.

Fundamental transformation of the form

$$\sum\sum a_{ij}x_ix_j + 2\sum \alpha_ix_i - \sum\sum a_{ij}l_il_j - 2\sum \alpha_il_i$$

73 By keeping the previous notations, let us indicate

$$F_{(L)}(x_1, x_2, \ldots, x_n) = \sum\sum a_{ij}x_ix_j + 2\sum \alpha_ix_i - A \qquad (1)$$

where one has admitted

$$A = \sum\sum a_{ij}l_{ik}l_{jk} + 2\sum \alpha_il_{ik}. \quad (k = 0, 1, 2, \ldots, n) \qquad (2)$$

By introducing the variables $\xi_0, \xi_1, \ldots, \xi_n$ after the conditions

$$x_i = \sum_{r=0}^{n} \xi_r l_{ir} \text{ where } \sum_{r=0}^{n} \xi_r = 1, \qquad (3)$$

one will present the function $F_{(L)}$ $(x_1, x_2, \ldots, x_n)$ under the following form:

$$F_{(L)}(x_1, x_2, \ldots, x_n) = \sum\sum a_{ij}x_ix_j - \sum_{r=0}^{n} \xi_r \sum\sum a_{ij}l_{ir}l_{jr}. \qquad (4)$$

One concludes that the function $F_{(L)}$ $(x_1, x_2, \ldots, x_n)$ is linear with regard to the coefficients $a_{ij}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$ of an arbitrary quadratic form $\sum\sum a_{ij}x_ix_j$.

By making in the formula (1) $x_i = l_{ik}$, one obtains, because of (2),

$$F_{(L)}(l_{1k}, l_{2k}, \ldots, l_{nk}) = 0. \quad (k = 0, 1, 2, \ldots, n)$$

The equalities obtained hold, whatever may be the values of a_{ij} , $i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$.

Let us indicate by L^0 a simplex congruent to the simplex L and characterised by the vertices

$$(l_{i0} + l_i), (l_{i1} + l_i), \ldots, (l_{in} + l_i),$$

$l_1, l_2, \ldots, l_n$ being arbitrary integers.

By noticing that because of (3)

$$x_i + l_i = \sum_{r=0}^{n} \xi_r (l_{ir} + l_i) \text{ where } \sum_{r=0}^{n} \xi_r = 1,$$

one will have an equality

$$F_{(L^0)}(x_1 + l_1, x_2 + l_2, \ldots, x_n + l_n) = \sum\sum a_{ij}(x_i + l_i)(x_j + l_j) - \sum_{r=0}^{n} \xi_r \sum\sum a_{ij}(l_{ir} + l_i)(l_{jr} + l_j)$$

and after the reductions, it becomes

$$F_{(L^0)}(x_1 + l_1, x_2 + l_2, \ldots, x_n + l_n) = \sum\sum a_{ij} x_i x_j - \sum_{r=0}^{n} \xi_r \sum\sum a_{ij} l_{ir} l_{jr},$$

therefore, because of (4) one will have

$$F_{(L^0)}(x_1 + l_1, x_2 + l_2, \ldots, x_n + l_n) = F_{(L)}(x_1, x_2, \ldots, x_n). \tag{5}$$

By virtue of the formula (18) of Number 71, one will determine the regulator ρ_k in the formula

$$2\rho_k \sum p_{ik}(l_{ih} - l'_{ik}) = F_{(L)}(l'_{1k}, l'_{2k}, \ldots, l'_{nk}). \quad (k = 0, 1, 2, \ldots, n) \tag{7}$$

74

Let us indicate

$$F_{L_k}(x_1, x_2, \ldots, x_n) = \sum\sum a_{ij} x_i x_j + 2\sum \alpha_{ik} x_i - A_k, \; (k = 0, 1, 2, \ldots, n)$$

L_k being a simplex contiguous to the simplex L by the face P_k $(k = 0, 1, 2, \ldots, n)$.

By substituting in this equality the expression of α_{ik} defined from the formula (8) of Number 70, one obtains

$$F_{L_k}(x_1, x_2, \ldots, x_n) = \sum\sum a_{ij} x_i x_j + 2\sum \alpha_i x_i + 2\rho_k \sum p_{ik} x_i - A_k$$

and, because of (1), one will have

$$F_{L_k}(x_1, x_2, \ldots, x_n) = F_{(L)}(x_1, x_2, \ldots, x_n) + 2\rho_k \sum p_{ik} x_i + A - A_k.$$

By substituting in this equality the expression of $A - A_k$ given by the formula (10) of Number 70, one finds

$$F_{L_k}(x_1, x_2, \ldots, x_n) = F_{(L)}(x_1, x_2, \ldots, x_n) + 2\rho_r \sum p_{ik}(x_i - l_{ih}). \; (h \neq k)$$

This formula can be written

$$F_{(L)}(x_1, x_2, \ldots, x_n) = F_{L_k}(x_1, x_2, \ldots, x_n) + 2\rho_k \sum p_{ik}(l_{ih} - x_i). \qquad (h \neq k, k = 0, 1, 2, \ldots, n) \tag{*}$$

The formula (*) obtained is capable of numerous and important applications.

75

Let us suppose that $x_1, x_2, \ldots, x_n$ be arbitrary integers and that the point (x_i) is not found among the vertices

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}) \tag{7}$$

of the simplex L. By admitting

$$x_i = \sum_{r=0}^{n} \xi_r l_{ir} \text{ where } \sum_{r=0}^{n} \xi_r = 1, \tag{8}$$

one will have by virtue of Theorem II of Number 62 among the numbers $\xi_0, \xi_1, \ldots, \xi_n$ at least one negative number. Let us suppose to fix the ideas that

$$\xi_k < 0. \tag{9}$$

By noticing that because of (3) and of the formula (4) in Number 69 one has

$$\sum p_{ik}(l_{ih} - x_i) = \xi_k \sum p_{ik}(l_{ih} - l_{ik})$$

and that

$$\sum p_{ik}(l_{ih} - l_{ik}) < 0, \quad (h \neq k)$$

one obtains, because of (9),

$$\sum p_{ik}(l_{ih} - x_i) > 0.$$

One concludes that the coefficient of $2\rho_k$ in the formula (*) is an integer and positive in the case considered. In the same manner, one will examine the function $F_{L_k}(x_1, x_2, \ldots, x_n)$ and so on.

76

Let us suppose that one have examined the simplex,

$$L, L', L'', \ldots, L^{(m)} \tag{10}$$

successively contiguous by the faces in $n-1$ dimensions the regulator of which present the function

$$\rho_1, \rho_2, \ldots, \rho_m.$$

Let us suppose that by applying the formula $(^*)$ to the simplexes (10) one have obtained the equalities

$$F_L(x_1, x_2, \ldots, x_n) = F_{L'}(x_1, x_2, \ldots, x_n) + 2h_1\rho_1 \text{ where } h_1 > 0,$$
$$F_{L'}(x_1, x_2, \ldots, x_n) = F_{L''}(x_1, x_2, \ldots, x_n) + 2h_2\rho_2 \text{ where } h_2 > 0,$$
$$\ldots$$
$$F_{L^{(m-1)}}(x_1, x_2, \ldots, x_n) = F_{L^{(m)}}(x_1, x_2, \ldots, x_n) + 2h_m\rho_m \text{ where } h_m > 0.$$

It follows that

$$F_L(x_1, x_2, \ldots, x_n) = 2\sum_{k=1}^{m} h_k\rho_k + F_{L^{(m)}}(x_1, x_2, \ldots, x_n). \tag{11}$$

The procedure shown can not be prolonged indefinitely and one will always arrive at a simplex $L^{(m)}$ among the vertices of which is found the point (x_i).

To demonstrate this, let us notice that the coefficients $a_{ij}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$ of the quadratic form $\sum\sum a_{ij}x_ix_j$ in the formulae obtained are arbitrary.

Let us suppose that one have chosen the positive quadratic form $\sum\sum a_{ij}x_ix_j$ which defines a set (R) of primitive parallelohedra belonging to the type characterised by the set (L) of simplexes.

We have seen in Number 70 that one will have the inequalities

$$\rho_k > 0. \quad (k = 1, 2, \ldots, m)$$

By virtue of the definition of the function $F_L(x_1, x_2, \ldots, x_n)$, one will have an inequality

$$F_L(x_1, x_2, \ldots, x_n) > 0,$$

whatever the integer values of $x_1, x_2, \ldots, x_n$ may be, abstraction made from vertices (7) of the simplex L. It results in

$$F_{(L^{(m)})}(x_1, x_2, \ldots, x_n) \geq 0$$

and the formula (11), in the case considered, gives

$$F_L(x_1, x_2, \ldots, x_n) \geq 2\sum_{k=1}^{m} h_k\rho_k.$$

As the coefficients h_k $(k = 1, 2, \ldots, m)$ are of positive integers and the regulators ρ_k $(k = 1, 2, \ldots, m)$ belong to a series of regulators corresponding to the incongruent faces of simplexes of the set (L), one concludes that the number m can not be increased indefinitely. As a result the series (10) will be terminated by a simplex $L^{(m)}$ among the vertices of which is found the point (x_i).

It follows that one will have indentically

$$F_{(L^{(m)})}(x_1, x_2, \ldots, x_n) = 0,$$

and the formula (11) becomes

$$F_L(x_1, x_2, \ldots, x_n) = 2\sum_{[k]=1}^{m} h_k\rho_k \text{ where } (k = 1, 2, \ldots, m) \tag{12}$$

Let us notice that the formula obtained presents an identity which holds, whatever the values of coefficients $a_{ij}, i = 1, 2, \ldots, n, j = 1, 2, \ldots, n$ may be, provided that the regulators ρ_k $(k = 1, 2, \ldots, m)$ are expressed by the formula (6).

77 *Fundamental theorem. Let us suppose that the regulators ρ_k $(k = 1, 2, \ldots, \sigma)$ corresponding to the various incongruent faces in $n-1$ dimensions of simplexes belonging to the set (L) be determined by the equations*

$$\rho_k = \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}^{(k)}a_{ij}. \quad (k = 1, 2, \ldots, \sigma)$$

For a quadratic form $\sum\sum a_{ij}x_ix_j$ to define a set (R) primitive parallelohedra belonging to the type characterised by the set (L) of simplexes, it is necessary and sufficient that the inequalities

$$\rho_k = \sum\sum p_{ij}^{(k)}a_{ij} > 0, \quad (k = 1, 2, \ldots, \sigma)$$

hold.

We have seen in Number 70 that the inequalities

$$\rho_k > 0, \quad (k = 1, 2, \ldots, \sigma) \tag{13}$$

present the necessary conditions. Let us suppose the coefficients of a quadratic form $\sum\sum a_{ij}x_ix_j$ verify the inequalities (13). By virtue of the formula (12), one will have the inequality

$$F_{(L)}(x_1, x_2, \ldots, x_n) > 0$$

so long as the point (x_i) the element s of which are integers is not found among the vertices of the simplex L. By virtue of the definition established, the simplex L is in this case correlative to a simplex vertex (α_i) of parallelohedra corresponding to the quadratic form examined $\sum\sum a_{ij}x_ix_j$.

The simplex L is chosen arbitrary in the set (L) of simplexes, therefore all the simplexes of the set (L) are correlative to the simple vertices of parallelohedra corresponding to the quadratic form $\sum\sum a_{ij}x_ix_j$.

I argue that these parallelohedra do not possess other vertices, it is that which one will verify without trouble.

78 Let us notice that any quadratic form $\sum\sum a_{ij}x_ix_j$ verifying the inequalities (13) is positive. To demonstrate this, let us examine a simplex L among the vertices in which is found the point (0). One will have in this case

$$F_{(L)}(x_1, x_2, \dots, x_n) = \sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i,$$

and consequently

$$F_{(L)}(x_1, x_2, \dots, x_n) + F_{(L)}(-x_1, -x_2, \dots, -x_n) = 2\sum\sum a_{ij}x_ix_j.$$

The two points (x_i) and $(-x_i)$ can not be the vertices of the simplex L, the point (0) being excluded. This results in

$$F_{(L)}(x_1, x_2, \dots, x_n) + F_{(L)}(-x_1, -x_2, \dots, -x_n) > 0,$$

therefore

$$\sum\sum a_{ij}x_ix_j > 0,$$

whatever may be the integer values of $x_1, x_2, \dots, x_n$, the system $x_1 = 0, x_2 = 0, \dots, x_n = 0$ being excluded.

Definition of quadratic forms with the help of regulators and corresponding characteristics.

79 Let us take any one quadratic form $\sum\sum a_{ij}x_ix_j$ in arbitrary coefficients. Let us choose n numbers $x_1, x_2, \dots, x_n$ which are subject to the only condition: *the equality*

$$h_1x_2 + h_2x_2 + \dots + h_nx_n = 0$$

is impossible so long as the numbers $h_1, h_2, \dots, h_n$ are integers.

Let us examine a vector g made up of points

$$\frac{l_1}{m} + ux_i \text{ where } 0 \le u \le 1,$$

$l_1, l_2, \dots, l_n$ being arbitrary integers and m being any one positive integer.

The vector g will traverse a certain number of simplexes belonging to the set (L). Let us indicate by

$$L_0, L_1, \dots, L_s \tag{1}$$

the simplexes of the set (L) which contain the various parts of the set g. On the ground of the supposition made, the simplexes (1) are well defined by the vector g and are successively contiguous by the faces in $n-1$ dimensions. In effect, two adjacent simplexes L_k and L_{k+1} of the series (1) possess a common point (ξ_{ik}) belonging to the vector g, therefore the simplexes L_k and L_{k+1} are contiguous by a face in any number of dimensions. Let us suppose that this face be characterised by the systems

$$(l_{i0}), (l_{i1}), \dots, (l_{i\nu}). \tag{2}$$

As

$$\xi_{ik} = \frac{l_i}{m} + u_kx_i \text{ where } 0 < u_k < 1,$$

one will have

$$\frac{l_i}{m} + u_kx_i = \sum_{r=0}^{\nu} \vartheta_r l_{ir} \text{ where } \sum \vartheta_r = 1 \text{ and } \vartheta_r > 0. \ (r = 0, 1, 2, \dots, \nu) \tag{3}$$

By supposing that $\nu < n-1$, one wil determine with the help of these equalities a system (h_i) of integers verifying the equation

$$h_1x_1 + h_2x_2 + \dots + h_nx_n = 0,$$

which is contrary to the hypothesis, therefore it is necessary that $\nu = n-1$ and that the point (ξ_{ik}) be interior to a face in $n-1$ dimensions which is common to the simplexes L_k and L_{k+1}.

Let us suppose that $\nu = n-1$. By indicating with (p_{ik}) the characteristic of the face (P_k) characterised by the systems (2) with regard to the simplex L_k, one will have, by virtue of the formula (4) of Number 69,

$$\sum p_{ik}l_{ir} = \delta_k, \quad (r = 0, 1, 2, \dots, n-1)$$

and the equalities (3) give

$$\sum p_i(\frac{l_i}{m} + u_kx_i) = \delta_k$$

and consequently

$$u_k \sum p_{ik}x_i = \delta - \sum p_{ik}\frac{l_i}{m}.$$

As $\sum p_{ik}x_i \neq 0$, on the ground of the supposition made, the equality obtained defines a point (ξ_{ik}) of the vector g which is interior to the face P_k. This results in that the vector g does not possess other points common to the face P_k. By attributing to the parameter u a negative variation δu sufficiently small, one will define a point $\frac{l_i}{m} + (u_k + \delta u)x_i$ of the vector g which is interior to the simplex L_k. By attributing to the parameter u a variable $\delta u > 0$, one obtains a point $\frac{l_i}{m} + (u_k + \delta u)x_i$ which is interior to the simplex L_{k+1}.

As in these cases one has

$$\sum p_{ik}(\frac{l_i}{m} + (u_k + \delta u)x_i) > \delta_k, \quad (\delta u < 0)$$

and

$$\sum p_{ik}(\frac{l_i}{m} + (u_k + \delta u)x_i) < \delta_k, \quad (\delta u > 0)$$

it becomes

$$\sum p_{ik}x_i < 0. \tag{4}$$

By indicating with ρ_k the regulator of the face P_k with regard to the chosen quadratic form $\sum a_{ij}x_ik_j$ $(k = 0, 1, 2, \ldots, s-1)$, let us apply the formula $(^*)$ of Number 74 to the simplexes (1). One will have the equalities

$$\begin{aligned} F_{(L_0)}(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n) &= F_{L_1}(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n) \\ &+ 2\rho_0 \sum p_{i0}(l_{ih_0} - \frac{l_i}{m} - x_i), \\ &\ldots \\ F_{L_{s-1}}(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n) &= F_{L_s}(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n) \\ &+ 2\rho_{s-1} \sum p_{i,s-1}(l_{ih_{s-1}} - \frac{l_i}{m} - x_i). \end{aligned}$$

It follows that

$$\begin{aligned} F_{(L_0)}(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n) &= 2\sum_{k=0}^{s-1} \rho_k \sum i = 1^n p_{ik}(l_{ih_k} - \frac{l_i}{m} - x_i) \\ &+ F_{L_s}(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n). \end{aligned} \tag{5}$$

Until now, the integers $l_1, l_2, \ldots, l_n$ had been arbitrary. Let us suppose that the integers $l_1, l_2, \ldots, l_n$ satisfy the conditions

$$0 \leq l_i < m. \ (i = 1, 2, \ldots, n) \tag{6}$$

Let us indicate by K the set of incongruent points $(\frac{l_i}{m})$ verifying these inequalities.

The number of points belonging to the set K is equal to m.

Let us apply the formula (5) to all the points of the set K and make the sum of equalities obtained. One will have a formula

$$\begin{aligned} &F_{(L_0)}(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n) = \\ &2\sum \rho_k \sum p_{ik}(l_{ih_k} - \frac{l_i}{m} - x_i) + \sum F_{(L_s)}(\frac{l_1}{m} + x_i, \ldots, \frac{l_n}{m} + x_n). \end{aligned} \tag{7}$$

All the sums which are formed in this formula can be determined with a certain approximation.

80

Let us suppose that the simplex L_0 be characterised by the systems

$$(l_{i0}), (l_{i1}), \ldots, (l_{in}).$$

On the ground of the supposition made, the point $\left(\frac{l_i}{m}\right)$ belongs to the simplex L_0. As there exist only a finite number of simplexes of the set L to which belong the points $\left(\frac{l_i}{m}\right)$ verifying the inequalities (6), one concludes that one can determine a positive parameter λ in such a manner that the inequalities

$$|l_{ik}| \leq \lambda \ (i = 1, 2, \ldots, n; k = 0, 1, 2, \ldots, n) \tag{8}$$

holds.

In this case, the corresponding value of the function

$$F_{(L_0)}\left(\frac{l_1}{m} + x_1, \frac{l_2}{m} + x_2, \ldots, \frac{l_n}{m} + x_n\right)$$

can be presented under the form

$$F_{(L_0)}\left(\frac{l_1}{m} + x_1, \ldots, \frac{l_n}{m} + x_n\right) = \sum\sum a_{ij}x_ix_j + \epsilon_0 + \sum \epsilon_i x_i$$

where the coefficients $\epsilon_0, \epsilon_1, \ldots, \epsilon_n$ do not exceed in numerical value a fixed limit ϵ which does not depend on coefficients of the quadratic form $\sum\sum a_{ij}x_ix_j$ and on the choice of the set (L) of simplexes.

Let us examine the function $F_{(L_s)}\left(\frac{l_1}{m}+x_1,\dots,\frac{l_n}{m}+x_n\right)$. After the proposition made, the point $\left(\frac{l_i}{m}+x_i\right)$ belongs to the simplex L_s. Let us determine the integers $t_1, t_2, \dots, t_n$ after the conditions

$$0 \le \frac{l_i}{m} + x_i + t_i < 1, \quad (i = 1, 2, \dots, m) \tag{9}$$

and indicae by L'_s the simplex congruent to the simplex L_s which obtains by a translation of the simplex L_s the length of the vector $[t_i]$.

By virtue of the formula (5) of Number 73, one will have

$$F_{(L_s)}\left(\frac{l_1}{m}+x_1,\dots,\frac{l_n}{m}+x_n\right) = F_{L'_s}\left(\frac{l_1}{m}+x_1+t_1,\dots,\frac{l_n}{m}+x_n+t_n\right). \tag{10}$$

By indicating with

$$(l'_{i0}), \dots, (l'_{in})$$

the vertices of the simplex L'_s, one will have, by virtue of (9), the inequalities (8):

$$|l'_{ik}| \le \lambda. \quad (i = 1, 2, \dots, n; k = 0, 1, 2, \dots, n)$$

It follows that the numerical value of the function

$$F_{L'_s}\left(\frac{l_1}{m}+x_1+t_1,\dots,\frac{l_n}{m}+x_n+t_n\right),$$

because of (8) and (9), does not exceed a fixed limit φ which depends only on coefficients of the quadratic form $\sum\sum a_{ij}x_ix_j$ and on the choice of the set (L) of simplexes. By virtue of (10), one can write

$$F_{(L_s)}\left(\frac{l_1}{m}+x_1,\dots,\frac{l_n}{m}+x_n\right) = \varphi_0 \text{ where } |\varphi_0| \le \varphi.$$

By substituting in the formula (7) the results obtained, one will present it under the following form

$$m_n\left(\sum\sum a_{ij}x_ix_j + \epsilon_0 + \sum \epsilon_i x_i\right) = 2\sum \rho_k \sum_{i=1}^{n} p_{ik}\left(l_{ih} - \frac{l_i}{m} - x_i\right) \tag{11}$$

In this formula, the coefficients $\epsilon_0, \epsilon_1, \dots, \epsilon_n$ do not exceed in numerical value a fixed limit which does not depend on numbers $x_1, x_2, \dots, x_n$.

81 Let us determine the coefficients of $2\rho_k$ in the formula obtained.

To that effect, let us choose any one face P in $n-1$ dimensions of simplexes of the set (L). Let us suppose that the face P is characterised by the systems

$$(0), (l_{(i1)}), \dots, (l_{(i,n-1)}).$$

Let us indicate by ρ the regulator and by $\pm(p_i)$ the characteristic of the face P with regard to the quadratic form $\sum\sum a_{ij}x_ix_j$. One will suppose, on the ground of (4), that

$$\sum p_i x_i < 0. \tag{12}$$

Let us suppose that the vector g is made up of points

$$\frac{l_i}{m} + ux_i \text{ where } 0 \le u \le 1$$

and corresponding to a point $\left(\frac{l_i}{m}\right)$ belonging to the set K possesses a point which is interior to a face P' congruent to the face P.

By supposing that the face P' is characterised by the systems

$$(g_i), (l_{i1} + g_i), \dots, (l_{i,n-1}, g_i),$$

one will have, on the ground of the supposition made,

$$\frac{l_i}{m} + ux_i = \sum_{k=0}^{n-1} \vartheta_k (g_i + l_{ik}) \text{ where } \sum_{k=0}^{n-1} \vartheta_k = 1 \text{ and } \vartheta_k > 0.$$

The corresponding value of the coefficient of 2ρ in the formula (11) is expressed by the sum

$$\sum\sum_{i=1}^{n} p_i\left(l_{ih} - \frac{l_i}{m} - x_i\right) \tag{14}$$

which extends to all the faces P' congruent to the face P verifying the equalities (13), provided that the points $\left(\frac{l_i}{m}\right)$ belong to the set K.

Let us indicate, to make short,

$$\sum_{i=1}^{n} p_i x_i = -\Delta \tag{15}$$

One will have, because of (12),

$$\Delta > 0.$$

As the system (l_{ih}) in the sum (14) indicate any one vertex of the face P', one can write down

$$l_{ih} = g_i,$$

and the equalities (13) and (15) give

$$\sum_{i=1}^{n} p_i \left(g_i - \frac{l_i}{m} - x_i \right) = (1-u)\Delta.$$

Therefore, the study of the coefficient of 2ρ in the formula (11) comes down to the evaluation of the sum

$$\sum (1-u)\Delta \text{ where } 0 < u < 1. \tag{16}$$

82 Let us designate, to summarise,

$$-mg_i + l_i = h_i. \tag{17}$$

The parameter u verifying the equalities (13) is expressed by the formula

$$u = \frac{1}{m\Delta} \sum p_i h_i, \tag{18}$$

and as $0 < u < 1$, it becomes

$$0 < \sum p_i h_i < m\Delta.$$

By indicating with τ the integer verifying the inequalities

$$0 < \tau < m\Delta, \tag{19}$$

let us write

$$\sum p_i h_i = \tau. \tag{20}$$

By virtue of (18), the corresponding value of the parameter u will be

$$u = \frac{\tau}{m\Delta}.$$

Let us substitute the expression found of the parameter u in the equalities (13), it will become, because of (17),

$$\frac{\tau}{\Delta} x_i + h_i = m \sum_{k=1}^{n-1} \vartheta_k l_{ik} \text{ where } \sum_{k=1}^{n-1} \vartheta_k < 1 \text{ and } \vartheta_k > 0. \tag{21}$$

$$(k = 1, 2, \ldots, n-1)$$

This stated, let us notice that one can attribute to the number τ an arbitrary value verifying the inequalities (19). For similar values of τ to exist, it is necessary that

$$m\Delta > 1,$$

Let us suppose that the positive integer m satisfies this condition. In this case, the finding of the sum (16) comes down to the solution of a sum

$$\sum (1-u)\Delta = \sum_{\tau > 0}^{\tau < m\Delta} m_\tau (\Delta - \frac{\tau}{m}) \tag{22}$$

where m_τ indicates the number of systems (h_i) of integers $h_1, h_2, \ldots,\ h_n$ verifying the equalities (20) and (21).

83 It is easy to determine the number m_τ.

Let us indicate by (h_i^0) a system of integers verifying the equality

$$\sum p_i h_i^0 = 1. \tag{23}$$

As, on the ground of the supposition made, the integers $p_1, p_2, \ldots, p_n$ have no common divisor, the systems of integers verifying this equality always exist.

One will determine all the systems (h_i) of integers verifying the equality (20) with the help of formulae

$$h_i = \tau h_i^0 + \sum_{k=1}^{n-1} \tau_k l_{ik} \tag{24}$$

where the rational numbers $\tau_1, \tau_2, \ldots, \tau_{n-1}$ belong to certain sets

$$\tau_k = \xi_{kr} + y_k, \quad (k = 1, 2, \ldots, n-1; r = 1, 2, \ldots, \omega) \tag{25}$$

ω being the greatest commondivisor of n determinants of the order $(n-1)^2$ which one can form from $n-1$ systems

$$(l_{i1}), (l_{i2}), \ldots, (l_{i,n-1}).$$

By substituting the expressions of $h_1, h_2, \ldots, h_n$ derived from equalities (24) in the equalities (21), one obtains

$$\frac{\tau}{\Delta} x_i + \tau h_i^0 = \sum_{k-1}^{n-1} (m\vartheta_k - \tau_k) l_{ik}. \tag{26}$$

Let us notice that the numerical value of the determinant of n systems

$$(x_i), (l_{i1}), \ldots, (l_{i,n-1})$$

is expressed by the formula

$$\pm \begin{vmatrix} x_1 & x_2 & \cdots & x_n \\ l_{11} & l_{21} & \cdots & l_{n1} \\ & \cdots & \cdots & \\ l_{1,n-1} & l_{2,n-1} & \cdots & l_{n,n-1} \end{vmatrix} = -\sum x_i p_i \omega = \omega\Delta.$$

Let us indicate by $\lambda_{1k}, \lambda_{2k}, \ldots, \lambda_{nk}$, $(k = 1, 2, \ldots, n-1)$ the minor determinants which are defined by the equalities

$$\sum_{i=1}^{n} \lambda_{ik} l_{ik} = \omega\Delta, \quad (k = 1, 2, \ldots, n-1)$$

$$\sum \lambda_{ik} x_i = 0, \ \sum \lambda_{ik} l_{ir} = 0. \quad (r = 1, 2, \ldots, n-1; r \neq k)$$

The equalities (26) give

$$\tau \sum \lambda_{ik} h_i^0 = \omega\Delta(m\vartheta_k - \tau_k), \quad (k = 1, 2, \ldots, n-1)$$

and as a result

$$m\vartheta_k = \tau_k + \frac{\tau}{\omega\Delta} \sum \lambda_{ik} h_i^0. \quad (k = 1, 2, \ldots, n-1)$$

By virtue of (21) one obtains the inequalities

$$\tau_k + \frac{\tau}{\omega\Delta} \sum \lambda_{ik} h_i^0 > 0, \quad (k = 1, 2, \ldots, n-1)$$

$$\sum_{k=1}^{n-1} (\tau_k + \frac{\tau}{\omega\Delta} \sum \lambda_{ik} h_i^0) < m.$$

Considering the set (25), one finds

$$\left\{ \begin{array}{l} y_k + \xi_{kr} + \dfrac{\tau}{\omega\Delta} \sum \lambda_{ik} h_i^0 > 0, \\ \displaystyle\sum_{k=1}^{n-1} \left(y_k \xi_{kr} + \frac{\tau}{\omega\Delta} \sum \lambda_{ik} h_i^0 \right) < m. \end{array} \right. \quad (k = 1, 2, \ldots, n-1) \tag{27}$$

Let us write

$$y_k \xi_{kr} + \frac{\tau}{\omega\Delta} \sum \lambda_{ik} h_i^0 = y'_k + \nu_k \ \text{ where } \ 0 < \nu_k \leq 1 \quad (k = 1, 2, \ldots, n-1)$$

and

$$\sum_{k=1}^{n-1} \nu_k = a_r^{(\tau)} + \nu \ \text{ where } \ 0 \leq \nu < 1, \tag{28}$$

$y'_1, \ldots, y'_{n-1}$ and $a_r^{(\tau)}$ being integers.

The inequalities (27) will be replaced by the following ones:

$$\sum_{k=1}^{n-1} y'_k < m - a_r^{(\tau)} - \nu, \ y'_k > -\nu_k, \quad (k = 1, 2, \ldots, n-1)$$

or differently

$$\sum_{k=1}^{n-1} y'_k \leq m - a_r^{(\tau)} - 1 \ \text{ and } \ y'_k \geq 0. \quad (k = 1, 2, \ldots, n-1)$$

The number of systems $(y'_1, y'_2, \ldots, y'_{n-1})$ of integers verifying these inequalities is equal to

$$\frac{(m - a_r^{(\tau)})(m + 1 - a_r^{(\tau)}) \cdots (m + n - 2 - a_r^{(\tau)})}{1 \cdot 2 \cdots (n-1)}.$$

One concludes that the symbol m_τ which expresses the number of solutions of equations (20) and (21) in integers is equal to the sum

$$m_\tau = \sum_{\tau=1}^{\omega} \frac{(m-a_r^{(\tau)})\cdots(m+n-2-a_r^{(\tau)})}{1\cdot 2\cdots(n-1)}.$$

84 By substituting in the sum (26), one obtains

$$\sum_{\tau>0}^{\tau<m\Delta}\left(\Delta-\frac{\tau}{m}\right)m_\tau = \sum_{\tau>0}^{\tau<m\Delta}\left(\Delta-\frac{\tau}{m}\right)\sum_{r=1}^{\omega}\frac{(m-a_r^{(\tau)})\cdots(m+n-2-a_r^{(\tau)})}{1\cdot 2\cdots(n-1)}. \tag{29}$$

Let us find a value approached by the sum obtained. By noticing that because of (28)

$$0 \le a_r^{(\tau)} \le n-1,$$

one can write

$$\sum_{r=1}^{\omega}\frac{(m-a_r^{(\tau)})\cdots(m+n-2-a_r^{(\tau)})}{1\cdot 2\cdots(n-1)} = \frac{\omega}{(n-1)!}m^{n-1}+\delta_\tau m^{n-2}$$

where $|\delta_t au|$ does not exceed a fixed limit which doesnot depend on the number m.

By substituting in the sum (27), one finds

$$\sum_{\tau>0}^{\tau<m\Delta}\left(\Delta-\frac{\tau}{m}\right)m_\tau = \frac{\omega}{(n-1)!}\sum_{\tau>0}^{\tau<m\Delta}\left(\Delta-\frac{\tau}{m}\right)+\delta\Delta^2 m^{n-1}$$

where $|\delta|$ does not exceed a fixed limit which does not depend on $x_1, x_2, \ldots, x_n$ and on the number m.

By noticing that

$$\sum_{\tau>0}^{\tau<m\Delta}\left(\Delta-\frac{\tau}{m}\right) = \Delta^2\frac{m}{2}-\frac{\Delta}{2}+\frac{\vartheta}{m} \quad \text{where} \quad 0 \le \vartheta < \frac{1}{8},$$

one can write

$$\sum_{\tau>0}^{\tau<m\Delta}\left(\Delta-\frac{\tau}{m}\right)m_\tau = \frac{1}{2}\frac{\omega\Delta^2}{(n-1)!}m^n + m^{n-1}(\delta\Delta^2+\delta'\Delta+\delta'') \tag{30}$$

where $\delta, \delta', \delta''$ do not exceed in numerical value a fixed limit.

By substituting in the formula (11) the coefficient found of 2ρ, one will have, because of (19),

$$\begin{aligned} m^n\Big(\sum\sum a_{ij}x_ix_j+\epsilon_0+\sum\epsilon_i x_i\Big) &= \frac{m^n}{(m-1)!}\sum_{k=1}^{\sigma}\rho_k\omega_k\Big(\sum p_i x_i\Big)^2 \\ &+ m^{n-1}\sum_{k=1}^{\sigma}2\rho_k\Big[\delta_k\Big(\sum p_{ik}x_i\Big)^2+\delta_k'\sum p_{ik}x_i+\delta_k''\Big]. \end{aligned} \tag{31}$$

In the formula obtained the coefficients $\epsilon_0, \epsilon_1, \ldots, \epsilon_n, \delta_k, \delta_k', \delta_k''$ $(k = 1, 2, \ldots, \sigma)$ do not exceed in numerical value a fixed limit which depend only on coefficients of the quadratic form $\sum\sum a_{ij}x_ix_j$ snf on the choice of the set (L) of simplexes.

Let us replace in the formula obtained the numbers $x_1, x_2, \ldots, x_n$ by the numbers $mx_1, mx_2, \ldots, mx_n$. As these numbers satisfy the conditions imposed on the numbers $x_1, x_2, \ldots, x_n$, the formula (31) is applicable and one obtains

$$\begin{aligned} m^n\Big(m^2\sum\sum a_{ij}x_ix_j+\epsilon_0+m\sum\epsilon_i x_i\Big) &= \frac{m^{n-2}}{(n-1)!}\sum_{k=1}^{\sigma}\rho_k\omega_k\Big(\sum p_{ik}x_i\Big)^2 \\ &+ m^{n-1}\sum_{k=1}^{\sigma}2\rho_k\Big[\delta_k m^2\Big(\sum p_{ik}x_i\Big)^2+\delta_k' m\sum p_{ik}x_i+\delta_k''\Big]. \end{aligned}$$

By dividing the two parts of the formula obtained by m^{n+2}, let us make the positive integer m increase indefinitely, it will become

$$\sum\sum a_{ij}x_ix_j = \frac{1}{(n-1)!}\sum_{k=1}^{\sigma}\rho_k\omega_k(p_{1k}x_1+p_{2k}x_2+\ldots+p_{nk}x_n)^2. \tag{32}$$

The sum which is found in the second member of the formula obtained extends to all the incongruent faces in $n-1$ in $n-1$ dimensions of simplexes of the set (L).

We have deduced the formula (34) by supposing that the numbers $x_1, x_2, \ldots, x_n$ form a irreducible basis. As the two parts of the formula (32) present two quadratic forms, one concludes that the formula (32) present an indentity. This results in that the formula (32) can be written

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a'_{ij} = \frac{1}{(n-1)!}\sum_{k=1}^{\sigma}\rho_k\omega_k\sum_{i=1}^{n}\sum_{j=1}^{n} a'_{ij}p_{ik}p_{jk} \qquad (I)$$

where one has written

$$\rho_k = \sum_{i=1}^{n}\sum_{j=1}^{n} p_{ij}^{(k)}a_{ij}, \quad (k = 1, 2, \ldots, \sigma)$$

the two quadratic forms $\sum\sum a_{ij}x_ix_j$ and $\sum\sum a'_{ij}x_ix_j$ being arbitrary.

Section V.

Properties of the set (Δ) of quadratic forms corresponding to the various types of primitive parallelohedra.

Definition of the domain of quadratic form corresponding to a type of primitive parallelohedra.

85 Let us suppose that a type of primitive parallelohedra is characterised by a set (L) of simplexes.

Let us indicate by

$$\rho_k = \sum\sum p_{ij}^{(k)}a_{ij} \quad (k = 1, 2, \ldots, \sigma)$$

the regulators which correspond to the various incongruent faces in $n-1$ dimensions of simplexes of the set (L).

Definition. One will call domain of quadratic forms corresponding to the type of primitive parallelohedra characterised by the set (L) *of simplexes a domain* Δ *in quadratic forms verifying the inequalities*

$$\rho_k = \sum\sum p_{ij}^{(k)}a_{ij} \geq 0. \ (k = 1, 2, \ldots, \sigma) \qquad (1)$$

On the ground of the fundamental theorem of Number 77, for a quadratic form f to define a set (R) of primitive parallelohedra belonging to the type characterised by the set (L) of simplexes, it is necessary and sufficient that the form f is interior to the domain Δ. This results in that the domain Δ is of $\frac{n(n+1)}{2}$ dimensions.

86 Among the inequalities (1) can be found dependent inequalities. Let us suppose that one has chosen a system of independent inequalities

$$\rho_1 \geq 0, \rho_2 \geq 0, \ldots, \rho_m \geq 0$$

which define the domain Δ. With the help of independent regulators $\rho_1, \rho_2, \ldots, \rho_m$ one will present all the regulators under the form

$$\rho_k = \sum_{r=1}^{m} h_r^{(k)}\rho_r \text{ where } h_r^{(k)} \geq 0. \ (r = 1, 2, \ldots, m; k = 1, 2, \ldots, \sigma) \qquad (2)$$

Let us observe that any one quadratic form $\sum\sum a_{ij}x_ix_j$ does not verify the equations

$$\rho_1 = 0, \rho_2 = 0, \ldots, \rho_m = 0$$

because the equalities (2) give

$$\rho_k = 0, \quad (k = 1, 2, \ldots, \sigma)$$

and, by virtue of the formula (I) of Number 84, one has

$$\sum\sum a_{ij}a'_{ij} = 0,$$

$\sum\sum a'_{ij}x_ix_j$ being an arbitrary form; it follows that

$$a_{ij} = 0. \quad (i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$$

To the domain Δ, therefore, the conclusion deduced in my first mémoire cited † are applicable.

Let us indicate by

$$\varphi_1, \varphi_2, \ldots, \varphi_s \qquad (3)$$

the quadratic forms which characterise the various edges of the domain Δ.

The domain Δ of quadratic forms will be determined by the equalities

$$\sum\sum a_{ij}x_ix_j = \sum_{u=1}^{s} u_k\varphi_k \text{ where } u_k \geq 0, \quad (k = 1, 2, \ldots, s)$$

$u_1, u_2, \ldots, u_s$ being positive arbitrary parameters or zeros.

---

† This journal V. 133, p. 97

Let us notice that by virtue of the formula (I) of Number (84), each form $\varphi_k (k = 1, 2, \dots, s)$ of the series (3) will have for expression

$$\varphi_k = \sum_{r=1}^{\sigma} \lambda_r^{(k)} (p_{1r}x_1 + p_{2r}x_2 + \dots + p_{nr}x_n)^2$$
$$\text{where } \lambda_r^{(4)} \geq 0. \quad (r = 1, 2, \dots, \sigma; k = 1, 2, \dots, \sigma)$$

Properties of independent regulators

87 By keeping the notations from Number 69–74 let us suppose that a simplex L of the set (L) is characterised by the systems

$$(l_{i0}), (l_{i1}), \dots, (l_{in}).$$

Let us suppose that among the regulators

$$\rho_0, \rho_1, \dots, \rho_n$$

which correspond to the various faces in $n-1$ dimensions of the simplex L, is found at least one independent regulator. Let us suppose, to fix the ideas, that ρ_0 is a similar regulator.

Let us indicate by L_k $(k = 0, 1, 2, \dots, n)$ the simplexes which are contiguous to the simplex (L) through the faces in $n-1$ dimensions characterised by the systems

$$(l_{ih}). \quad (h = 0, 1, 2, \dots, n; h \neq k; k = 0, 1, 2, \dots, n)$$

Let us suppose that by replacing in the simplex L the vertex (l_{ik}) by a vertex (l'_{ik}), one obtains the simplex L_k $(k = 0, 1, 2, \dots, n)$.

By virtue of the formula (6) of Number 73, one will have

$$2\rho_k \sum p_{ik}(l_{ih} - l'_{ik}) = F_{(L)}(l'_{1k}, l'_{2k}, \dots, l'_{nk}). \quad (k = 0, 1, 2, \dots, n; h \neq k) \tag{1}$$

Let us admit

$$l'_{ik} = \sum_{k=0}^{n} \vartheta_r^{(k)} l_{ik} \text{ where } \sum_{r=0}^{n} \vartheta_r^{(k)} = 1. \quad (k = 0, 1, 2, \dots, n) \tag{2}$$

As, because of the inequality (14) of Number 70,

$$\sum p_{ik}(l_{ih} - l'_{ik}) > 0, \quad (h \neq k)$$

it becomes, by virtue of (2)

$$\vartheta_k^{(k)} < 0. \quad (k = 0, 1, 2, \dots, n) \tag{3}$$

Let us examine the numbers

$$\vartheta_0^0, \vartheta_1^0, \dots, \vartheta_n^0 \tag{4}$$

which correspond to the independent regulator ρ_0. One will have, because of (3),

$$\vartheta_0^0 = 0.$$

I say that among the numbers $\vartheta_1^0, \vartheta_2^0, \dots, \vartheta_n^0$ at least two numbers are positive. As $\sum_{k=0}^{n} \vartheta_k^0 = 1$, it is evident that at least one number, for example ϑ_n^0, will be positive. Let us suppose that ϑ_n^0 is the only positive number in the series (4).

Let us indicate, to fix the ideas,

$$\vartheta_0^0 < 0, \vartheta_1^0, \dots, \vartheta_\lambda^0 < 0, \vartheta_{\lambda[+1]}^0 = 0, \dots, \vartheta_{n-1}^0 = 0, \vartheta_n^0 > 0. \tag{5}$$

The corresponding value of the function $F_{(L)}(l'_{10}, \dots, l'_{n0})$, by virtue of the formula (4) of Number 78, can be presented under the form

$$\begin{aligned} F_{(L)}(l'_{10}, \dots, l'_{n0}) &= \sum \sum a_{ij}(l'_{i0} - l_{in})(l'_{j0} - l_{jn}) \\ &- \sum_{k=0}^{\lambda} \vartheta_k \sum \sum a_{ij}(l_{ik} - l_{in})(l_{jk} - l_{jn}). \end{aligned} \tag{6}$$

By virtue of the formula (I) of Number 84 and of inequalities (5), one can present this equality under the form

$$F_{(L)}(l'_{10}, \dots, l'_{n0}) = \sum_{r=1}^{\sigma} h_r \rho_r \text{ where } h_r \geq 0,$$

and as on the other hand, because of (1)

$$2\rho_0 \sum p_{i0}(l_{ih} - l'_{i0}) = F_{(L)}(l'_{i0}, \dots, l'_{n0}), \tag{7}$$

it becomes

$$\rho_0 = \sum_{r=1}^{\sigma} g_r \rho_r \text{ where } g_r \geq 0. \quad (r = 1, 2, \dots, \sigma)$$

We have supposed that ρ_0 is an independent regulator, therefore it is necessary that

$$g_2 = 0 \text{ so long as a regulator } \rho_r \text{ is not proportional to } \rho_0.$$

The formula (6) gives

$$\sum\sum a_{ij}(l'_{i0} - l_{in})(l'_{j0} - l_{jn}) = \delta\rho_0 \text{ where } \delta > 0,$$
$$\sum\sum a_{ij}(l_{ik} - l_{in})(l_{jk} - l_{jn}) = \delta_k\rho_0 \text{ where } \rho_k > 0.$$
$$(k = 0, 1, 2, \ldots, \lambda)$$

It follows that one has identically,

$$\sum\sum a_{ij}(l'_{i0} - l_{in})(l'_{j0} - l_{jn}) = \frac{\delta}{\delta_k}\sum\sum a_{ij}(l_{ik} - l_{in})(l_{jk} - l_{jn}).$$

For this identity to hold, it is necessary and sufficient that

$$l'_{i0} - l_{in} = \sqrt{\frac{\delta}{\delta_k}}(l_{ik} - l_{in}).$$

By virtue of Theorem I of Number 51, the numbers $l'_{i0} - l_{in}$ $(i = 1, 2, \ldots, n)$, and $l_{ik} - l_{in}$ $(i = 1, 2, \ldots, n)$, do not have common divisor, one concludes that

$$l'_{i0} = l_{ik},$$

which is impossible.

Let us indicae, to fix the ideas,

$$\vartheta_0^0 < 0, \vartheta_1^0 < 0, \ldots, \vartheta_\lambda^0 < 0, \vartheta_{\lambda+1}^0 = 0, \ldots, \vartheta_\mu^0 = 0, \vartheta_{\mu+1}^0 > 0, \ldots, \vartheta_n^0 > 0 \tag{8}$$

where $\lambda \geq 0$ and $\mu \leq n - 2$.

88 *Theorem: By replacing in the simplex L the vertices $(l_{ik}), k = 0, 1, 2, \ldots, \lambda$, successively by the vertex (l'_{i0}) one will obtain the simplexes*

$$L_0, L_1, \ldots, L_\lambda \tag{9}$$

which are contiguous to the simplex L and one to one by the faces in $n - 1$ dimensions the regulators of which are proportional to the regulator ρ_0.

Let us apply the formula (∗) of Number 74 to the simplex L and L_k $(k = 0, 1, 2, \ldots, \lambda)$; one will have

$$F_{(L)}(l'_{10}, \ldots, l'_{n0}) = F_{(L_k)}(l'_{10}, \ldots, l'_{n0}) + 2\rho_k\sum p_{ik}(l_{ih} - l'_{i0}).$$
$$(h \neq k; n = 0, 1, 2, \ldots, \lambda)$$

By virtue of (8), one obtains,

$$\sum p_{ik}(l_{ih} - l'_{i0}) > 0. \quad (h \neq k; k = 0, 1, 2, \ldots, \lambda)$$

In view of (8), one finds

$$F_{(L_k)}(l'_{10}, \ldots, l'_{n0}) = \delta_k\rho_0 \text{ where } \delta_k \geq 0 \quad (k = 0, 1, 2, \ldots, \lambda)$$

and

$$\rho_k = u_k\rho_0 \text{ where } u_k > 0. \ (k = 0, 1, 2, \ldots, \lambda) \tag{10}$$

On the grounds of (1) and (7), it becomes

$$F_{(L)}(l'_{1k}, \ldots, l'_{nk}) = \omega_k F_{(L)}(l'_{10}, \ldots, l'_{n0}) \text{ where } \omega_k > 0. \ (k = 0, 1, 2, \ldots, \lambda)$$

The equality obtained presents an identity with regard to the coefficients of the quadratic form $\sum\sum a_{ij}x_ix_j$. One derives, because of (2), the equalities

$$\left(\vartheta_i^{(k)}\right)^2 - \vartheta_i^{(k)} = \omega_k\left((\vartheta_i^0)^2 - \vartheta_i^0\right), \quad (i = 0, 1, 2, \ldots, n)$$
$$\vartheta_i^{(k)}\vartheta_j^{(k)} = \omega_k\vartheta_i^0\vartheta_j^0. \quad (i = 0, 1, 2, \ldots, n; j = 0, 1, 2, \ldots, n; i \neq j)$$

As $\sum_{i=0}^n \vartheta_i^{(k)} = 1$ and $\sum_{i=0}^n \vartheta_i^0 = 1$, it is necessary that $\omega_k = 1$ and

$$\vartheta_i^{(k)} = \vartheta_i^0, \quad (i = 0, 1, 2, \ldots, n)$$

therefore

$$l'_{ik} = l'_{i0}. \quad (k = 0, 1, 2, \ldots, \lambda)$$

The formula (1) becomes in this case

$$2\rho_k\sum p_{ik}(l_{ih} - l'_{i0}) = F_{(L)}(l'_{10}, \ldots, l'_{n0}). \quad (h \neq k; k = 0, 1, 2, \ldots, \lambda)$$

Let us notice that the simplexes

$$L, L_0, L_1, \ldots, L_\lambda$$

make up a group of perfectly determined simplexes corresponding to the independent regulator ρ_0. That which we have mentioned concerning the simplex L can be related back to all the simplexes of the series (9). All the simplexes which remain $L_{\lambda+1}, \ldots, L_n$ are contiguous to the simplex L through the faces the regulators of which are not proportional to ρ_0.

89 Let us notice that the simplexes (9) make up a convex polyhedron K having $n+2$ vertices

$$(l'_{i0}), (l_{i0}), \ldots, (l_{in}).$$

In effect, all the points of simplexes (9) belong to polyhedron K made up of points determined by the equalities

$$x_i = ul'_{i0} + \sum_{k=0}^{n} u_k l_{ik} \text{ where } u + \sum_{u=0}^{n} u_k = 1 \text{ and } u \geq 0, u_k \geq 0. \quad (k = 0, 1, 2, \ldots, n) \tag{11}$$

I argue that any point (x_i) determined by these equalities belongs to at least one of simplex (9).

Let us suppose, in the first place, that one has the inequalities

$$u_k + u\vartheta_k^0 \geq 0. \quad (k = 0, 1, 2, \ldots, n)$$

One will present the equation (11), because of (2), under the following form:

$$x_i = \sum_{k=0}^{n} (u_k + u\vartheta_k^0) l_{ik}$$

and, as $\sum_{k=0}^{n}(u_k + u\vartheta_k^0) = 1$, one concludes that the point (x_i) belongs to the simplex L.

This laid down, let us suppose that at least one of numbers $u_k + u\vartheta_k^0$, $k = 0, 1, 2, \ldots, n$, is negative. Let us choose among the numbers

$$\frac{u_0}{\vartheta_0^0}, \frac{u_1}{\vartheta_1^0}, \ldots, \frac{u_\lambda}{\vartheta_\lambda^0}$$

which are all negatives or zeros, because of (8) and (11), a number $\frac{u_k}{\vartheta_\lambda^0}$ the numerical value of which is the smallest. The point (x_i) determined by the equalities (11) belongs in this case to the simplex L_k. To demonstrate this, one will present the equalities (11) under the form

$$x_i = \left(u + \frac{u_k}{\vartheta_k}\right) l'_{i0} + \sum_r \left(u_r - u_k \frac{\vartheta_r}{\vartheta_k}\right) l_{ir}. \quad (r = 0, 1, 2, \ldots, n; r \neq k)$$

On the ground of suppositions made, one will have the inequalities

$$u + \frac{u_k}{\vartheta_k} > 0, \quad u_r - u_k \frac{\vartheta_r}{\vartheta_k} \geq 0, \quad (r = 0, 1, 2, \ldots, n; r \neq k)$$

and, as

$$u + \frac{u_k}{\vartheta_k} + \sum_r \left(u_r - u_k \frac{\vartheta_r}{\vartheta_k}\right) = 1,$$

one concludes that the point (x_i) belongs to the simplex (L_k) $(u = 0, 1, 2, \ldots, \lambda)$.

Let us examine the faces in $n-1$ dimensions of the polyhedron K. On the ground of conditions (8), the polyhedron K possesses $\mu - \lambda$ faces in $n-1$ dimensions Q_k which are characterised by $n+1$ vertices

$$(l'_{i0}), (l_{ih}). \quad (h = 0, 1, 2, \ldots, n; h \neq k; k = \lambda + 1, \ldots, \mu)$$

The vertex (l_{ik}) where $k = \lambda + 1, \ldots, \mu$ is opposite to the face $\theta_k (k = \lambda + 1, \ldots, \mu)$.

All the faces in $n-1$ dimensions of the polyhedron K which remain are characterised by n vertices. One will characterise them in the polyhedron K by two opposite vertices.

One obtains in this way $n - \mu$ faces P_k $(k = \mu + 1, \ldots, n)$ of the polyhedron K characterised by two opposite vertices (l'_{i0}) and l_{ik} $(k = \mu+1, \ldots, n)$ and one obtains $(\lambda+1)(n-\mu)$ faces $P_{kh} (h = 0, 1, 2, \ldots, \lambda; k = \mu + 1, \ldots, n)$ characterised by two opposite vertices l_{ik} and l_{ih}.

90 Let us notice that the polyhedron K is contiguous through the faces Q_k $(k = \lambda + 1, \ldots, \mu)$ to other independent regulator ρ_0.

to demonstrate this, let us examine the simplex L_k $(k = \lambda+1, \ldots, \mu)$ contiguous to the simplex L through the face in $n-1$ dimensions characterised by the vertices

$$(l_{ih}). \quad (h = 0, 1, 2, \ldots, n; h \neq k; k = \lambda + 1, \ldots, \mu)$$

This face presents a part of the corresponding face Q_k of the polyhedron K.

By applying the formula (*) of Number 74 to the simplexes L and L_k, one obtains

$$F_{(L)}(l'_{10}, \ldots, l'_{n0}) = F_{(L_k)}(l'_{10}, \ldots, l'_{n0}) + 2\rho_k \sum p_{ik}(l_{ih} - l'_{i0})$$
$$\text{where } h \neq k \text{ and } n = \lambda + 1, \ldots, \mu.$$

On the ground of conditions (8), one will have

$$\sum p_{ik}(l_{ih} - l'_{i0}) = 0, \quad (k = \lambda + 1, \dots, \mu)$$

therefore, because of (7),

$$F_{(L_k)}(l'_{10}, \dots, l'_{n0}) = F_{(L)}(l'_{10}, \dots, l'_{n0}) = 2\rho_0 \sum p_{i0}(l_{ih} - l'_{i0}).$$
$$(h \neq 0; k = \lambda + 1, \dots, \mu)$$

As the point (l'_{10}) is not found among the vertices of the simplex L_k, it is necessary that among the regulators of faces of the simplex L_k are found, by virtue of the equation obtained, regulators which are proportional to ρ_0.

By noticing that

$$l'_{i0} = \sum \vartheta^0_h l_{ih} + \vartheta^0_k l'_{ik}, \text{ where } \sum \vartheta^0_h + \vartheta^0_k = 1$$
$$(h = 0, 1, 2, \dots, n; h \neq k; k = \lambda + 1, \dots, \mu)$$

since because of (8), $\vartheta^0_k = 0$ $(k = \lambda + 1, \dots, \mu)$, one concludes, on the ground of the previous theorem, that by replacing in the simplex L_k the vertices (l_{ih}) $(h = 0, 1, 2, \dots, \lambda)$ by the vertex (l'_{i0}) one will obtain a group of simplexes

$$L_k, L_k^{(0)}, L_k^{(1)}, \dots, L_k^{(\lambda)}, \quad (k = \lambda + 1, \dots, \mu) \tag{12}$$

which are contiguous one to one by faces in $n - 1$ dimensions the regulators of which are proportional to ρ_0.

Let us indicate by K_h the convex polyhedron made up of simplexes (12). One obtains the polyhedron K_k by replacing in the polyhedron K the vertex (l_{ik}) by the vertex l'_{ik} $(k = \lambda + 1, \dots, \mu)$. One concludes that the polyhedra K and K_k are contiguous through the face Q_k.

Let us examine other faces of the polyhedra K. The face P_k $(k = \mu + 1, \dots, n)$ belongs to the simplex L which is contiguous through the face P_k to the simplex L_k. The regulator ρ_k $(k = \mu + 1, \dots, n)$ of this face can not be proportional to the independent regulator ρ_0.

It may turn out that any one of regulators corresponding to the various faces of the simplex L_k is not proportional to the regulator ρ_0. In this case, the polyhedron K will not be contiguous through the face P_k to any one analogous polyhedron corresponding to the independent regulator ρ_0.

It may also turn out that among the regulators of faces of the simplex L_k are found regulators which are proportional to ρ_0; in this case, the simplex L_k belongs to a convex polyhedron K_k which is contiguous to K through the face P_k $(k = \mu + 1, \dots, n)$.

In the same way, one will examine the analogous faces P_{hk} of the polyhedron K $(h = 0, 1, 2, \dots, \lambda; k = \mu + 1, \dots, n)$.

By applying the procedure shown to the various incongruent simplexes of the set (L), one will determine the incongruent convex polyhedron

$$K, K_1, \dots, K_{\omega-1}$$

which are made up of corresponding groups of simplexes belonging to the set (L).

Reconstruction of the set (L) of simplexes by another set (L') of simplexes.

91 One can partition the convex polyhedra

$$K, K_1, \dots, K_{\omega-1} \tag{1}$$

corresponding to an independent regulator ρ into new simplexes.

By keeping the previous notations, let us examine the convex polyhedron K made up of simplexes

$$L, L_0, \dots, L_\lambda. \tag{2}$$

The polyhedron K possesses $n + 2$ vertices

$$(l'_i), (l_{i0}), \dots, (l_{in}).$$

it Theorem. By replacing in the simplex L characterised by the vertices

$$(l_{i0}), (l_{i1}), \dots, (l_{in})$$

a value (l_{ik}) by the vertex (l'_i) where $k = \mu + 1, \dots, n$, one obtains $n - \mu$ simplexes

$$L_{\mu+1}, \dots, L'_n \tag{3}$$

which also make up the polyhedron K. The simplexes obtained do not belong to the set (L) of simplexes.

Let us write, as we have done in Number 87,

$$l'_i = \sum_{k=0}^{n} \vartheta_k l_{ik} \text{ where } \sum_{k=0}^{n} \vartheta_k = 1 \tag{4}$$

and

$$\vartheta_0 < 0, \vartheta_1 < 0, \dots, \vartheta_\lambda < 0, \vartheta_{\lambda+1} = 0, \dots, \vartheta_\mu = 0, \vartheta_{\mu+1} > 0, \dots, \vartheta_n > 0. \tag{5}$$

It is clear that each point of simplex (3) belongs to the polyhedron K.

Let (x_i) be any one point of the polyhedron K determined with the help of equalities

$$x_i = ul'_i + \sum_{k=0}^{n} u_k l_{ik} \text{ where } u + \sum u_k = 1, u \geq 0, u_k \geq 0. \; (k = 0, 1, \ldots, n) \tag{6}$$

Let us choose among the numbers

$$\frac{u_{\mu+1}}{\vartheta_{\mu+1}}, \ldots, \frac{u_n}{\vartheta_n}$$

the one which is the smallest. Let us suppose, to fix the ideas, that

$$\frac{u_r}{\vartheta_r} \geq \frac{u_k}{\vartheta_k}. \quad (r = \mu + 1, \ldots, n)$$

I argue that the point (x_i) belongs to the simplex L'_k. In effect, the equality (6) can be speculated, because of (4), under the form

$$x_i = \left(u + \frac{u_k}{\vartheta_k}\right) l'_i + \sum_h \left(u_h - u_k \frac{\vartheta_h}{\vartheta_k}\right) l_{ih}. \quad (h = 0, 1, 2, \ldots, n; h \neq k)$$

By observing that

$$u + \frac{u_k}{\vartheta_k} \geq 0, \; u_h - u_k \frac{\vartheta_h}{\vartheta_k} \geq 0, \quad (h = 0, 1, 2, \ldots, n; h \neq k)$$

and that

$$u + \frac{u_k}{\vartheta_k} + \sum_h \left(u_k - u_k \frac{\vartheta_h}{\vartheta_k}\right) = 1 \quad (h = 0, 1, 2, \ldots, n; h \neq k)$$

one concludes that the point (x_i) belongs to the simplex L'_k $(k = \mu + 1, \ldots, n)$.

The simplexes (3) can not belong to the set (L), because this set, by virtue of Theorem I of Number 61, uniformly partition the space in n dimensions.

92

Let us suppose that one has replaced in the set (L) the group of simplexes (2) by the corresponding group (3). Let us suppose that one has effected this reconstruction of simplexes of the set (L) with regard to all the polyhedra which are congruent to the polyhedra (1). One obtains in this way a new set (L') of simplexes which enjoy the following properties.

1. The set (L') of simplexes uniformly fills the space in n dimensions.

2. The set (L') can be divided into classes of congruent simplexes and the number of different classes is finite.

Let us find the regulators and the characteristics of faces in $n - 1$ dimensions of simplexes belonging to the set (L').

Let L' and L'_0 be any two simplexes of the set (L') which are contiguous through a face P in $n - 1$ dimensions. Suppose that the two simplexes L' and L'_0 also belong to the set (L'). In this case the regulator and the characteristic of the face P in the set (L') do not change.

Let us suppose that at least one of the simplexes examined does not belong to the set (L) of simplexes. This simplex will belong in this case to a polyhedron which is congruent to a polyhedron of the series (1). Let us suppose to fix the ideas that this is the polyhedron K.

By noticing that the simplex examined is found among the simplexes (3) let us choose one of these simplexes L'_k $(k = \mu + 1, \ldots, n)$ and examine the regulators and the characteristics of all these faces in $n - 1$ dimensions.

By virtue of the definition established, the simplex L'_k is characterised by the vertices

$$(l_{i0}), \ldots, (l_{i,k-1}), (l'_i), (l_{i,k+1}), \ldots, (l_{in}). \quad (k = \mu + 1, \ldots, n)$$

Let us indicate by P_{hk} a face in $n - 1$ dimensions of the simplex L'_k which is opposite to the vertex (l_{ih}) $(h = 0, 1, 2, \ldots; h \neq k)$. By P'_k let us indicate the faces of simplex L'_k which is opposite to the vertex l'_i.

All the faces in $n - 1$ dimensions of the simplex L'_k can be divided into three groups:

1. $P_{0k}, P_{1k}, \ldots, P_{\lambda k}$ and P_k;
2. $P_{\lambda+1,k}, \ldots, P_{\mu k}$;
3. $P_{\mu+1,k}, \ldots, P_{k-1,k}, P_{k+1,k}, \ldots, P_{nk}$.

93

Let us find the regulators of faces of the simplex L'_k belonging to the first group.

Let us examine, in the first place, the face P_k. As the face P_k is characterised by the vertices

$$(l_{i0}), \ldots, (l_{i,k-1}), (l_{i,k+1}), \ldots, (l_{in}),$$

it presents a face of the polyhedron K.

In the set (L) the face P_k would belong to two simplexes L and L_k. Two cases to distinguish:

First case: the simplex L_k belongs to the set (L').

Let us indicate by ρ_k the regulator and by (P_{ik}) the characteristic of the face P_k in the set (L) with regard to the simplex L.

Let us indicate by ρ'_k the regulator of the face P_k in the set (L'). The characteristics of the face P_k in the set (L') with regard to the simplex L'_k will be (P_{ik}).

By virtue of the definition established in Number 73, one can declare

$$F_{(L_k)}(l'_1,\dots,l'_n) = 2\rho'_k \sum(-p_{ik})(l_{ih}-l'_i). \quad (h \neq k)$$

By applying the formula (∗) of Number 74 to the simplex L and L_k, one obtains

$$F_{(L)}(l'_1,\dots,l'_n) = F_{(L_k)}(l'_1,\dots,l'_n) + 2\rho_k \sum p_{ik}(l_{ih}-l'_i). \quad (h \neq k) \tag{7}$$

It follows that

$$\rho'_k = \rho_k + \frac{F_{(L)}(l'_1,\dots,l'_n)}{2\sum p_{ik}(-l_{ih}+l'_i)}. \quad (k = \mu+1,\dots,n) \tag{8}$$

We have seen in Number 87 that the function $F_{(L)}(l'_1,\dots,l'_n)$ is proportional to the independent regulator ρ. As, by virtue of (5),

$$\sum p_{ik}(-l_{ih}+l'_i) > 0, \quad (h \neq k; k = \lambda+1,\dots,\mu)$$

the formula (8) can be written

$$\rho'_k = \rho_k + \delta_k\rho \text{ where } \delta_k > 0.$$

Second case: the simplex L_k does not belong to the set (L').

In this case the simplex L_k belongs to a convex polyhedron K_k and the face P_k will belong in the set (L') to two new simplexes.

The face P_k in the set (L') belongs to the simplex L'_k and to a simplex which one obtains by replacing the vertex l'_{ik} of the simplex L_k by a new vertex which one will indicate by (l^0_i). Let us indicae by L^0_k the simplex which one obtains by replacing in the simplex L_k the vertex l'_{ik} by the vertex (l^0_i).

By virtue of the definition established in Number 73, one will have

$$F_{(L'_k)}(l^0_1,\dots,l^0_n) = 2\rho'_k \sum p_{ik}(l_{ih}-l^0_i). \quad (h \neq k)$$

By applying the fundamental formula (∗) of Number 74 to the simplexes L_k and L'_k which are contiguous through the face P_k, one obtains

$$F_{(L_k)}(l^0_1,\dots,l^0_n) = F_{(L'_k)}(l^0_1,\dots,l^0_n) + \sum(-p_{ik})(l_{ih}-l^0_i)\frac{F_{(L_k)}(l'_1,\dots,l'_n)}{\sum(-p_{ik})(l_{ih}-l'_i)}$$

and, because of (7), it becomes

$$\rho'_k = \rho_k + \frac{F_{(L)}(l'_1,\dots,l'_n)}{2\sum p_{ik}(-l_{ih}+l'_i)} + \frac{F_{(L_k)}(l^0_1,\dots,l^0_n)}{2\sum p_{ik}(l_{ih}-l^0_i)}. \tag{9}$$

On the ground of the supposition made, the functions

$$F_{(L)}(l'_1,\dots,l'_n) \text{ and } F_{(L_k)}(l^0_1,\dots,l^0_n)$$

are proportional to the independent regulator ρ. As

$$\sum p_{ik}(-l_{ih}+l'_i) > 0 \text{ and } \sum p_{ik}(l_{ih}-l^0_i) > 0, \quad (k = \mu+1,\dots,n)$$

the formula (9) can be written

$$\rho'_k = \rho_k + \delta_k\rho \text{ where } \delta_k > 0. \quad (k = \mu+1,\dots,n)$$

In the same way, one will examine the regulator of the face P_{hk}; $(h = 0,1,2,\dots,\lambda)$.

As the face P_{hk} belongs in the set (L) to the simplex L_h $(h = 0,1,2,\dots,\lambda)$, one concludes that by designating the simplex L_h with the simplex L one will return to one of the two previous cases.

94 Let us find the regulators of faces of the simplex L'_k belonging to the second group. Choose one face P_{hk} $(h = \lambda+1,\dots,\mu; k = \mu+1,\dots,n)$ in this group.

The face P_{hk} presents a part of the face Q_k of the polyhedron K.

We have indicated in Number 90 by K_h the polyhedron which is contiguous to the polyhedron K through the face Q_h. The polyhedron K_h posses $n+1$ vertices

$$(l'_i),(l_{io}),\dots,(l_{i,h-1}),(l'_{ih}),(l_{i,h+1}),\dots,(l_{in}).$$

In the set (L'), the polyhedron K_h is partitioned into simplexes

$$L'_{\mu+1,h},\dots,l'_{n,h}$$

which one obtains by replacing in the simplexes

$$L'_{\mu+1},\dots,L'_n$$

the vertex (l_{ih}) by the vertex (l'_{ih}) $(h = \lambda+1,\dots,\mu)$.

One concludes that the face P_{hk} belongs to the set (L') to two simplexes

$$L'_k \text{ and } L'_{kh}.$$

Let us indicate by ρ'_{hk} the regulator corresponding to the face P_{hk} in the set (L'). Notice that the characteristic (P'_{ih}) will be the same for all the faces P_{hk} where $k=\mu+1,\dots,n$ because these faces make up the face Q_k of the polyhedron K.

In the set (L), the face Q_h is partitioned into faces

$$P_h, P_{h0}, \dots, P_{h\lambda}$$

of simplexes $L, L_0, \dots, L_\lambda$ which have the same characteristic (p_{ih}). One concludes that

$$p'_{ih} = p_{ih},$$

provided that the characteristic (p_{ih}) is chosen with regard to the simplexes (2).

By virtue of the definition established in Number 73, one can write

$$F_{(L_k)}(l'_{ih},\dots,L'_{nh}) = 2\rho'_{hk}\sum p_{ih}(l_{ir}-l'_{ih}). \quad (r\neq h)$$

By applying the formula (*) of Number 74 to the simplexes L and L'_k, one will have

$$F_{(L)}(l'_{ih},\dots,L'_{nh}) = F_{(L'_k)}(l'_{ih},\dots,L'_{nh}) + \sum p_{ik}(l_{ir}-l'_{ih})\frac{F_{(L)}(l'_1,\dots,l'_n)}{\sum p_{ik}(l_{ir}-l'_i)}.$$

Besides, one has

$$F_{(L)}(l'_{ih},\dots,L'_{nh}) = 2\rho_h\sum p_{ih}(l_{ir}-l'_{ih}), \quad (r\neq h)$$

and consequently

$$\rho'_{hk} = \rho_h + \frac{\sum p_{ik}(l_{ir}-l'_{ih})}{\sum p_{ih}(l_{ir}-l'_{ih})}\cdot\frac{F_{(L)}(l'_1,\dots,l'_n)}{2\sum(p_{ik}-l_{ir}+l'_i)}.$$

One can thus write

$$\rho'_{hk} = \rho_h + \delta_{hk}\rho. \quad (h=\lambda+1,\dots,\mu;\, k=\mu+1,\dots,n)$$

In the formula obtained, the number δ_{hk} can be positive, negative or zero.

95 Let us find the regulators of faces of the simplex L'_k belonging to the third group. Let P_{hk} be a face belonging to this group, $h=\mu+1,\dots,n$, $h\neq k$, $k=\mu+1,\dots,n$. The face P_{hk} belongs in the set (L') to two simplexes L'_k and L'_h of the series (3). By replacing in the simplex L'_k the vertices (l_{ih}) by the vertex (l_{ik}), one obtains the simplex L'_h. This results in that by indicating with ρ_{hk} the regulator and with $(p_i^{(hk)})$ the characteristic of the face P_{hk} with respect to the simplex L'_k, one will have

$$F_{(L'_k)}(l_{1k},\dots,l_{nk}) = 2\rho'_{hk}\sum p_i^{(hk)}(l_{ir}-l_{ik}). \quad (r\neq h; r\neq k) \tag{10}$$

The equality (4) can be written

$$l_{ik} = \frac{1}{\vartheta_k}l'_i + \sum_r\left(-\frac{\vartheta_r}{\vartheta_k}\right)l_{ir}. \quad (r=0,1,2,\dots,n; r\neq k)$$

By noticing that

$$\frac{1}{\vartheta_k} + \sum_r\left(-\frac{\vartheta_r}{\vartheta_k}\right) = 1,$$

one will determine the value of the function $F_{(L'_k)}(l_{1k},\dots,l_{nk})$, by virtue of the formula (4) of Number 73, by the equality

$$\begin{aligned} F_{(L'_k)}(l_{1k},\dots,l_{nk}) &= \sum\sum a_{ij}l_{ik}l_{jk} - \frac{1}{\vartheta_k}\sum\sum a_{ij}l'_il'_j \\ &+ \sum_r\frac{\vartheta_r}{\vartheta_k}\sum\sum a_{ij}l_{ir}l_{jr}. \quad (r=0,1,2,\dots,n; r\neq k) \end{aligned}$$

By recalling that because of (4)

$$F_{(L)}(l'_1,\dots,l'_n) = \sum\sum a_{ij}l'_il'_j - \sum_{k=0}^{n}\vartheta_k\sum\sum a_{ij}l_{ik}l_{jk},$$

and by comparing the two equalities obtained, one finds

$$F_{(L'_k)}(l_{1k},\dots,l_{nk}) = -\frac{1}{\vartheta_k}F_{(L)}(l'_1,\dots,l'_n).$$

By substituting in the formula (10) the expression found of the function $F_{(L'_k)}(l_{1k},\dots,l_{nk})$, one obtains

$$\rho'_{hk} = -\frac{1}{\vartheta_k}\cdot\frac{F_{(L)}(l'_1,\dots,l'_n)}{2\sum p_i^{(hk)}(l_{ir}-l_{ik})}.$$

$$(h=\mu+1,\dots,n; k=\mu+1,\dots,n; h\neq k; r\neq k)$$

One concludes that by admitting

$$\rho'_{hk} = -\delta_{hk}\rho, \quad (h = \mu+1, \dots, n; k = \mu+1, \dots, n; h \neq k)$$

one will have $\delta_{hk} > 0$.

Algorithm for the study of domains of quadratic forms which are contiguous to a given domain through the faces in $\frac{n(n+1)}{2} - 1$ dimensions.

96 Let us suppose that a domain Δ of quadratic form corresponding to a type of primitive parallelohedra which is characterised by the set (L) of simplexes is defined by the independent inequalities

$$\rho_k \geq 0. \quad (k = 1, 2, \dots, m)$$

Let us suppose that one of these regulators is proportional to an independent regulator ρ and construct the set (L) of simplexes in another set (L') with the help of the procedure shown in Number 91–92.

Let us indicate by

$$\rho_1, \rho_2, \dots, \rho_\sigma$$

all the regulators of incongruent faces of simplexes belonging to the set (L) and indicate by

$$\rho'_1, \rho'_2, \dots, \rho'_\tau$$

all the regulators of faces of simplexes belonging to the set (L').

We have seen in Number 93–95 that all these regulators can be presented under the form

$$\begin{cases} \text{either} \quad \rho'_k = -\delta_k\rho \quad \text{where } \delta_k > 0, \\ \text{or} \quad \rho'_k = \rho_k + \delta_{kh}\rho \end{cases} \tag{1}$$

so long as a regulator ρ'_k is not proportional to ρ.

Let us examine the domain D' of quadratic forms determined by the inequalities

$$\rho'_k \geq 0. \quad (k = 1, 2, \dots, \tau) \tag{2}$$

I argue that these inequalities define a domain of quadratic forms in $\frac{n(n+1)}{2}$ dimensions. By supposing the contrary, one will find the parameters u_k $(k = 1, 2, \dots, \tau)$ positive or zero which reduce into an identity the equality

$$\sum_{k=1}^{\tau} u_k\rho'_k = 0 \text{ where } u_k \geq 0. \ (k = 1, 2, \dots, \tau) \tag{3}$$

By virtue of formulae (1), this identity can be written

$$\sum_{k=1}^{\sigma} v_k\rho_k \pm v\rho = 0 \text{ where } v_k \geq 0. \ (k = 1, 2, \dots, \sigma)$$

As the regulator ρ is independent it is necessary that $v_k = 0$ as long as a regulator ρ_k is not proportional to ρ. This results in that within the identity (3) one also has $u_k = 0$ so long as a regulator ρ'_k is not proportional to ρ. By virtue of (1), the identity (3) takes the form

$$\sum u_k(-\delta_k\rho) = 0 \text{ where } u_k \geq 0 \text{ and } \delta_k > 0,$$

which is impossible.

The domain Δ' defined by the inequalities (2) corresponds to a new type of primitive parallelohedra characterised by the set (L') of simplexes.

Let us notice that by virtue of inequalities (1), any quadratic form which is interior to the face of the domain Δ determined by the equation

$$\rho = 0 \tag{4}$$

belongs to the domain Δ' and vice-versa. One concludes that the two domain Δ and Δ' are contiguous through the face in $\frac{n(n+1)}{2} - 1$ dimensions determined by the equation (4).

Set (Δ) of domains of quadratic forms corresponding to the different types of primitive parallelohedra.

97 With the help of the algorithm explained in the previous Number, one can determine the domains of quadratic forms

$$\Delta_1, \Delta_2, \dots, \Delta_m \tag{1}$$

which are contiguous to the domain Δ by their faces in $\frac{n(n+1)}{2} - 1$ dimensions, then one will determine the domains which are contiguous to the domains (1) and so on.

Let us indicate by (Δ) the set composed of all the domains of quadratic terms which correspond to the various types of primitive parallelohedra.

Theorem I. The set (Δ) of domains of quadratic forms uniformly divides the set of all the positive quadratic forms in n variables.

Let $\varphi(x_1, x_2, \dots, x_n)$ be an arbitrary positive quadratic form. Let us choose a form $\varphi_0(x_1, x_2, \dots, x_n)$ which is interior to the domain Δ and let us examine a vector g made up of forms

$$f = \varphi_0 + u(\varphi - \varphi_0) \text{ where } 0 \leq u \leq 1. \tag{2}$$

By making the parameter u grow in a continuous manner in the interval $0 \leq u \leq 1$, one will determine a series of domains

$$\Delta, \Delta', \dots, \Delta^{(k)} \tag{3}$$

which are successively contiguous through the faces in $\frac{n(n+1)}{2} - 1$ dimensions and which contain the various forms of the vector g.

I argue that the series of domains (3) will always be terminated by a domain to which belong the given quadratic form φ.

To demonstrate this, let us indicate by

$$(l_{i1}), (l_{i2}), \dots, (l_{i\tau}) \text{ where } \tau = 2^n - 1 \tag{4}$$

the systems which characterise the faces in $n-1$ dimensions of primitive parallelohedra belonging to the type which corresponds to the domain Δ of quadratic form.

Let us indicate by the symbol $N(f)$ a sum

$$N(f) = \sum_{h=1}^{\tau} f(l_{1h}, \dots, l_{nh})$$

of values of a form $f(x_1, x_2, \dots, x_n)$ corresponding to the systems (4).

Let us indicate, similarly, by

$$(l_{i1}^{(k)}), (l_{i2}^{(k)}), \dots, (l_{i\tau}^{(k)}) \text{ where } \tau = 2^n - 1 \tag{5}$$

the systems which characterise the faces in $n-1$ dimensions of primitive parallelohedra belonging to the type which corresponds to a domain $\Delta^{(k)}$ of the series (3) and declare

$$N^{(k)}(f) = \sum_{h=1}^{\tau} f(l_{1h}^{(k)}, \dots, l_{nh}^{(k)}). \quad (k = 1, 2, \dots)$$

We have seen in Number 95 that the systems (4) and (5) are congruent with respect to the modulus 2. By virtue of the theorem of Number 48, one will have an inequality

$$N(f) < N^{(k)} f, \quad (k = 1, 2, \dots)$$

as long as a quadratic form f is interior to the domain Δ. This results in that the inequality

$$N(f) \leq N^{(k)}(f) \quad (k = 1, 2, \dots)$$

holds providing that a form f belongs to the domain Δ.

This stated, let us notice that by virtue of the supposition made, the form φ_0 is interior to the domain Δ, therefore one will have the inequality

$$N(\varphi_0) < N^{(k)}(\varphi_0). \ (k = 1, 2, \dots) \tag{6}$$

Let f be a form of the vector g which belongs to the domain $\Delta^{(k)}$ of te series (3).

One will have an inequality

$$N(f) \geq N^{(k)}(f). \tag{7}$$

By noticing that because of (2)

$$N(f) = (1-u)N(\varphi_0) + uN(\varphi),$$
$$N^{(k)}(f) = (1-u)N^{(k)}(\varphi_0) + uN^{(k)}(\varphi),$$

the inequality (7) can be written

$$u\left[N(\varphi) - N^{(k)}(\varphi)\right] \geq (1-u)\left[N^{(k)}(\varphi_0) - N(\varphi_0)\right].$$

As $0 < u \leq 1$, this inequality gives, because of (6),

$$N^{(k)}(\varphi) \leq N(\varphi).$$

The quadratic form φ being positive, there exist only a limited number of different systems (5) verifying this inequality. Besides, there exist only a limited number of domains of forms belonging to the set (Δ) which are characterised by the same systems (5). One concludes this that the series (3) will always be terminated by a domain to which belong the given quadratic form φ.

Let us notice that a quadratic form φ which is interior to a domain Δ does not belong to any other domain of the set (Δ), since the primitive parallelohedron corresponding to the quadratic form φ will belong to the type characterised by the domain (Δ) and can not belong to any other type of parallelohedra.

Suppose that a positive quadratic form φ is interior to a face P in a certain number of dimensions of the domain Δ. The set of all the quadratic forms belonging to the face P will be perfectly determined by a certain type of nonprimitive parallelohedra. One concludes that the form φ can not belong to the domains which are contiguous through the face P.

98

By effecting the various transformation of the set (Δ) of quadratic forms with the help of substitutions of integer coefficients and of the determinant which is equal to ±1, one will do only the permutation of domains of the set (Δ).

One concludes that the set (Δ) of domains of forms can be divided into classes of domains composed of equivalent domains.

Theorem II. The number of various classes of domains belonging to the set (Δ) *is finite.*

Let us choose any one domain Δ of the set (Δ) and let φ be a form which is interior to the domain Δ. We have seen in Number 54 that the positive quadratic form can be transformed into another equivalent form φ' which enjoys the property that the system (4) corresponding to the form φ' are made up of integers which do not exceed in numerical value a fixed limit.

The form φ' is interior to a domain Δ' which is equivalent to the domain Δ.

As the domain Δ' is characterised by the systems of integers which do not exceed in numerical value a fixed limit, there exist only a limited number of identical domains in the set (Δ).

99

With the help of the algorithm introduced in Number 96, one can successively determine all the representatives

$$\Delta, \Delta_1, \ldots, \Delta_{\mu-1} \tag{8}$$

of different classes of domains belonging to the set (Δ).

The domains obtained enjoy the same property as the domains of quadratic forms which have been studied in my first mémoire cited. † It results in that the domains (8) can serve in the reduction of positive quadratic forms. By calling reduced the positive quadratic forms which belong to the domains (8), one obtains a new reduction method of positive quadratic forms which is entirely analogous to a reduction method of positive quadratic forms introduced in the cited mémoire.

On the nonprimitive parallelohedra corresponding to positive quadratic forms.

100

Let us suppose that a positive quadratic form φ defines a primitive parallelohedron R.

By virtue of Theorem I of Number 97, the form φ belongs at least to the domain of the set (Δ). The form φ can not be interior to any one domain of the set (Δ) because otherwise the parallelohedron R would be primitive.

Therefore the form φ belongs to one face of domains of the set (Δ).

It results in that the coefficients of the form φ verify one of many linear equations

$$\sum\sum p_{ij}a_{ij} = 0$$

to rational coefficients p_{ij} $(i = 1, 2, \ldots, n; j = 1, 2, \ldots, n)$

One concludes that a positive quadratic form $\sum\sum a_{ij}x_ix_j$ the coefficients of which present an irreducible basis can define only one primitive parallelohedron.

Let us suppose that the examined form φ is interior to a face P to any one number of dimensions of domains belonging to the set (Δ).

Let us indicate by

$$\Delta, \Delta', \ldots, \Delta^{(m)}$$

the domains of the set (Δ) which are contiguous through the face P.

By virtue of that which has been stated in Number 97, one will have the equalities

$$N(\varphi) = N'(\varphi) = \ldots = N^{(m)}(\varphi).$$

One concludes that a positive quadratic form φ can belong to only a finite number of domains of the set (Δ).

101

Let us suppose that an infinite series of quadratic forms

$$f_1, f_2, \ldots \tag{1}$$

is made up of forms which are interior to the domain Δ. Suppose that the forms of this series tend towards a limit φ.

The forms (1) define an infinite series of primitive parallelohedra

$$R_1, R_2, \ldots$$

belonging to the one type characterised by the domain Δ which tend towards a limit R.

One concludes that any nonprimitive parallelohedron R corresponding to a positive quadratic form φ can be considered as a limit of primitive parallelohedra (2).

Let us indicate by the symbol S_ν the number of faces in ν dimensions of the nonprimitive parallelohedron R and by S_ν^0 let us indicate the number of faces in ν dimensions of primitive parallelohedron (2) $(\nu = 0, 1, 2, \ldots, n-1)$.

As the faces of the nonprimitive parallelohedron R are made up of boundaries of faces of primitive parallelohedra belonging to the series (2), one concludes that

$$S_\nu \leq S_\nu^0. \quad (\nu = 0, 1, 2, \ldots, n-1)$$

We have seen in Number 65 that

$$S_\nu^0 \leq (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}, \quad (\nu = 0, 1, 2, \ldots, n)$$

and consequently

$$S_\nu \leq (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}, \quad (\nu = 0, 1, 2, \ldots, n)$$

Principal domain of the set (Δ).

---

† This Journal, V. 133

102

Let us apply the general theory introduced in this mémoire to a positive quadratic form

$$f = nx_1^2 + nx_2^2 + \ldots + nx_n^2 - 2x_1x_2 - 2x_1x_3 \ldots - 2x_{n-1}x_n$$

where one has admitted

$$a_{11} = n \text{ and } a_{ij} = -1. \ (i \neq j; i = 1, 2, \ldots, n; j = 1, 2, \ldots, n) \tag{1}$$

Let us find all the representations of the minimum of the form f in a set composed of all the systems of integers which are congruous to a system $(l_1, l_2, \ldots, l_n)$ with respect to the modulus 2. Let us admit

$$l_1 = 1, \ i = 1, 2, \ldots, \lambda \text{ and } l_i = 0, i = \lambda + 1, \ldots, n. \ (\lambda = 1, 2, \ldots, n) \tag{2}$$

The problem described reduces to the study of the minimum of the form

$$f(l_1 + 2x_1, l_2 + 2x_2, \ldots, l_n + 2x_n)$$

in the set E composed of all the systems (x_i) of integers $x_1, x_2, \ldots, x_n$.

Let us notice that the form f, by virtue of equalities (1), can be written

$$f = \sum_{i=1}^{n} x_i^2 + \sum\sum_{i<j} (x_i - x_j)^2. \tag{3}$$

Each form

$$x_i^2, \ (i = 1, 2, \ldots, n) \ (x_i - x_j)^2, \ (i = 1, 2, \ldots, n; i < j; j = 1, 2, \ldots, n)$$

satisfied, by (2), the condition

$$\begin{cases} (l_i + 2x_i)^2 \geq l_i^2, & (i = 1, 2, \ldots, n) \\ (l_i - l_j + 2(x_i - x_j))^2 \geq (l_i - l_j)^2, & (i = 1, 2, \ldots, n; i < j; j = 1, 2, \ldots, n) \end{cases} \tag{4}$$

Whatever may be the integer values of $x_1, x_2, \ldots, x_n$. It follows, by (3), that

$$f(l_1 + 2x_1, l_2 + 2x_2, \ldots, l_n + 2x_n) \geq f(l_1, l_2, \ldots, l_n).$$

For the equality

$$\varphi(l_1 + 2x_1, l_2 + 2x_2, \ldots, l_n + 2x_n) = \varphi(l_1, l_2, \ldots, l_n)$$

to holds, it is necessary, by (3) and (4), that one had the equalities

$$(l_i + 2x_i)^2 = l_i^2, \ \ l_i - l_j + 2(x_i - x_j) = (l_i - l_j)^2.$$
$$(i = 1, 2, \ldots, n; i < j; j = 1, 2, \ldots, n)$$

By virtue of (2), one obtains

$$x_i = 0 \text{ or } x_i = -l_i, \quad (i = 1, 2, \ldots, n)$$

therefore the form f possesses only two representations of the minimum $(l_1, l_2, \ldots, l_n)$ and $(-l_1, -l_2, \ldots, -l_n)$ in the set examined.

By attributing to the index λ in the inequalities (2) the values $\lambda = 1, 2 \ldots, n$ and by permuting the numbers $l_1, l_2, \ldots, l_n$, one obtains $2^n - 1$ systems which characterise, by virtue of the theorem of Number 48, the faces in $n - 1$ dimensions of the parallelohedron R corresponding to the positive quadratic form f.

The parallelohedron R will be defined by $2(2^n - 1)$ independent inequalities

$$1 \cdot n \pm 2x_{k_1} \geq 0,$$
$$2 \cdot (n - 1) \pm 2(x_{k_1} + x_{k_2}) \geq 0, \quad (k_1 < k_2)$$
$$\ldots$$
$$\lambda(n - \lambda + 1) \pm 2(x_{k_1} + \ldots + x_{k_\lambda}) \geq 0, \qquad (k_1 < k_2 < \cdots < k_\lambda)$$
$$\ldots$$
$$n \cdot 1 \pm 2(x_{k_1} + x_{k_2} + \ldots + x_{k_n}) \geq 0, \qquad (k_1 < k_2 < \cdots < k_n)$$

where $k_1 = 1, 2, \ldots, n, \ k_2 = 2, \ldots, n, \ldots, \ k_\lambda = \lambda, \ldots, n, \ k_n = n.$

To have more convenience in the subsequent notations, let us write

$$u_0 = x_1 + x_2 \ldots + x_n, \ u_1 = -x_1, \ u_2 = -x_2, \ldots, \ u_n = -x_n \tag{5}$$

and notice that all the sums

$$\pm x_{k_1}, \ \pm(x_{k_1} + x_{k_2}), \ldots, \ \pm(x_{k_1} + x_{k_2} + \ldots + x_{k_n})$$

are expressed by the sums

$$u_{h_0}, \ u_{h_0} + u_{h_1}, \ldots, \ u_{h_0} + u_{h_1} + \ldots + u_{h_{n-1}}$$

where $h_0 < h_1 < h_2 < \cdots < h_{n-1}$ and $h_0 = 0, 1, 2, \ldots, n, \ h_1 = 1, 2, \ldots, n, \ldots, \ h_{n-1} = n - 1, \ n.$

The inequalities which define the parallelohedron R can be written

$$\begin{cases} 1\cdot n + 2u_{h_0} \geq 0, \\ 2\cdot (n-1) + 2(u_{h_0} + u_{h_1}) \geq 0, \ (h_0 < h_1) \\ \quad\dots \\ n\cdot 1 + 2(u_{h_0} + u_{h_1} \cdots + u_{h_{n-1}}) \geq 0, \ (h_0 < h_1 < \cdots < h_{n-1}) \end{cases} \tag{6}$$

where $h_0 = 0, 1, 2, \dots, n,\ h_1 = 1, 2, \dots, n, \dots,\ h_{n-1} = n-1, n.$

103

Let us find the vertices of the parallelohedron R. To this effect, let us examine a point (α_i) verifying the equations

$$n + 2u_1 = 0,\ 2(n-1) + 2(u_2 + u_2) = 0, \dots, n\cdot 1 + 2(u_1 + u_2 + \dots + u_n) = 0 \tag{7}$$

By virtue of (5), one obtains

$$\alpha_1 = \frac{1}{2}n,\ \alpha_2 = \frac{1}{2}(n-2),\ \alpha_k = \frac{1}{2}(n-2k+2), \dots, \alpha_n = \frac{1}{2}(-n+2). \tag{8}$$

I argue that the point obtained (α_i) presents a vertex of the parallelohedron R. To demonstrate this, let us examine the form

$$f(x_1, x_2, \dots, x_n) + 2\sum_{i=1}^{n} \alpha_i x_i$$

or, by (8), the form

$$f(x_1, x_2, \dots, x_n) + \sum_{i=1}^{n} (n - 2i + 2)x_i.$$

For the point (α_i) determined by the equalities (8) to be a vertex of the parallelohedron R, it is necessary and sufficient that the inequality

$$f(x_1, x_2, \dots, x_n) + \sum_{i=1}^{n} (n - 2i + 2)x_i \geq 0 \tag{9}$$

holds in the set E. By noticing that

$$f(x_1, x_2, \dots, x_n) + \sum_{i=1}^{n} (n - 2i + 2)x_i =$$
$$\sum_{i=1}^{n} (x_i^2 + x_i) + \sum_{i<j}\sum \left[(x_i - x_j)^2 + x_i - x_j\right],$$

one obtains the inequalities (9) because the inequalities

$$x_i^2 + x_i \geq 0,\ (x_i - x_j)^2 + x_i - x_j \geq 0, \quad (i = 1, 2, \dots, n; j = 2, 3, \dots, n)$$

take place within the set E.

For the equality

$$f(x_1, x_2, \dots, x_n) + \sum_{i=1}^{n} (n - 2i + 2)x_i = 0 \tag{10}$$

to hold, it is necessary and sufficient that one had the equality

$$x_i^2 + x_i = 0,\ (x_i - x_j)^2 + x_i - x_j = 0.(\mathrm{i} = 1, 2, \dots, \mathrm{n}; \mathrm{i} < \mathrm{j}; \mathrm{j} = 2, 3, \dots, \mathrm{n})$$

One declares that

$$x_i = -1,\ (i = 1, 2, \dots, \lambda) \quad x_i = 0.\ (i = \lambda + 1, \dots, n)$$

By attributing to the index λ the values $\lambda = 0, 1, 2 \dots, n$ one obtains $n + 1$ systems verifying the equality (10).

$$(0, 0, \dots, 0), (-1, 0, \dots, 0), (-1, -1, 0, \dots, 0), \dots, (-1, -1, \dots, -1).$$

It is thus demonstrated that the point (α_i) determined by the equations (7) presents a simple vertex of the parallelohedron R.

Let us introduce in our studies symbol

$$(h_0, h_1, h_2, \dots, h_n) \tag{11}$$

in which the indices $h_0, h_1, h_2, \dots, h_n$ present a permutation of numbers $0, 1, 2, \dots, n$ and let us agee to indicate by this symbol a point which verifies the equations

$$\begin{gathered} n + 2u_{h_0} = 0,\ 2(n-1) + 2\left(u_{h_0} + u_{h_1}\right) = 0, \dots, \\ n + 2\left(u_{h_0} + u_{h_1} + \dots, u_{h_{n-1}}\right) = 0. \end{gathered} \tag{12}$$

By virtue of the definition established of the symbol (11), the vertex (α_i) of the parallelohedron R determined by the equations (7) will be characterised by the symbol

$$(1, 2, \dots, n, 0).$$

I argue that each symbol (11) characterises a vertex of parallelohedron R.

To demonstrate this, let us effect a transformation of the parallelohedron R with the help of a substitution

$$u_1 = u'_{h_0},\ u_2 = u'_{h_1}, \dots, u_n = u'_{h_{n-1}},\ u_0 = u'_{h_n}, \tag{13}$$

where one has admitted

$$u'_0 = x'_1 + x'_2 + \ldots + x'_n,\ u'_1 = -x'_1, \dots,\ u'_n = -x'_n.$$

The inequalities (6) which define the parallelohedron R will be permuted by the substitution considered, therefore the parallelohedron R will be transformed into itself.

To vertex (α_i) of the parallelohedron R determined by the equations (7) will be transformed, by virtue of (13), into a vertex of the parallelohedron R determined by the equations (12), therefore the vertex will be characterised by the symbol (11).

We have demonstrated the existence of $(n+1)!$ simple vertices of the parallelohedron R corresponding to the positive quadratic form φ. As the number of vertices of any one parallelohedron corresponding to a positive quadratic form does not exceed a limit $(n+1)!$, by virtue of the formula (3) of Number 101, one concludes that the parallelohedron R does not possess vertices other than those which are characterised by the symbol

$$(h_0, h_1, \dots, h_n)$$

in which one permutes the indices $0, 1, 2, \dots, n$ in every possible ways.

All the vertices of the parallelohedron R are simple, therefore the parallelohedron R is primitive. By noticing that the number of vertices of the parallelohedra R is expressed by the formula

$$S_0 = (n+1)! = (n+1)\Delta^{(m)}(m^n)_{m=1}, \tag{14}$$

one concludes, by virtue of that which has been said in Number 66, that the number S_ν of faces in ν dimensions of the parallelohedron R is expressed by the formula

$$S_\nu = (n+1-\nu)\Delta^{(n-\nu)}(m^n)_{m=1}. \quad (\nu = 0, 1, 2, \dots, n)$$

104 Let us find the regulators and the characteristics of faces in $n-1$ dimensions of simplexes of the set (L) which defines the type of primitive parallelohedra to which belongs the parallelohedron R examined.

Any symbol $(h_0, h_1, \dots, h_n)$ defines a simplex characterised by the linear functions

$$u_{h_0}, u_{h_0} + u_{h_1}, \dots, u_{h_0} + u_{h_1} + \ldots + u_{h_n}.$$

By virtue of (5), one will have identically

$$u_{h_0} + u_{h_1} + \ldots + u_{h_n} = 0. \tag{15}$$

Notice that $n+1$ simplexes which one obtains by carrying out the circular permutations of indices $h_0, h_1, \dots, h_n$

$$(h_0, h_1, \dots, h_n), (h_1, h_2, \dots, h_0), \dots, (h_n, h_0, \dots, h_{n-1})$$

are congruent. By choosing a representative among these simplexes, one will determine in this manner $n!$ incongruent simplexes of the set (L).

Let us examine two simplexes determined by two symbols

$$(h_0, h_1, h_2, \dots, h_n) \quad \text{and} \quad (h_1, h_0, h_2, \dots, h_n)$$

which differ only by a transposition of indices h_0 and h_1.

By virtue of the definition established, these simplexes are characterised by the functions

$$[u_{h_0}, u_{h_0} + u_{h_1}, u_{h_0} + u_{h_1} + u_{h_2}, \dots, u_{h_0} + u_{h_1} + \ldots + u_{h_n}] \tag{16}$$

and

$$[u_{h_1}, u_{h_1} + u_{h_0}, u_{h_1} + u_{h_0} + u_{h_2}, \dots, u_{h_1} + u_{h_0} + \ldots + u_{h_n}]. \tag{17}$$

These two simplexes differ only by the vertices which are characterised by the function u_{h_0} and u_{h_1}.

One concludes that these two simplexes are contiguous by a face in $n-1$ dimensions which is characterised by the functions

$$[u_{h_0} + u_{h_1}, u_{h_0} + u_{h_1} + u_{h_2}, \dots, u_{h_0} + u_{h_1} + \ldots + u_{h_n}]. \qquad ([1]8)$$

Let us determine the characteristic $\pm(p_i)$ of this face. By declaring, as that which we have done in Number 72,

$$\overset{0}{u}_{h_0} + \overset{0}{u}_{h_1} = \delta, \overset{0}{u}_{h_0} + \overset{0}{u}_{h_1} + \overset{0}{u}_{h_2} = \delta, \dots, \overset{0}{u}_{h_0} + \dots, + \overset{0}{u}_{h_1} + \overset{0}{u}_{h_n} = \delta,$$

one obtains, by (15), $\delta = 0$ and consequently

$$\overset{0}{u}_{h_0} = -\overset{0}{u}_{h_1}, \overset{0}{u}_{h_2} = 0, \dots, \overset{0}{u}_{h_n} = 0. \tag{19}$$

By indicating with (p_i) the characteristic of the face (18) with regard to the simplex (16), one will have a supplementary condition

$$\overset{0}{u}_{h_0} > 0 \tag{20}$$

which, added to the equalities (19), well defines the characteristic (p_i).

Let us indicate, to make short,

$$h_0 = i \quad \text{and} \quad h_1 = j \tag{21}$$

and suppose that $i \neq 0$ and $j \neq 0$. By virtue of equalities (5), one will have

$$u^0_{h_0} = u^0_i = -p_i, u^0_{h_1} = u^0_j = -p_j.$$

By virtue of (19) and (20), one obtains

$$\begin{cases} p_k = 0, \\ p_i = -1, p_j = 1. \end{cases} \quad (k = 1, 2, \ldots, n; k \neq i; k \neq j) \tag{22}$$

One can therefore characterise the characteristic (p_i) by a corresponding function

$$\sum p_i x_i = -x_i + x_j.$$

Let us suppose that $j = 0$. One will have in this case the equalities

$$p_1 + p_2 + \ldots + p_n = p_i, p_k = 0 \quad (k = 1, 2, \ldots, n; k \neq i)$$

and consequently, by (20),

$$\sum p_i x_i = -x_i.$$

In the same way one will examine the case $i = 0$. One can bring together the three cases examined by indicating the characteristic of the face (18) by the function $-x_i + x_j$, provided that $x_0 = 0$.

Let us find the regulator ρ_{ij} of the face examined. To that effect, let us determine the number $\vartheta_0, \vartheta_1, \ldots, \vartheta_n$ after the conditions

$$u_{h_1} = \sum_{\lambda=0}^{n} \vartheta_\lambda (u_{h_0} + \ldots + u_{h_\lambda}) \text{ where } \sum \vartheta_\lambda = 1.$$

One obtains

$$\vartheta_0 = -1, \vartheta_1 = 1, \vartheta_2 = 0, \ldots, \vartheta_{n-1} = 0, \vartheta_n = 1.$$

By applying the formula (20) of Number 72, one finds

$$\begin{aligned} 2\rho_{ij} \left[(u^0_{h_j} + \ldots + u^0_{h_\lambda}) - u^0_{h_1} \right] &= (u_{h_1})^2 + (u_{h_0})^2 - (u_{h_0} + u_{h_1})^2 \\ &- (u_{h_0} + u_{h_1} + \ldots + u_{h_n})^2 \text{ where } \lambda > 0. \end{aligned}$$

By virtue of equalities (15) and (19), this formula comes down to the one here

$$2\rho_{ij} = (u_{h_1})^2 + (u_{h_0})^2 - (u_{h_0} + u_{h_1})^2,$$

and consequently

$$\rho_{ij} = -u_{h_0} u_{h_1}$$

or, by (21)

$$\rho_i = -u_i u_j. \tag{23}$$

Let us suppose that $j = 0$; the formulae (5) give

$$\rho_{i0} = x_i(x_1 + x_2 + \ldots + x_n) = x_1 x_i + x_2 x_i + \ldots + x_n x_i.$$

By replacing in this formula $x_i x_j$ by a_{ij}, one obtains the sought-for expression of the regulator ρ_{i0}

$$\rho_{i0} = \sum_{k=1}^{n} a_{ki} \quad (i = 1, 2, \ldots, n) \tag{24}$$

By supposing that $i \neq 0$ and $j \neq 0$, one will have

$$\rho_{ij} = -x_i x_j$$

and consequently

$$\rho_{ij} = -a_{ij} \quad (i = 1, 2, \ldots, \nu; i \neq j; j = 1, 2, \ldots, n) \tag{25}$$

Observe that the face (18) possesses the regulator ρ_{ij} and the characteristic $-x_i + x_j$ in addition to values of indices $h_2, \ldots, h_n$. One concludes that there exist $(n-1)!$ inconcruent faces of simplexes of the set (L) which possess the same regulator ρ_{ij} and the same characteristic

$$-x_0 + x_j. \quad (i = 0, 1, 2, \ldots, n; i \neq j; j = 0, 1, 2, \ldots, n)$$

By applying the formula (I.) from Number 84, one obtains

$$\sum \sum a_{ij} x_i x_j = \sum_{i<j} \sum \rho_{ij} (x_i - x_j)^2,$$
$$(i = 0, 1, 2, \ldots, n; i < j; j = 1, 2, \ldots, n)$$

where one has admitted $x_0 = 0$, or differently

$$\sum\sum a_{ij}x_ix_j = \sum_{i=1}^{n} \rho_{i0}x_i^2 + \sum\sum_{i<j} \rho_{ij}(x_i - x_j)^2. \tag{26}$$

$$(i = 1, 2, \ldots, n; i < j; j = 2, 3, \ldots, n)$$

The domain Δ of quadratic forms corresponding to the type of primitive parallelohedra examined will be determined by the inequalities

$$\rho_{ij} \geq 0 \quad (i = 0, 1, 2, \ldots, n; j = 0, 1, 2, \ldots, n)$$

or differently, according to (24) and (25), by the inequalities

$$\sum_{k=1}^{n} a_{ki} \geq 0, \quad -a_{ij} \geq 0. \ (i = 1, 2, \ldots, n; i < j; j = 2, 3, \ldots, n) \tag{27}$$

The number of these inequalities is equal to $\frac{n(n+1)}{2}$, thus the domain of quadratic forms defined by these inequalities is a simple domain.

By attributing to the parameters ρ_{ij} $(i = 0, 1, 2, \ldots, n; j = 0, 1, 2, \ldots, n; i < j)$ in the formula (26) the positive arbitrary values or zeros, one will determine all the quadratic forms belonging to the domain Δ.

One remarkable coincidence is signalling. The domain of quadratic forms (27) has been studied in my first mémoire cited [†] where it has been called principal domain. This domain corresponds to a principal perfect positive quadratic form

$$\varphi = x_1^2 + x_2^2 + \ldots + x_n^2 + x_1x_2 + \ldots + x_{n-1}x_n.$$

It is remarkable that the set of characteristics found

$$\pm x_i, \ \pm(x_i - x_j), \ (i = 1, 2, \ldots, n; i < j; j = 2, 3, \ldots, n)$$

coincides with the set of representations of the minimum of the principal perfect form φ.

Domains of quadratic forms contiguous to the principal domain.

105 All the faces in $\frac{n(n+1)}{2} - 1$ dimensions of the principal domain Δ are equivalent. [†]

It follows that all the domains of forms belonging to the set (Δ) which are contiguous to the principal domain Δ by the faces in $\frac{n(n+1)}{2} - 1$ dimensions are equivalent.

In the case $n = 2$ and $n = 3$, the set (Δ) of domains of quadratic forms is made up of a single class of domains equivalent to the principal domain.

One concludes that in the space of 2 and of 3 dimensions there is only a single type of primitive parallelohedra, provided that one does not consider as different the equivalent types which correspond to the equivalent domains of quadratic forms.

Let us suppose that $n \geq 4$ and find the domain Δ' which is contiguous to the principal domain Δ by the face determined with the help of the equation

$$\rho = -a_{12} = 0.$$

By applying the algorithm explained in Number 96, let us determine the incongruent convex polyhedra which correspond to the independent regulator ρ.

We have seen in Number 104 that the regulator $\rho = \rho_{12}$ corresponds to the common faces of simplexes defined by the symbols

$$(1, 2, h_1, \ldots, h_n), (2, 1, h_2, \ldots, h_n)$$

where $h_2, h_3, \ldots, h_n$ present an arbitrary permutation of indices $0, 3, 4, \ldots, n$.

The two corresponding simplexes are characterised by the functions

$$[u_1, u_1 + u_2, u_1 + u_2 + u_{h_2}, \ldots, u_1 + u_2 + \ldots + u_{h_n}]$$

and

$$[u_2, u_2 + u_1, u_2 + u_1 + u_{h_2}, \ldots, u_2 + u_1 + \ldots + u_{h_n}].$$

By declaring

$$u_2 = \vartheta_0 u_1 + \vartheta_1(u_1 + u_2) + \vartheta_2(u_1 + u_2 + u_{h_2}) + \ldots + \vartheta_n(u_1 + u_2 + \ldots + u_{h_n})$$

where $\sum_{k=0}^{n} = 1$, one obtains

$$\vartheta_0 = -1, \vartheta_1' = 1, \vartheta_2 = 0, \ldots, \vartheta_{n-1} = 0, \vartheta_n = 1. \tag{3}$$

As among the numbers obtained is found only one negative number ϑ_0, one concludes that the two simplexes (1) and (2) make up a polyhedron K corresponding to the independent regulator ρ.

Let us indicate by (L') the set of simplexes which characterise the domain Δ' of quadratic forms. By virtue of that which has been said in Number 91, the polyhedron K in the set (L') will be made up from simplexes which one obtains by replacing the vertices of the simplex (1) which correspond to the positive values of numbers (3) by the vertex characterised by the function u_2. As in the series (3) only two positive numbers ϑ_1 and ϑ_n are found, one obtains two simplexes characterised by the functions

---

[†] This Journal, V. 133

[†] See my mémoire cited

$$[u_1, u_2, u_1 + u_2 + u_{h_2}, \dots, u_1 + u_2 + \dots + u_{h_n}] \tag{4}$$

and

$$\left[u_1, u_1 + u_2, \dots, u_1 + u_2 + \dots + u_{h_{n-1}}, u_2\right]. \tag{5}$$

These two simplexes make up the polyhedron K and replace the two simplexes (1) and (2) in the set (L').

By effecting all the permutation of indices $h_2, \dots, h_n$, one obtains $(n-1)!$ incongruent convex polyhedra which correspond to the independent regulator ρ.

By replacing in the set (L) the simplexes congruent to the simplexes (1) and (2) by the simplexes which are congruent to the simplexes (4) and (5), one will reconstruct the set (L) of simplexes into a set (L') which characterises the domain Δ'.

Notice that the number of incongruent simplexes of the set (L') is equal to $n!$ also. It follows that the number of faces in ν dimensions of primitive parallelohedra belonging to the type characterised by the domain of form Δ' is expressed by the formula

$$S_\nu = (n + 1 - \nu)\Delta^{(n-\nu)}(m^n)_{m=1}. \quad (\nu = 0, 1, 2, \dots, n)$$

106 Let us find the regulators and the characteristic of faces in $n-1$ dimensions of simplexes belonging to the set (L').

Let us examine in the first place the simplexes contiguous to a face in $n-1$ dimensions which belong to the set (L) and to the set (L').

The condition necessary and sufficient for which the two simplexes characterised by the symbols

$$(h_0, h_1, h_2, \dots, h_n) \text{ and } (h_1, h_0, h_2, \dots, h_n), \tag{6}$$

which are contiguous by a face in $n-1$ dimensions, also belong to the set (L'), consists in so long as within the two series

$$h_0, h_1, h_2, \dots, h_n, h_0 \quad \text{and} \quad h_1, h_0, h_2, \dots, h_n, h_1$$

the indices 1 and 2 are not adjacent. By declaring

$$h_0 = i \quad \text{and} \quad h_1 = j,$$

one obtains $(n-1)! - 2(n-2)!$ pairs of symbols (6) which satisfy the condition assumed.

By indicating with ρ_{ij} the regulator and with $\pm(x_i - x_j)$ the characteristic of the face common to the simplexes (6) determined in the set (L), one will have for the set (L') the same regulator

$$\rho'_{ij} = \rho_{ij}, \quad (i = 0, 1, 2, \dots, n; i < j; j = 1, 2, \dots, n)$$

and the same characteristic $\pm(x_i - x_j)$, the regulator ρ_{12} being excluded. The regulator obtained ρ'_{ij} and the characteristic $\pm(x_i - x_j)$ belong to $(n-1)! - 2(n-2)!$ incongruent faces in $n-1$ dimensions of the set (L') of simplexes.

107 This declared, let us examine the regulators and the characteristics of faces of simplexes (4) and (5) which make up the polyhedron K in the set (L').

The first group of faces of the simplex (4) is composed of two faces which are opposite to the vertices u_1 and u_2. The first face is characterised by the functions

$$u_2, u_1 + u_2 + u_{h_2}, \dots, u_1, u_2, \dots, u_{h_n}. \tag{7}$$

This face belongs in the set (L) to the simplexes characterised by the symbols

$$(2, 1, h_2, h_3, \dots, h_n) \quad \text{and} \quad (2, h_2, 1, h_3, \dots, h_n).$$

The second simplex also belongs to the set (L'). It follows that the simplex (4) is contiguous to the simplex $(2, h_2, 1, h_3, \dots, h_n)$ by the face (7).

Let us declare $h_2 = i$ where $i = 0, 3, \dots, n$ and indicate by ρ'_{i1} the regulator and by $\pm(x_1 - x_i)$ the characteristic of the face (7) in the set (L'). By applying the formula (8) of Number 93, one obtains

$$\rho'_{i1} = \rho_{i1} + \rho. \quad (i = 0, 3, \dots, n)$$

In the same manner, one will examine the regulators of the face of the simplex (4) which is opposite to the vertex u_2. By putting $h_2 = i$, one will have

$$\rho'_{i2} = \rho_{i2} + \rho. \quad (i = 0, 3, \dots, n) \tag{9}$$

Examine the first group of faces of the simplex (5). This group is made up of two faces which are opposite to the vertices u_1 and u_2. The first face is characterised by the functions.

$$u_1 + u_2, u_1 + u_2 + u_{h_2}, \dots, u_1 + u_2 + \dots + u_{h_{n-1}}, u_2.$$

This face belongs in the set (L) to the simplex $(2, 1, h_2, \dots, h_n)$ and to the simplex congruent to the simplex $(h_n, 1, h_2, \dots, h_{n-1}, 2)$ and which is characterised by the functions

$$u_2, u_2 + u_1, u_2 + u_1 + u_{h_2}, \dots, u_2 + u_1 + \dots + u_{h_{n-1}}, u_2 - u_{h_n}.$$

This simplex also belongs to the set (L'). By putting $h_n = i$ and by applying the formula (8) of Number (93), one finds

$$\rho'_{i2} = \rho_{i2} + \rho$$

The characteristic of this face will be $\pm(x_2 - x_i)$ $(i = 0, 3, \ldots, n)$.

In the same way one will examine the face of the simplex (5) which is opposite to the vertex u_2.

One will obtain by letting $h_n = i$

$$\rho'_{i1} = \rho_{i1} + \rho$$

and the characteristic will be $\pm(x_1 - x_i)$, $(i = 0, 3, \ldots, n)$.

Let us notice that the number of incongruent faces, belonging to the first group of simplexes of the set (L'), which possess the regulator determined by the formula (8) or by the formula (9), is equal to $2(n-2)!$.

108 The second group of faces in $n-1$ dimensions of the simplex (4) is composed of $n-2$ faces which are opposite to the vertices

$$u_1 + u_2 + u_{h_2}, u_1 + u_2 + u_{h_2} + u_{h_3}, \ldots, u_1 + u_2 + \ldots, u_{h_{n-1}}.$$

Let us examine a face which is opposite to the vertex $u_1 + u_2 + \ldots + u_{h_k}$ $(k = 2, 3, \ldots, n-1)$.

A transposition of indices h_k and h_{k+1} in the symbol $(*)$ leads to the symbol

$$[u_1, u_2, \ldots, u_1 + u_2 + \ldots + u_{h_{k-1}} + u_{h_{k+1}}, u_1 + u_2 + \ldots + u_{h_{k+1}} + u_{h_k}, \\ \ldots, u_1 + u_2 + \ldots + u_{h_n}]$$

which defines a simplex belonging to the set (L') and which is contiguous to the simplex (4) by the face in $n-1$ dimensions examined.

Let us write $h_k = i, h_{k+1} = j$ and indicate by ρ'_{ij} the corresponding regulator. The corresponding characteristic will be determined by the equations

$$u_1^0 = 0, u_2^0 = 0, \ldots, u_1^0 + u_2^0 + \ldots + u_{h_{k-1}}^0 = 0, \\ u_1^0 + u_2^0 + \ldots + u_{h_{k+1}}^0 = 0, u_1^0 + u_2^0 + \ldots + u_{h_n}^0 = 0.$$

One obtains

$$u_i^0 = -u_j^0 \text{ and } u_r^0 = 0. \quad (r = 0, 1, 2, \ldots, n; r \neq i; r \neq j)$$

It follows that the characteristic will be represented by the function $\pm(x_i - x_j)$.

By declaring

$$\begin{aligned} u_1 + u_2 + \ldots + u_{h_{k-1}} + u_{h_{k+1}} = {} & \vartheta_0 u_1 + \vartheta_1 u_2 + \vartheta_2(u_1 + u_2 + u_{h_2}) + \ldots \\ & + \vartheta_k(u_1 + u_2 + \ldots + u_{h_{k-1}} + u_{h_k}) + \ldots + \vartheta_n(u_1 + u_2 + \ldots + u_{h_n}) \end{aligned} \tag{10}$$

where $\sum_{r=0}^{n} \vartheta_r = 1$, one obtains

$$\vartheta_0 = 0, \vartheta_1 = 0, \ldots, \vartheta_{k-2} = 0, \vartheta_{k-1} = 1, \vartheta_k = -1, \vartheta_{k+1} = 1, \vartheta_{k+2} = 0, \\ \ldots, \vartheta_n = 0,$$

provided that $k > 2$.

The regulator ρ'_{ij} will be determined by the formula

$$\begin{aligned} 2\rho'_{ij} = {} & (u_1 + u_2 + \ldots + u_{h_{k-1}} + u_{h_{k+1}})^2 - (u_1 + u_2 + \ldots + u_{h_{k-1}})^2 + \\ & (u_1 + u_2 + \ldots + u_{h_k})^2 - (u_1 + u_2 + \ldots + u_{h_{k+1}})^2. \end{aligned}$$

After the reductions, one finds

$$\rho'_{ij} = -u_{h_k} u_{h_{k+1}}$$

or differently

$$\rho'_{ij} = -u_i u_j,$$

thus, by virtue of the formula (23) of Number 104, one will have

$$\rho'_{ij} = \rho_{ij}. \quad (i = 0, 3, \ldots, n; j = 0, 3, \ldots, n; k = 3, 4, \ldots, n-1) \tag{11}$$

Let us examine the case $k = 2$. The equality (10) gives in this case

$$\vartheta_0 = 1, \vartheta_1 = 1, \vartheta_2 = -1, \vartheta_3 = 1, \vartheta_4 = 0, \ldots, \vartheta_{n-1} = 0, \vartheta_n = -1,$$

and consequently

$$\begin{aligned} 2\rho'_{ij} = {} & (u_1 + u_2 + u_{h_3})^2 - u_1^2 - u_2^2 + (u_1 + u_2 + u_{h_2})^2 \\ & - (u_1 + u_2 + u_{h_2} + u_{h_3})^2 + (u_1 + u_2 + \ldots + u_{h_n})^2. \end{aligned}$$

As $u_{h_2} = i$, $u_{h_3} = j$ and $u_1 + u_2 + \ldots + u_{h_n} = 0$, one obtains

$$\rho'_{ij} = u_1 u_2 - u_i u_j,$$

thus one will have in the case examined

$$\rho'_{ij} = \rho_{ij} - \rho. \quad (i = 0, 3, \ldots, n; j = 0, 3, \ldots, n) \tag{12}$$

In the same way, one will examine the faces of the simplex (5) belonging to the second group and one will obtain the same formulae (11) and (12).

Let us notice that the number of incongruent faces of simplexes of the set (L') which belong to the second group and the regulator of which is determined by the formula (11) is equal to $(n-3)!2(n-3)$. The number of regulators which are determined by the formula (12) is equal to $2(n-3)!$.

109 The third group of faces of simplexes (4) and (5) is composed of a single face

$$\left[u_1, u_2, u_1+u_2+u_{h_2}, \ldots, u_1+u_2+\ldots+u_{h_{n-1}}\right]$$

which is common to these two simplexes.

The characteristic of this face is determined by the equations

$$u_1^0=\delta, u_2^0=\delta, u_1^0+u_2^0+u_{h_2}^0=\delta, \ldots, u_1^0+u_2^0+\ldots+u_{h_{n-1}}^0=\delta.$$

It results in that

$$u_1^0=u_2^0=\delta, u_2^0=-\delta, u_{h_3}^0=0, \ldots, u_{h_{n-1}}^0=0, u_{h_n}^0=\delta.$$

One concludes that $\delta=\pm 1$. By admitting

$$h_2=i \quad \text{and} \quad h_n=j,$$

one obtains the characteristic $\pm(x_1+x_2-x_i-x_j)$.

Let us indicate the corresponding regulator by ρ'_{ij}. with the help of equalities

$$\begin{aligned} u+u_2 =& \vartheta_0 u_1+\vartheta_1 u_2+\vartheta_2(u_1+u_2+u_{h_2})+\ldots \\ & +\vartheta_n(u_1+u_2+\ldots+u_{h_n}) \text{ where } \sum_{k=1}^{n}\vartheta_k=1, \end{aligned}$$

one obtains

$$\vartheta_0=1, \vartheta_1=1, \vartheta_2=0, \ldots, \vartheta_{n-1}=0, \vartheta_n=-1.$$

The regulator ρ'_{ij} will be determined by the formula

$$2\rho'_{ij}=(u_1+u_2)^2-u_1^2-u_2^2-(u_1+u_2+\ldots+u_{h_n}),$$

and it becomes

$$\rho'_{ij}=u_1u_2,$$

thus

$$\rho'_{ij}=-\rho. \quad (i=0,3,\ldots,n; j=0,3,\ldots,n)$$

The number of incongruent faces belonging to the third group having the characteristic $\pm(x_1+x_2-x_i-x_j)$ is equal to $2(n-3)!$.

110 With the help of deduced formulae, one can determine all the independent regulators. Let us admit

$$\rho'_{12}=-\rho,\ \rho'_{i1}=\rho i1+\rho,\ \rho'_{i2}=\rho i2+\rho,\ \rho'_{ij}=\rho_{ij}. \tag{13}$$

$$(i=0,3,\ldots,n; i<j; j=3,\ldots,n)$$

One obtains in this manner $\frac{n(n+1)}{2}$ independent regulators ρ'_{ij} $(i=0,1,2,\ldots,n; i<j; j=1,2,\ldots,n)$.

The results of our studies can be gathered in the following table:

1. Regulator ρ'_{12}, characteristic $\pm(x_1+x_2-x_i-x_j)$, their number $2(n-3)!$
2. Regulator ρ'_{i1}, characteristic $\pm(x_1-x_i)$, their number $2(n-2)!$
3. Regulator ρ'_{i2}, characteristic $\pm(x_2-x_i)$, their number $2(n-2)!$
4. Regulator $\rho'_{i1}+\rho'_{12}$, characteristic $\pm(x_1-x_i)$, their number $(n-1)!-2(n-2)!$
5. Regulator $\rho'_{i2}+\rho'_{12}$, characteristic $\pm(x_2-x_i)$, their number $(n-1)!-2(n-2)!$
6. Regulator ρ'_{ij}, characteristic $\pm(x_i-x_j)$, their number $(n-1)!-2(n-3)!$
7. Regulator $\rho'_{ij}+\rho'_{12}$, characteristic $\pm(x_i-x_j)$, their number $2(n-3)!$

The indices i and j are the values $0,3,\ldots,n$ and one has admitted $x_0=0$.

The domain Δ' of quadratic forms corresponding to the set (L') of simplexes is determined by $\frac{n(n+1)}{2}$ independent inequalities

$$\rho'_{ij}\geq 0. \quad (i=0,1,2,\ldots,n; j=0,1,2,\ldots,n; i\neq j)$$

As a result, the domain Δ' is simple.

By applying the formula (I) of Number 84, one will determine, by (13), any quadratic form $\sum\sum a_{ij}x_ix_j$ by the following formula

$$\sum\sum a_{ij}x_ix_j = \sum_{i<j}\sum\rho'_{ij}(x_i-x_j)^2+\rho'_{12}\omega, \tag{14}$$

$$(i=0,1,2,\ldots,n; j=1,2,\ldots,n)$$

where one has admitted $x_0 = 0$ and

$$\omega = (n-2)x_1^2 + (n-2)x_2^2 + 2x_3^2 + \ldots + 2x_n^2 + 2x_1x_2 + 2x_1x_3 - \ldots \\ - 2x_1x_n - 2x_2x_3 - \ldots - 2x_2x_n.$$

By attributing to the independent parameters ρ'_{ij} $(i = 0, 1, 2, \ldots, n;\ j = 0, 1, 2, \ldots, n)$ all the positive values or null, one will determine by the formula (14) all the quadratic forms belonging to the domain Δ'.

One coincidence is to be pointed out: the domain Δ' presents a part of the domain R, corresponding to the perfect form φ, which has been determined in my mémoire cited. The set composed of linear forms

$$\pm(x_i - x_j) \quad (i = 0, 1, 2, \ldots, n; j = 0, 1, 2, \ldots, n)$$

(the form $\pm(x_1 - x_2)$ being excluded) and of forms

$$(x_1 + x_2 - x_i - x_j) \quad (i = 0, 3, \ldots, n; j = 0, 3, \ldots, n)$$

where one has put $x_0 = 0$, coincides with the linear forms which define all the representations of the minimum of the perfect form φ_1.

Parallelohedra in two dimensions

111 The set (Δ) of domains of binary quadratic forms is composed of a single class of domains which are equivalent to the principal domain Δ determined by the inequalities

$$a + b \geq 0, \quad -b \geq 0, \quad c + b \geq 0.$$

Here are the conditions of reduction of binary positive quadratic forms $ax^2 + 2bxy + cy^2$ due to Selling. [‡] Any quadratic form belonging to the principal domain Δ can be determined by the equalities

$$ax^2 + 2bxy + cy^2 = \lambda x^2 + \mu y^2 + \nu(x-y)^2 \text{ where } \lambda > 0,\ \mu \geq 0,\ \nu \geq 0.$$

The parameters λ, μ and ν present the regulators of the hexagon of Lejeune Dirichlet [‡] defined by the inequalities

$$\begin{aligned} -\frac{1}{2}(\lambda + \nu) \leq x \leq \frac{1}{2}(\lambda + \nu), \\ -\frac{1}{2}(\mu + \nu) \leq y \leq \frac{1}{2}(\mu + \nu), \\ -\frac{1}{2}(\lambda + \mu) \leq x + y \leq \frac{1}{2}(\lambda + \mu), \end{aligned} \quad .$$

By attributing to the arbitrary parameters λ, μ, ν the positive values, one will determine by these inequalities a primitive parallelohedron in two dimensions, that is to say a hexagon of Lejeune Dirichlet.

By nullifying one of the parameters λ, μ, ν, for example ν, one will obtain four independent inequalities.

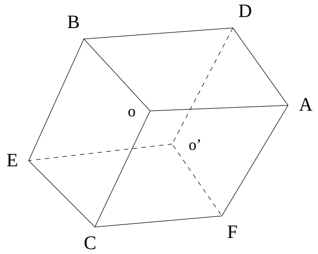
$$\begin{aligned} -\frac{1}{2}\lambda \leq x \leq \frac{1}{2}\lambda, \\ -\frac{1}{2}\mu \leq y \leq \frac{1}{2}\mu, \end{aligned}$$

which define a nonprimitive parallelohedron in two dimensions, it is a parallelogram.

It is easy to demonstrate that other nonprimitives of the space in two dimensions do not exist.

Each hexagon of Lejeune Dirichlet can be constructed from three parallelograms as is indicated in Fig. 1.

Fig. 1



One of the three parallelograms $OADB$, $OBEC$ and $OCFA$ which form the hexagon $ADBECF$ can be arbitrarily chosen. By choosing, for example, an arbitrary parallelogram $OADB$, one will determine the two remaining parellograms $OBEC$ and $OCFA$ by taking an arbitrary vector OC, provided that by extending this vector in the inverse direction one passes through the chosen parallelogram $OADB$.

Observe that in general the point O does not present the centre of the hexagon $ADBECF$.

One can make up the same hexagon of three parallelograms $O'DBE$, $O'ECF$ and $O'FAD$. One concludes that the hexagon of Lejeune Dirichlet does not present anything other than projection of a parallelepiped on the plane.

Parallelohedra in three dimensions

112

[‡] See the Introduction

The set (Δ) of domains of ternary quadratic forms is composed of a single class of domains equivalent to the principal domain Δ determined by the inequalities

$$a+b'+b''\geq 0, a'+b''+b\geq 0, a''+b+b'\geq 0, -b\geq 0, -b'\geq 0, -b''\geq 0.$$

Here are the conditions of reduction of ternary positive quadratic forms $ax^2+a'y^2+a''z^2+2byz+2b'zx+2b''xy$ due to Selling.

Any ternary quadratic form belonging to the principal domain Δ can be determined by the equalities

$$\begin{aligned} ax^2+a'y^2+a''z^2+2byz+2b'zx+2b''xy = \\ \lambda x^2+\lambda' y^2+\lambda'' z^2+\mu(y-z)^2+\mu'(z-x)^2+\mu''(x-y)^2. \end{aligned}$$

All the primitive parallelohedra in three dimensions can be transformed with the help of linear substitutions into primitive parallelohedra determined by 14 independent inequalities

$$\left\{ \begin{array}{lrcll} (1') & -\frac{1}{2}(\lambda+\mu'+\mu'') & \leq x \leq & \frac{1}{2}(\lambda+\mu'+\mu''), & (1'), \\ (2') & -\frac{1}{2}(\lambda'+\mu''+\mu) & \leq y \leq & \frac{1}{2}(\lambda'+\mu''+\mu), & (2'), \\ (3') & -\frac{1}{2}(\lambda''+\mu+\mu') & \leq z \leq & \frac{1}{2}(\lambda''+\mu+\mu'), & (3'), \\ (4') & -\frac{1}{2}(\lambda'+\lambda''+\mu'+\mu'') & \leq y+z \leq & \frac{1}{2}(\lambda'+\lambda''+\mu'+\mu''), & (4'), \\ (5') & -\frac{1}{2}(\lambda_{\prime\prime}+\lambda+\mu_{\prime\prime}+\mu) & \leq z+x \leq & \frac{1}{2}(\lambda_{\prime\prime}+\lambda+\mu_{\prime\prime}+\mu), & (5'), \\ (6') & -\frac{1}{2}(\lambda+\lambda_{\prime}+\mu+\mu') & \leq x+y \leq & \frac{1}{2}(\lambda+\lambda_{\prime}+\mu+\mu'), & (6'), \\ (7') & -\frac{1}{2}(\lambda+\lambda'+\lambda'') & \leq x+y+z \leq & \frac{1}{2}(\lambda+\lambda'+\lambda''), & (7') \end{array} \right. \qquad (2)$$

The parameter $\lambda, \lambda_{\prime}, \lambda_{\prime\prime}, \mu, \mu_{\prime}, \mu_{\prime\prime}$ present the independent regulators of the primitive parallelohedron defined by these inequalities and corresponding to a ternary positive quadratic form (1).

By virtue of the formula (14) of Number 103, any primitive parallelohedron of the space in three dimensions possesses 24 vertices which can be characterised by three numbers corresponding to the different faces in two dimensions of the parallelohedron defined by the inequalities (2).

One will divide these (24) vertices into three groups I, II and III:

I	$1\ 6\ 7$	$1'\ 2\ 4$	$2'\ 6'\ 3$	$4'\ 3'\ 7'$
	$1'\ 6'\ 7'$	$1\ 2'\ 4'$	$2\ 6\ 3'$	$4\ 3\ 7$
II	$1\ 7\ 5$	$1'\ 4\ 3$	$4'\ 7'\ 2'$	$3'\ 2\ 5'$
	$1'\ 7'\ 5'$	$1\ 4'\ 3'$	$4\ 7\ 2$	$3\ 2'\ 5$
III	$1\ 3'\ 6$	$1'\ 5'\ 2$	$5\ 3\ 7$	$2'\ 7'\ 6'$
	$1'\ 3\ 6'$	$1\ 5\ 2'$	$5'\ 3'\ 7'$	$2\ 7\ 6$

Each line of this table is composed of four congruent vertices. In each group the second line is formed from vertices opposite to those which are found in the first line.

Let us examine the regulators and the characteristics of edges of the primitive parallelohedron in three dimensions. It suffices to examine the regulators and the characteristics of faces in two dimensions belonging to three simplexes

$$(0,1,6,7),\ \ (0,1,7,5),\ \ (0,1,3',6).$$

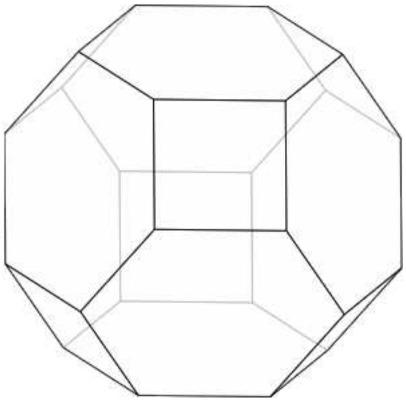
The results of these studies can be brought together in the following table:

I	$0\ 1\ 6\ 7$	$-1\ 0\ \ 0$	λ	$1\ -1\ \ 0$	μ''	$0\ \ 1\ -1$	μ	$0\ \ 0\ \ 1$	λ''
		$2\ 1\ \ 1$		$0\ \ 1\ \ 0$		$1\ \ 0\ \ 1$		$0\ \ 0\ -1$	
II	$0\ 1\ 7\ 5$	$-1\ 0\ \ 0$	λ	$1\ \ 0\ -1$	μ'	$0\ \ 1\ \ 0$	λ'	$0\ -1\ \ 1$	μ
		$2\ 1\ \ 1$		$0\ \ 0\ \ 1$		$0\ -1\ \ 0$		$1\ \ 1\ \ 0$	
III	$0\ 1\ 3'\ 6$	$-1\ 0\ \ 1$	μ'	$1\ -1\ \ 0$	μ''	$0\ \ 0\ -1$	$\lambda_{\prime\prime}$	$0\ \ 1\ \ 0$	$\lambda_{\prime}$
		$1\ 0\ -1$		$0\ \ 1\ \ 9$		$1\ \ 1\ \ 1$		$0\ -1\ -1$	

The first line contains the characteristics and the regulators of different faces of the corresponding simplexes. The second line is composed of vertices which define the simplexes contiguous to the corresponding simplex by the faces the characteristics of which are found above, in the first line.

The faces of primitive parallelohedron R in three dimensions (Fig. 2) are divided into 8 hexagons of Lejeune Dirichlet and into 6 parallelograms.

Fig. 2



The hexagonal faces of the primitive parallelohedron R are characterised by the numbers

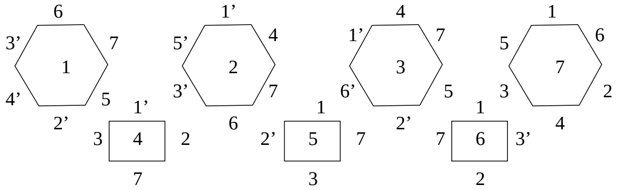
$$1, 2, 3, 7, 1', 2', 3', 7'.$$

The parallelogrammatic faces are characterised by the numbers

$$4, 5, 6, 4', 5', 6'.$$

The faces, the edges and the vertices of the primitive parallelohedron are systematically characterised in Fig. 3.

Fig. 3



One has indicated in this figure the numbers of faces which are contiguous to one of 7 incongruent faces. Each edge is characterised by two adjacent numbers and each vertex by three numbers.

113 By nullifying one or more parameters $\lambda, \lambda', \lambda'', \mu, \mu', \mu''$ in the inequalities (2), one will determine the nonprimitive parallelohedra in three dimensions. It is easy to see that the nonprimitive parallelohedra obtained are divided into four different spaces. (Fig. 4–7)

Fig. 4

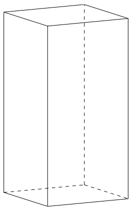


Fig. 5

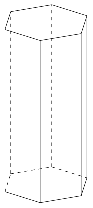


Fig. 6

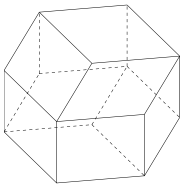
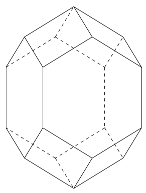


Fig. 7



Nonprimitive parallelohedra of the 1^{st} *space.* By making $\mu = 0, \mu' = 0, \mu'' = 0$ in the inequalities (2), one will obtain 6 independent inequalities

$$\begin{array}{rcl} -\frac{1}{2}\lambda & \leq x \leq & \frac{1}{2}\lambda, \\ -\frac{1}{2}\lambda' & \leq y \leq & \frac{1}{2}\lambda', \\ -\frac{1}{2}\lambda'' & \leq z \leq & \frac{1}{2}\lambda'', \end{array}$$

which define a parallelepiped (Fig. 4).

Nonprimirive parallelohedra of the 2^{nd} *space.* By making $\mu' = 0, \mu'' = 0$ in the inequalities (2), one will obtain 8 independent inequalities

$$\begin{array}{rcl} -\frac{1}{2}\lambda & \leq x \leq & \frac{1}{2}\lambda, \\ -\frac{1}{2}(\lambda' + \mu) & \leq y \leq & \frac{1}{2}(\lambda' + \mu), \\ -\frac{1}{2}(\lambda'' + \mu) & \leq z \leq & \frac{1}{2}(\lambda'' + \mu), \\ -\frac{1}{2}(\lambda' + \lambda'') & \leq y + z \leq & \frac{1}{2}(\lambda' + \lambda''), \end{array}$$

which define a prism with hexagonal base (Fig. 5).

Nonprimirive parallelohedra of the 3^{rd} *space.* By making $\lambda'' = 0, \mu'' = 0$ in the inequalities (2), one will obtain 12 independent inequalities

$$\begin{array}{rcl}
-\frac{1}{2}(\lambda+\mu') & \le x \le & \frac{1}{2}(\lambda+\mu'), \\
-\frac{1}{2}(\lambda'+\mu) & \le y \le & \frac{1}{2}(\lambda'+\mu), \\
-\frac{1}{2}(\mu+\mu') & \le z \le & \frac{1}{2}(\mu+\mu'), \\
-\frac{1}{2}(\lambda'+\mu') & \le y+z \le & \frac{1}{2}(\lambda'+\mu'), \\
-\frac{1}{2}(\lambda+\mu) & \le z+x \le & \frac{1}{2}(\lambda+\mu), \\
-\frac{1}{2}(\lambda+\lambda') & \le x+y+z \le & \frac{1}{2}(\lambda+\lambda'),
\end{array}$$

which define a parallelogrammatic dodecahedron (Fig. 6).

Nonprimirive parallelohedra of the 4^{th} *space.* By making $\mu''=0$ in the inequalities (2), one will obtain 12 independent inequalities

$$\begin{array}{rcl}
-\frac{1}{2}(\lambda+\mu') & \le x \le & \frac{1}{2}(\lambda+\mu'), \\
-\frac{1}{2}(\lambda'+\mu) & \le y \le & \frac{1}{2}(\lambda'+\mu), \\
-\frac{1}{2}(\lambda''+\mu+\mu') & \le z \le & \frac{1}{2}(\lambda''+\mu+\mu'), \\
-\frac{1}{2}(\lambda'+\lambda''+\mu') & \le y+z \le & \frac{1}{2}(\lambda'+\lambda''+\mu'), \\
-\frac{1}{2}(\lambda''+\lambda+\mu) & \le z+x \le & \frac{1}{2}(\lambda''+\lambda+\mu), \\
-\frac{1}{2}(\lambda+\lambda'+\lambda'') & \le x+y+z \le & \frac{1}{2}(\lambda+\lambda'+\lambda''),
\end{array}$$

which define a dodecahedron in 4 hexagonal faces and in 8 parallelogrammatic face (Fig. 7). Mr. Fedorow has demonstrated that other parallelohedra in three dimensions do not exist. ¶

Parallelohedra in four dimensions

114 The first type of primitive parallelohedra in four dimensions is characterised by the principal domain Δ of quaternary quadratic forms which is determined by the independent inequalities

$$\begin{gathered}
\lambda_1 = a_{11}+a_{12}+a_{13}+a_{14} \ge 0, \quad \lambda_2 = a_{21}+a_{22}+a_{23}+a_{24} \ge 0, \\
\lambda_3 = a_{31}+a_{32}+a_{33}+a_{34} \ge 0, \quad \lambda_4 = a_{41}+a_{42}+a_{43}+a_{44} \ge 0, \\
\mu_1 = -a_{12} \ge 0, \quad \mu_2 = -a_{13} \ge 0, \quad \mu_3 = -a_{14} \ge 0, \\
\mu_4 = -a_{23} \ge 0, \quad \mu_5 = -a_{24} \ge 0, \quad \mu_6 = -a_{34} \ge 0.
\end{gathered}$$

Any quaternary quadratic form

$$f(x_1,x_2,x_3,x_4)$$

belonging to the domain Δ can be determined by the equalities

$$\begin{aligned}
f(x_1,x_2,x_3,x_4) = {} & \lambda_1 x_1^2+\lambda_2 x_2^2+\lambda_3 x_3^2+\lambda_4 x_4^2+\mu_1(x_1-x_2)^2+ \\
& \mu_2(x_1-x_3)^2+\mu_3(x_1-x_4)^2+\mu_4(x_2-x_3)^2+ \qquad (1) \\
& \mu_5(x_2-x_4)^2+\mu_6(x_3-x_4)^2.
\end{aligned}$$

The corresponding parallelohedron is determined by 30 inequalities which one will write down in the form

$$\pm(l_1x_1+l_2x_2+l_3x_3+l_4x_4) \le \frac{1}{2}f(l_1,l_2,l_3,l_4). \qquad (2)$$

The systems $\pm(l_1,l_2,l_3,l_4)$ and the corresponding values

$$f(l_1,l_2,l_3,l_4)$$

of the quadratic form (1) are given in the following table:

Ist *type of parallelohedra*

N	l_1	l_2	l_3	l_4	$f(l_1,l_2,l_3,l_4)$	$-l_1$	$-l_2$	$-l_3$	$-l_4$	$2'$
1	1	0	0	0	$\lambda_1+\mu_1+\mu_2+\mu_3$	−1	0	0	0	$1'$
2	0	2	0	0	$\lambda_2+\mu_1+\mu_4+\mu_5$	0	−1	0	0	$2'$
3	0	0	1	0	$\lambda_3+\mu_2+\mu_4+\mu_6$	0	0	−1	0	$3'$
4	0	0	0	1	$\lambda_4+\mu_3+\mu_5+\mu_6$	0	0	0	−1	$4'$
5	1	1	0	0	$\lambda_1+\lambda_2+\mu_2+\mu_3+\mu_4+\mu_5$	−1	−1	0	0	$5'$
6	1	0	1	0	$\lambda_1+\lambda_3+\mu_1+\mu_3+\mu_4+\mu_6$	−1	0	−1	0	$6'$
7	1	0	0	1	$\lambda_1+\lambda_4+\mu_1+\mu_2+\mu_5+\mu_6$	−1	0	0	−1	$7'$
8	0	1	1	0	$\lambda_2+\lambda_3+\mu_1+\mu_2+\mu_5+\mu_6$	0	−1	−1	0	$8'$
9	0	1	0	1	$\lambda_2+\lambda_4+\mu_1+\mu_3+\mu_4+\mu_6$	0	−1	0	−1	$9'$
10	0	0	1	1	$\lambda_3+\lambda_4+\mu_2+\mu_3+\mu_4+\mu_5$	0	0	−1	−1	$10'$
11	1	1	1	0	$\lambda_1+\lambda_2+\lambda_3+\mu_3+\mu_5+\mu_6$	−1	−1	−1	0	$11'$
12	1	1	0	1	$\lambda_1+\lambda_2+\lambda_4+\mu_2+\mu_4+\mu_6$	−1	−1	0	−1	$12'$
13	1	0	1	1	$\lambda_1+\lambda_3+\lambda_4+\mu_1+\mu_4+\mu_5$	−1	0	−1	−1	$13'$
14	0	1	1	1	$\lambda_1+\lambda_3+\lambda_4+\mu_1+\mu_2+\mu_3$	0	−1	−1	−1	$14'$
15	1	1	1	1	$\lambda_1+\lambda_2+\lambda_3+\lambda_4$	−1	−1	−1	−1	$15'$

By attributing to the parameters

$$\lambda_1,\ \lambda_2,\ldots,\ \mu_6$$

the arbitrary positive values, one will determine with the help of inequalities (2) all the primitive parallelohedra of the first type.

¶ See the Introduction

The primitive parallelohedra of the first type possess 120 vertices which can be divided into 12 groups composed of congruent vertices and of opposite vertices.

All these vertices are put together in the following table:

Vertices of primitive parallelohedron of Ist *type*

I	$1\ 5\ 11\ 15$	$1'\ 2\ 8\ 14$	$2'\ 5'\ 3\ 10$	$8'\ 3'\ 11'\ 4$	$14'\ 10'\ 4'\ 15'$
	$1'\ 5'\ 11'\ 15'$	$1\ 2'\ 8'\ 4'$	$2\ 5\ 3'\ 10'$	$8\ 3\ 11\ 4'$	$14\ 10\ 4\ 15$
II	$1\ 5\ 3'\ 10'$	$1'\ 2\ 6'\ 13'$	$2'\ 5'\ 11'\ 15'$	$6\ 11\ 3\ 4'$	$13\ 15\ 4\ 10$
	$1'\ 5'\ 3\ 10$	$1\ 2'\ 6\ 13$	$2\ 5\ 11\ 15$	$6'\ 11'\ 3'\ 4$	$13'\ 15'\ 4'\ 10'$
III	$1\ 6\ 11\ 4'$	$1'\ 3\ 8\ 7'$	$3'\ 6'\ 2\ 13'$	$8'\ 2'\ 11'\ 15'$	$7\ 13\ 15\ 4$
	$1'\ 6'\ 11'\ 4$	$1\ 3'\ 8'\ 7$	$3\ 6\ 2'\ 13$	$8\ 2\ 11\ 15$	$7'\ 13'\ 15'\ 4'$
IV	$1\ 7\ 13\ 15$	$1'\ 4\ 10\ 14$	$4'\ 7'\ 3\ 8$	$10'\ 3'\ 13'\ 2$	$14'\ 8'\ 2'\ 15'$
	$1'\ 7'\ 13'\ 15'$	$1\ 4'\ 10'\ 14'$	$4\ 7\ 3'\ 8'$	$10\ 3\ 13\ 2'$	$14\ 8\ 2\ 15$
V	$1\ 5\ 12\ 15$	$1'\ 2\ 9\ 14$	$2'\ 5'\ 4\ 10$	$9'\ 4'\ 12'\ 3$	$14'\ 10'\ 3'\ 15'$
	$1'\ 5'\ 12'\ 15'$	$1\ 2'\ 9'\ 14'$	$2\ 5\ 4'\ 10'$	$9\ 4\ 12\ 3'$	$14\ 10\ 3\ 15$
VI	$1\ 5\ 4'\ 10'$	$1'\ 2\ 7'\ 13'$	$2'\ 5'\ 12'\ 15'$	$7\ 12\ 4\ 3'$	$13\ 15\ 3\ 10$
	$1'\ 5'\ 4\ 10$	$1\ 2'\ 7\ 13$	$2\ 5\ 12\ 15$	$7'\ 12'\ 4'\ 3$	$13'\ 15'\ 3'\ 10'$
VII	$1\ 7\ 12\ 3'$	$1'\ 4\ 9\ 6'$	$4'\ 7'\ 2\ 13'$	$9'\ 2'\ 12'\ 15'$	$6\ 13\ 15\ 3$
	$1'\ 7'\ 12'\ 3$	$1\ 4'\ 9'\ 6$	$4\ 7\ 2'\ 13$	$9\ 2\ 12\ 15$	$6'\ 13'\ 15'\ 3'$
VIII	$1\ 6\ 13\ 15$	$1'\ 3\ 10\ 14$	$3'\ 6'\ 4\ 9$	$10'\ 4'\ 13'\ 2$	$14'\ 9'\ 2'\ 15'$
	$1'\ 6'\ 13'\ 15'$	$1\ 3'\ 10'\ 14'$	$3\ 6\ 4'\ 9'$	$10\ 4\ 13\ 2'$	$14\ 9\ 2\ 15$
IX	$1\ 5\ 11\ 4'$	$1'\ 2\ 8\ 7'$	$2'\ 5'\ 3\ 12'$	$8'\ 3'\ 11'\ 15'$	$7\ 12\ 15\ 4$
	$1'\ 5'\ 11'\ 4'$	$1\ 2'\ 8'\ 7$	$2\ 5\ 3'\ 12$	$8\ 3\ 11\ 15$	$7'\ 12'\ 15'\ 4'$
X	$1\ 5\ 12\ 3'$	$1'\ 2\ 9\ 6'$	$2'\ 5'\ 4\ 11'$	$9'\ 4'\ 12'\ 15'$	$6\ 11\ 15\ 3$
	$1'\ 5'\ 12'\ 3$	$1\ 2'\ 9'\ 6$	$2\ 5\ 4'\ 11'$	$9\ 4\ 12\ 15$	$6'\ 11'\ 15'\ 3'$
XI	$1\ 6\ 11\ 15$	$1'\ 3\ 8\ 14$	$3'\ 6'\ 2\ 9$	$8'\ 2'\ 11'\ 4$	$14'\ 9'\ 4'\ 15'$
	$1'\ 6'\ 11'\ 15'$	$1\ 3'\ 8'\ 14'$	$3\ 6\ 2'\ 9'$	$8\ 2\ 11\ 4'$	$14\ 9\ 4\ 15$
XII	$1\ 7\ 12\ 15$	$1'\ 4\ 9\ 14$	$4'\ 7'\ 2\ 8$	$9'\ 2'\ 12'\ 3$	$14'\ 8'\ 3'\ 15'$
	$1'\ 7'\ 12'\ 15'$	$1\ 4'\ 9'\ 14'$	$4\ 7\ 2'\ 8'$	$9\ 2\ 12\ 3'$	$14\ 8\ 3\ 15$

Regulators and characteristics corresponding to the Ist *type of parallelohedra.*

I	$0\ 1\ 5\ 11\ 15$	$-1\ 0\ 0\ 0$	λ_1	$1\ -1\ 0\ 0$	$0\ 1\ -1\ 0$	$0\ 0\ 1\ -1$	$0\ 0\ 0\ 1$
		$2\ 1\ 1\ 1$		$(2)\ \mu_1$	$(6)\ \mu_4$	$(12)\ \mu_6$	$(4')\ \lambda_4$
II	$0\ 1\ 5\ 3'\ 10'$	$-1\ 0\ 1\ 0$	μ_2	$1\ -1\ 0\ 0$	$0\ 1\ 0\ 0$	$0\ 0\ -1\ 1$	$0\ 0\ 0\ 1$
		$1\ 0\ -1\ 0$		$(2)\ \mu_1$	$(14')\ \lambda_2$	$(4')\ \mu_6$	$(12)\ \lambda_4$
III	$0\ 1\ 6\ 11\ 4'$	$-1\ 0\ 0\ 1$	μ_3	$1\ 0\ -1\ 0$	$0\ -1\ 1\ 0$	$0\ 1\ 0\ 0$	$0\ 0\ 0\ -1$
		$1\ 0\ 0\ -1$		$(3)\ \mu_2$	$(5)\ \mu_4$	$(9')\ \lambda_2$	$(15)\ \lambda_4$
IV	$0\ 1\ 7\ 13\ 15$	$-1\ 0\ 0\ 0$	λ_1	$1\ 0\ 0\ -1$	$0\ 0\ -1\ 1$	$0\ -1\ 1\ 0$	$0\ 1\ 0\ 0$
		$2\ 1\ 1\ 1$		$(4)\ \mu_3$	$(6)\ \mu_6$	$(12)\ \mu_4$	$(2')\ \lambda_2$
V	$0\ 1\ 5\ 12\ 15$	$-1\ 0\ 0\ 0$	λ_1	$1\ -1\ 0\ 0$	$0\ 1\ 0\ -1$	$0\ 0\ -1\ 1$	$0\ 0\ 1\ 0$
		$2\ 1\ 1\ 1$		$(2)\ \mu_1$	$(7)\ \mu_5$	$(11)\ \mu_6$	$(3')\ \lambda_3$
VI	$0\ 1\ 5\ 4'\ 10'$	$-1\ 0\ 0\ 1$	μ_3	$1\ -1\ 0\ 0$	$0\ 1\ 0\ 0$	$0\ 0\ 1\ -1$	$0\ 0\ -1\ 0$
		$1\ 0\ 0\ -1$		$(2)\ \mu_1$	$(14')\ \lambda_2$	$(3')\ \mu_6$	$(11)\ \lambda_3$
VII	$0\ 1\ 7\ 12\ 3'$	$-1\ 0\ 1\ 0$	μ_2	$1\ 0\ 0\ -1$	$0\ -1\ 0\ 1$	$0\ 1\ 0\ 0$	$0\ 0\ -1\ 0$
		$1\ 0\ -1\ 0$		$(4)\ \mu_3$	$(5)\ \mu_5$	$(8')\ \lambda_2$	$(15)\ \lambda_3$
VIII	$0\ 1\ 6\ 13\ 15$	$-1\ 0\ 0\ 0$	λ_1	$1\ 0\ -1\ 0$	$0\ 0\ 1\ -1$	$0\ -1\ 0\ 1$	$0\ 1\ 0\ 0$
		$2\ 1\ 1\ 1$		$(3)\ \mu_2$	$(7)\ \mu_6$	$(11)\ \mu_5$	$(2')\ \lambda_2$
IX	$0\ 1\ 5\ 11\ 4'$	$-1\ 0\ 0\ 1$	μ_3	$1\ -1\ 0\ 0$	$0\ 1\ -1\ 0$	$0\ 0\ 1\ 0$	$0\ 0\ 0\ -1$
		$1\ 0\ 0\ -1$		$(2)\ \mu_1$	$(6)\ \mu_4$	$(10')\ \lambda_3$	$(15)\ \lambda_4$
X	$0\ 1\ 5\ 12\ 3'$	$-1\ 0\ 1\ 0$	μ_2	$1\ -1\ 0\ 0$	$0\ 1\ 0\ -1$	$0\ 0\ 0\ 1$	$0\ 0\ -1\ 0$
		$1\ 0\ -1\ 0$		$(2)\ \mu_1$	$(7)\ \mu_5$	$(10')\ \lambda_4$	$(15)\ \lambda_3$
XI	$0\ 1\ 6\ 11\ 15$	$-1\ 0\ 0\ 0$	λ_1	$1\ 0\ -1\ 0$	$0\ -1\ 1\ 0$	$0\ 1\ 0\ -1$	$0\ 0\ 0\ 1$
		$2\ 1\ 1\ 1$		$(3)\ \mu_2$	$(5)\ \mu_4$	$(13)\ \mu_5$	$(4')\ \lambda_4$
XII	$0\ 1\ 7\ 12\ 15$	$-1\ 0\ 0\ 0$	λ_1	$1\ 0\ 0\ -1$	$0\ -1\ 0\ 1$	$0\ 1\ -1\ 0$	$0\ 0\ 1\ 0$
		$2\ 1\ 1\ 1$		$(4)\ \mu_3$	$(5)\ \mu_5$	$(13)\ \mu_4$	$(3')\ \lambda_3$

In this table, the first line of each group contains the characteristics of faces in three dimensions corresponding to the simplexes $I, II, \ldots, XII$.

The second line contains the vertices of simplexes which are contiguous to the simplexes $I, II, \ldots, XII$ by the faces, the characteristic of which are indicated above in the first line, and the regulators are indicated near by in the second line.

Let us examine the parallelohedra in four dimensions which belong to the second type of primitive parallelohedra defined by the domain Δ' of quaternary quadratic forms. The domain Δ' is contiguous to the principal domain Δ by the face in a dimensions defined by the equation

$$\mu_1 = 0.$$

The independent regulator μ_1 corresponds to the faces of simplexes

$$I,\ II,\ V,\ VI,\ IX,\ ,X.$$

All these simplexes have to be reconstructed with the help of the algorithm explained in Number 91.

One will determine the numbers $\vartheta_0, \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$ after the conditions

$$(2) = \vartheta_1(1) + \vartheta_2(5) + \vartheta_3(11) + \vartheta_4(15) \text{ and } \vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_4 = 1;$$

it becomes

$$\vartheta_0 = 1,\ \vartheta_1 = -1,\ \vartheta_2 = 1,\ \vartheta_3 = 0,\ \vartheta_4 = 0.$$

It follows that one will replace the three pairs of simplexes

$$\begin{gathered}(0,1,5,11,15) \text{ and } (0,2,5,11,15),\\ (0,1,5,12,15) \text{ and } (0,2,5,12,15),\\ (0,1,5,11,4') \text{ and } (0,2,5,11,4')\end{gathered}$$

by the simplexes

$$\begin{gathered}(0,1,5,11,15) \text{ and } (0,1,2,11,15),\\ (2,1,5,12,15) \text{ and } (0,1,2,12,15),\\ (2,1,5,11,4') \text{ and } (0,1,2,11,4').\end{gathered} \tag{3}$$

By designating the system $(1,-1,0,0)$ by the symbol (5) and the system $(-1,1,0,0)$ by the symbol $(5')$, one will designate

$$\begin{gathered}\text{I} - (0,5,1,6,13),\quad \text{II} - (0,1,2,11,15),\quad \text{V} - (0,5,1,7,13),\\ \text{VI} - (0,1,2,12,15),\quad \text{IX} - (0,5,1,6,9'),\quad \text{X} - (0,1,2,11,4').\end{gathered}$$

These simplexes are congruent to the new simplexes (3).

The primitive parallelohedra of the II^{nd} type possess 120 vertices which are brought together in the following table:

Vertices of the primitive parallelohedron of II^{nd} *type*

Vertices of primitive parallelohedron of I^{st} *type*

I	$1\ 5\ 6\ 13$	$1'\ 2'\ 3\ 10$	$2\ 5'\ 8\ 14$	$3'\ 8'\ 6'\ 4$	$10'\ 14'\ 4'\ 13'$
	$1'\ 5'\ 6'\ 13'$	$1\ 2\ 3'\ 10'$	$2'\ 5\ 8'\ 14'$	$3\ 8\ 6\ 4'$	$10\ 14\ 4\ 13$
II	$1\ 2\ 11\ 15$	$1'\ 5'\ 8\ 14$	$5\ 2'\ 6\ 13$	$8'\ 6'\ 11'\ 4$	$14'\ 13'\ 4'\ 15'$
	$1'\ 2'\ 11'\ 15'$	$1\ 5\ 8'\ 14'$	$5'\ 2\ 6'\ 13'$	$8\ 6\ 11\ 4'$	$14\ 13\ 4\ 15$
III	$1\ 6\ 11\ 4'$	$1'\ 3\ 8\ 7'$	$3'\ 6'\ 2\ 13'$	$8'\ 2'\ 11'\ 15'$	$7\ 13\ 15\ 4$
	$1'\ 6'\ 11'\ 4$	$1\ 3'\ 8'\ 7$	$3\ 6\ 2'\ 13$	$8\ 2\ 11\ 15$	$7'\ 13'\ 15'\ 4'$
IV	$1\ 7\ 13\ 15$	$1'\ 4\ 10\ 14$	$4'\ 7'\ 3\ 8$	$10'\ 3'\ 13'\ 2$	$14'\ 8'\ 2'\ 15'$
	$1'\ 7'\ 13'\ 15'$	$1\ 4'\ 10'\ 14'$	$4\ 7\ 3'\ 8'$	$10\ 3\ 13\ 2'$	$14\ 8\ 2\ 15$
V	$1\ 5\ 7\ 13$	$1'\ 2'\ 4\ 10$	$2\ 5'\ 9\ 14$	$4'\ 9'\ 7'\ 3$	$10'\ 14'\ 3'\ 13'$
	$1'\ 5'\ 7'\ 13'$	$1\ 2\ 4'\ 10'$	$2'\ 5\ 9'\ 14'$	$4\ 9\ 7\ 3'$	$10\ 14\ 3\ 13$
VI	$1\ 2\ 12\ 15$	$1'\ 5'\ 9\ 14$	$5\ 2'\ 7\ 13$	$9'\ 7'\ 12'\ 3$	$14'\ 13'\ 3'\ 15'$
	$1'\ 2'\ 12'\ 15'$	$1\ 5\ 9'\ 14'$	$5'\ 2\ 7'\ 13'$	$9\ 7\ 12\ 3'$	$14\ 13\ 3\ 15$
VII	$1\ 7\ 12\ 3'$	$1'\ 4\ 9\ 6'$	$4'\ 7'\ 2\ 13'$	$9'\ 2'\ 12'\ 15'$	$6\ 13\ 15\ 3$
	$1'\ 7'\ 12'\ 3$	$1\ 4'\ 9'\ 6$	$4\ 7\ 2'\ 13$	$9\ 2\ 12\ 15$	$6'\ 13'\ 15'\ 3'$
VIII	$1\ 6\ 13\ 15$	$1'\ 3\ 10\ 14$	$3'\ 6'\ 4\ 9$	$10'\ 4'\ 13'\ 2$	$14'\ 9'\ 2'\ 15'$
	$1'\ 6'\ 13'\ 15'$	$1\ 3'\ 10'\ 14'$	$3\ 6\ 4'\ 9'$	$10\ 4\ 13\ 2'$	$14\ 9\ 2\ 15$
IX	$1\ 5\ 6\ 9'$	$1'\ 2'\ 3\ 12'$	$2\ 5'\ 8\ 7'$	$3'\ 8'\ 6'\ 15'$	$12\ 7\ 15\ 9$
	$1'\ 5'\ 6'\ 9$	$1\ 2\ 3'\ 12$	$2'\ 5\ 8'\ 7$	$3\ 8\ 6\ 15$	$12'\ 7'\ 15'\ 9'$
X	$1\ 2\ 11\ 4'$	$1'\ 5'\ 8\ 7'$	$5\ 2'\ 6\ 9'$	$8'\ 6'\ 11'\ 15'$	$7\ 9\ 15\ 4$
	$1'\ 2'\ 11'\ 4$	$1\ 5\ 8'\ 7$	$5'\ 2\ 6'\ 9$	$8\ 6\ 11\ 15$	$7'\ 9'\ 15'\ 4'$
XI	$1\ 6\ 11\ 15$	$1'\ 3\ 8\ 14$	$3'\ 6'\ 2\ 9$	$8'\ 2'\ 11'\ 4$	$14'\ 9'\ 4'\ 15'$
	$1'\ 6'\ 11'\ 15'$	$1\ 3'\ 8'\ 14'$	$3\ 6\ 2'\ 9'$	$8\ 2\ 11\ 4'$	$14\ 9\ 4\ 15$
XII	$1\ 7\ 12\ 15$	$1'\ 4\ 9\ 14$	$4'\ 7'\ 2\ 8$	$9'\ 2'\ 12'\ 3$	$14'\ 8'\ 3'\ 15'$
	$1'\ 7'\ 12'\ 15'$	$1\ 4'\ 9'\ 14'$	$4\ 7\ 2'\ 8'$	$9\ 2\ 12\ 3'$	$14\ 8\ 3\ 15$

Regulators and characteristic corresponding to the II^{nd}-type of parallelohedra

I	0 1 5 6 13	−1 0 0 0 2 0 1 1	λ_1	1 1 −1 0 (2′) μ_1	0 −1 0 0 (15) λ_2	0 0 1 −1 (7) μ_6	0 0 0 1 (9′) $\lambda_4+\mu_1$
II	0 1 2 11 15	−1 −1 1 0 1 1 0 0	μ_1	1 0 −1 0 (8) μ_2	0 1 −1 0 (6) μ_4	0 0 1 −1 (12) $\mu_6+\mu_1$	0 0 0 1 (4′) λ_4
III	0 1 6 11 4′	−1 0 0 1 1 0 0 −1	$\mu_3+\mu_1$	1 0 −1 0 (8) μ_2	0 −1 1 0 (2) μ_4	0 1 0 0 (9′) $\lambda_2+\mu_1$	0 0 0 −1 (15) λ_4
IV	0 1 7 13 15	−1 0 0 0 2 0 1 1	λ_1	1 0 0 −1 (4) $\mu_3+\mu_1$	0 0 −1 1 (6) μ_6	0 −1 1 0 (12) $\mu_4+\mu_1$	0 1 0 0 (5) λ_2
V	0 1 5 7 13	−1 0 0 0 2 0 1 1	λ_1	1 1 0 −1 (2′) μ_1	0 −1 0 0 (15) λ_2	0 0 −1 1 (6) μ_6	0 0 1 0 (8′) $\lambda_3+\mu_1$
VI	0 1 2 12 15	−1 −1 0 1 1 1 0 0	μ_1	1 0 0 −1 (9) μ_3	0 1 0 −1 (7) μ_5	0 0 −1 1 (11) $\mu_6+\mu_1$	0 0 1 0 (3′) λ_3
VII	0 1 7 12 3′	−1 0 1 0 1 0 −1 0	$\mu_2+\mu_1$	1 0 0 −1 (9) μ_3	0 −1 0 1 (2) μ_5	0 1 0 0 (8′) $\lambda_2+\mu_1$	0 0 −1 0 (15) λ_3
VIII	0 1 6 13 15	−1 0 0 0 2 0 1 1	λ_1	1 0 −1 0 (3) $\mu_2+\mu_1$	0 0 1 −1 (7) μ_6	0 −1 0 1 (11) $\mu_5+\mu_1$	0 1 0 0 (5) λ_2
IX	0 1 5 6 9′	−1 0 0 1 1 −1 0 −1	μ_3	1 1 −1 −1 (2′) μ_1	0 −1 0 1 (4′) μ_5	0 0 1 0 (14′) λ_3	0 0 0 −1 (13) $\lambda_4+\mu_1$
X	0 1 2 11 4′	−1 −1 1 1 1 1 0 0	μ_1	1 0 −1 0 (8) μ_2	0 1 −1 0 (6) μ_4	0 0 1 0 (10′) $\lambda_3+\mu_1$	0 0 0 −1 (15) λ_4
XI	0 1 6 11 15	−1 0 0 0 2 1 1 1	$\lambda_1+\mu_1$	1 0 −1 0 (8) μ_2	0 −1 1 0 (2) μ_4	0 1 0 −1 (13) $\mu_5+\mu_1$	0 0 0 1 (4′) λ_4
XII	0 1 7 12 15	−1 0 0 0 2 1 1 1	$\lambda_1+\mu_1$	1 0 0 −1 (9) μ_3	0 −1 0 1 (2) μ_5	0 1 −1 0 (13) $\mu_4+\mu_1$	0 0 1 0 (3′) λ_3

The domain Δ' of quaternary quadratic forms which define the second type of primitive parallelohedra in four dimensions is determined by 10 independent inequalities

$$\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0,$$
$$\mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0, \mu_4 \geq 0, \mu_5 \geq 0, \mu_6 \geq 0.$$

Any quaternary quadratic form belonging to the domain Δ' can be written

$$f(x_1,x_2,x_3,x_4) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \mu_1\omega + \mu_2(x_1-x_3)^2 +$$
$$\mu_3(x_1-x_4)^2 + \mu_4(x_2-x_3)^2 + \mu_5(x_2-x_4)^2 + \mu_6(x_3-x_4)^2$$

where

$$\omega = 2x_1^2 + 2x_2^2 + 2x_3^2 + 2x_4^2 + 2x_1x_2 - 2x_1x_3 - 2x_1x_4 - 2x_2x_3 - 2x_2x_4. \tag{4}$$

The parallelohedra belonging to the II^{nd} type are determined by 30 inequalities of form (2) which are symbolically presented in the following table:

II^{nd}*-type of parallelohedra*

N	l_1	l_2	l_3	l_4	$f(l_1,l_2,l_3,l_4)$	$-l_1$	$-l_2$	$-l_3$	$-l_4$	N
1	1	0	0	0	$\lambda_1 + 2\mu_1 + \mu_2 + \mu_3$	−1	0	0	0	1′
2	0	1	0	0	$\lambda_2 + 2\mu 1_{+}\mu_4 + \mu_5$	0	−1	0	0	2′
3	0	0	1	0	$\lambda_3 + 2\mu_1 + \mu_2 + \mu_4 + \mu_6$	0	0	−1	0	3′
4	0	0	0	1	$\lambda_4 + 2\mu_1 + \mu_3 + \mu_5 + \mu_6$	0	0	0	−1	4′
5	1	−1	0	0	$\lambda_1 + \lambda_2 + 2\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5$	−1	1	0	0	5′
6	1	0	1	0	$\lambda_1 + \lambda_3 + 2\mu_1 + \mu_3 + \mu_4 + \mu_6$	−1	0	−1	0	6′
7	1	0	0	1	$\lambda_1 + \lambda_4 + 2\mu_1 + \mu_2 + \mu_5 + \mu_6$	−1	0	0	−1	7′
8	0	1	1	0	$\lambda_2 + \lambda_3 + 2\mu_1 + \mu_2 + \mu_5 + \mu_6$	0	−1	−1	0	8′
9	0	1	0	1	$\lambda_2 + \lambda_4 + 2\mu_1 + \mu_3 + \mu_4 + \mu_6$	0	−1	0	−1	9′
10	0	0	1	1	$\lambda_3 + \lambda_4 + 4\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5$	0	0	−1	−1	10′
11	1	1	1	0	$\lambda_1 + \lambda_2 + \lambda_3 + 4\mu_1 + \mu_3 + \mu_5 + \mu_6$	−1	−1	−1	0	11′
12	1	1	0	1	$\lambda_1 + \lambda_2 + \lambda_4 + 4\mu_1 + \mu_2 + \mu_4 + \mu_6$	−1	−1	0	−1	12′
13	1	0	1	1	$\lambda_1 + \lambda_3 + \lambda_4 + 2\mu_1 + \mu_4 + \mu_5$	−1	0	−1	−1	13′
14	0	1	1	1	$\lambda_2 + \lambda_3 + \lambda_4 + 2\mu_1 + \mu_2 + \mu_3$	0	−1	−1	−1	14′
15	1	1	1	1	$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2\mu_1$	−1	−1	−1	−1	15′

116 Let us find the domain Δ'' of quaternary quadratic forms which is contiguous to the domain Δ' by the face in 9 dimensions defined by the equation

$$\mu_6 = 0.$$

The independent regulator μ_6 corresponds to the faces of simplexes

I, IV, V, VIII.

All these simplexes have to be reconstructed after the algorithm explained in Number 91. One will determine, to this effect, the number

$$\vartheta_0, \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4$$

after the condition

$$\vartheta_1(1) + \vartheta_2(5) + \vartheta_3(6) + \vartheta_4(13) \quad \text{and} \quad \vartheta_0 + \vartheta_1 + \vartheta_2 + \vartheta_3 + \vartheta_4; \tag{5}$$

it becomes

$$\vartheta_0 = 0, \quad \vartheta_1 = 1, \quad \vartheta_2 = 0, \quad \vartheta_3 = -1, \quad \vartheta_4 = 1.$$

One concludes that the two pairs of simplexes

$$(0,1,5,6,13) \text{ and } (0,1,5,7,13) \text{ and}$$
$$(0,1,7,13,15) \text{ and } (0,1,6,13,15)$$

have to be replaced by the new simplexes

$$\text{I} - (0,7,5,6,13) \quad \text{and} \quad \text{V} - (0,1,5,6,7)$$
$$\text{IV} - (0,6,7,13,15) \quad \text{and} \quad \text{VIII} - (0,1,7,6,15).$$

By designating the system $(0,0,1,-1)$ by the symbol (10) and the system $(0,0,-1,1)$ by the symbol $(10')$ one will determine all the vertices of primitive parallelohedra belonging to the new type as follows.

Vertices of primitive parallelohedron of III^{rd}-type.

I	7 5 6 13 $7'\ 5'\ 6'\ 13'$	$7'\ 9'\ 10\ 3$ $7\ 9\ 10'\ 3'$	$9\ 5'\ 8\ 14$ $9'\ 5\ 8'\ 14'$	$10'\ 8'\ 6'\ 4$ $10\ 8\ 6\ 4'$	$3'\ 14'\ 4'\ 13'$ 3 14 4 13
II	1 2 11 15 $1'\ 2'\ 11'\ 15'$	$1'\ 5'\ 8\ 14$ $1\ 5\ 8'\ 14'$	$5\ 2'\ 6\ 13$ $5'\ 2\ 6'\ 13'$	$8'\ 6'\ 11'\ 4$ $8\ 6\ 11\ 4'$	$14'\ 13'\ 4'\ 15'$ 14 13 4 15
III	$1\ 6\ 11\ 4'$ $1'\ 6'\ 11'\ 4$	$1'\ 3\ 8\ 7'$ $1\ 3'\ 8'\ 7$	$3'\ 6'\ 2\ 13'$ $3\ 6\ 2'\ 13$	$8'\ 2'\ 11'\ 15'$ 8 2 11 15	7 13 15 4 $7'\ 13'\ 15'\ 4'$
IV	6 7 13 15 $6'\ 7'\ 13'\ 15'$	$6'\ 10'\ 4\ 9$ $6\ 10\ 4'\ 9'$	$10\ 7'\ 3\ 8$ $10'\ 7\ 3'\ 8'$	$4'\ 3'\ 13'\ 2$ $4\ 3\ 13\ 2'$	$9'\ 8'\ 2'\ 15'$ 9 8 2 15
V	1 5 6 7 $1'\ 5'\ 6'\ 7'$	$1'\ 2'\ 3\ 4$ $1\ 2\ 3'\ 4'$	$2\ 5'\ 8\ 9$ $2'\ 5\ 8'\ 9'$	$3'\ 8'\ 6'\ 10'$ 3 8 6 10	$4'\ 9'\ 10\ 7'$ $4\ 9\ 10'\ 7$
VI	1 2 12 15 $1'\ 2'\ 12'\ 15'$	$1'\ 5'\ 9\ 14$ $1\ 5\ 9'\ 14'$	$5\ 2'\ 7\ 13$ $5'\ 2\ 7'\ 13'$	$9'\ 7'\ 12'\ 3$ $9\ 7\ 12\ 3'$	$14'\ 13'\ 3'\ 15'$ 14 13 3 15
VII	$1\ 7\ 12\ 3'$ 6 13 15 3	$1'\ 4\ 9\ 6'$ $1'\ 7'\ 12'\ 3$	$1\ 4'\ 9'\ 6$ 4 7 2 13	$4'\ 7'\ 2\ 13'$ 9 2 12 15	$9'\ 2'\ 12'\ 15'$ $6'\ 13'\ 15'\ 3'$
VIII	1 7 6 15 $1'\ 7'\ 6'\ 15'$	$1'\ 4\ 3\ 14$ $1\ 4'\ 3'\ 14'$	$4'\ 7'\ 10\ 8$ $4\ 7\ 10'\ 8'$	$3'\ 10'\ 6'\ 9$ $3\ 10\ 6\ 9'$	$14'\ 8'\ 9'\ 15'$ 14 8 9 15
IX	$1\ 5\ 6\ 9'$ $1'\ 5'\ 6'\ 9$	$1'\ 2'\ 3\ 12'$ $1\ 2\ 3'\ 12$	$2\ 5'\ 8\ 7'$ $2'\ 5\ 8'\ 7$	$3'\ 8'\ 6'\ 15'$ 3 8 6 15	12 7 15 9 $12'\ 7'\ 15'\ 9'$
X	$1\ 2\ 11\ 4'$ $1'\ 2'\ 11'\ 4$	$1'\ 5'\ 8\ 7'$ $1\ 5\ 8'\ 7$	$5\ 2'\ 6\ 9'$ $5'\ 2\ 6'\ 9$	$8'\ 6'\ 11'\ 15'$ 8 6 11 15	7 9 15 4 $7'\ 9'\ 15'\ 4'$
XI	1 6 11 15 $1'\ 6'\ 11'\ 15'$	$1'\ 3\ 8\ 14$ $1\ 3'\ 8'\ 14'$	$3'\ 6'\ 2\ 9$ $3\ 6\ 2'\ 9'$	$8'\ 2'\ 11'\ 4$ $8\ 2\ 11\ 4'$	$14'\ 9'\ 4'\ 15'$ 14 9 4 14
XII	1 7 12 15 $1'\ 7'\ 12'\ 15'$	$1'\ 4\ 9\ 14$ $1\ 4'\ 9'\ 14'$	$4'\ 7'\ 2\ 8$ $4\ 7\ 2'\ 8'$	$9'\ 2'\ 12'\ 3$ $9\ 2\ 12\ 3'$	$14'\ 8'\ 3'\ 15'$ 14 8 3 15

Regulators and characteristics corresponding to the III^{rd}-type parallelohedra

I	0 7 5 6 13	$-1\ 0\ 0\ 0$ 2 0 1 1	λ_1	$1\ 1\ -1\ 0$ $(2')\ \mu_1$	$0\ -1\ 0\ 0$ $(15)\ \lambda_2$	$1\ 1\ 0\ -1$ $(2')\ \mu_1$	$-1\ -1\ 1\ 1$ $(1)\ \mu_6$
II	0 1 2 11 15	$-1\ -1\ 1\ 0$ 1 1 0 1	μ_1	$1\ 0\ -1\ 0$ $(8)\ \mu_2$	$0\ 1\ -1\ 0$ $(6)\ \mu_4$	$0\ 0\ 1\ -1$ $(12)\ \mu_1$	0 0 0 1 $(4')\ \lambda_4$
III	$0\ 1\ 6\ 11\ 4'$	$-1\ 0\ 0\ 1$ $1\ 0\ 1\ -1$	$\mu_3+\mu_1$	$1\ 0\ -1\ 0$ $(8)\ \mu_2$	$0\ -1\ 1\ 0$ $(2)\ \mu_4$	0 1 0 0 $(9')\ \lambda_2+\mu_1$	$0\ 0\ 0\ -1$ $(15)\ \lambda_4$
IV	0 6 7 13 15	$-1\ 0\ 0\ 0$ 2 0 1 1	λ_1	$1\ 0\ 0\ -1$ $(4)\ \mu_3+\mu_1$	$1\ 0\ -1\ 0$ $(3)\ \mu_2+\mu_1$	$-1\ -1\ 1\ 1$ $(1)\ \mu_6$	0 1 0 0 $(5)\ \lambda_2$
V	0 1 5 6 7	$-1\ 0\ 0\ 0$ 2 0 1 1	λ_1	$1\ 1\ -1\ -1$ $(13)\ \mu_6$	$0\ -1\ 0\ 0$ $(15)\ \lambda_2$	0 0 1 0 $(8')\ \lambda_3+\mu_1$	0 0 0 1 $(9')\ \lambda_4+\mu_1$
VI	0 1 2 12 15	$-1\ -1\ 0\ 1$ 1 1 1 0	μ_1	$1\ 0\ 0\ -1$ $(9)\ \mu_3$	$1\ 1\ 0\ -1$ $(7)\ \mu_5$	$0\ 0\ -1\ 1$ $(11)\ \mu_1$	0 0 1 0 $(3')\ \lambda_3$
VII	$0\ 1\ 7\ 12\ 3'$	$-1\ 0\ 0\ 0$ $1\ 0\ -1\ 1$	$\mu_2+\mu_1$	$1\ 0\ 0\ -1$ $(9)\ \mu_3$	$0\ -1\ 0\ 1$ $(2)\ \mu_5$	0 1 0 0 $(8')\ \lambda_2+\mu_1$	$0\ 0\ -1\ 0$ $(15)\ \lambda_3$
VIII	0 1 7 6 15	$-1\ 0\ 0\ 0$ 2 0 1 1	λ_1	$1\ 1\ -1\ -1$ $(13)\ \mu_6$	$0\ -1\ 0\ 1$ $(11)\ \mu_5+\mu_1$	$0\ -1\ 1\ 0$ $(12)\ \mu_4+\mu_1$	0 1 0 0 $(5)\ \lambda_2$
IX	$0\ 1\ 5\ 6\ 9'$	$-1\ 0\ 0\ 1$ $1\ -1\ 0\ -1$	μ_3	$1\ 1\ -1\ -1$ $(2')\ \mu_1+\mu_6$	$0\ -1\ 0\ 1$ $(4')\ \mu_5$	0 0 1 0 $(14')\ \lambda_3$	$0\ 0\ 0\ -1$ $(7)\ \lambda_4+\mu_1$
X	$0\ 1\ 2\ 11\ 4'$	$-1\ -1\ 1\ 1$ 1 1 0 0	$\mu_1+\mu_6$	$1\ 0\ -1\ 0$ $(8)\ \mu_2$	$0\ 1\ -1\ 0$ $(6)\ \mu_4$	0 0 1 0 $(3')\ \lambda_3+\mu_1$	$0\ 0\ 0\ -1$ $(15)\ \lambda_4$
XI	0 1 6 11 15	$-1\ 0\ 0\ 0$ 2 1 1 1	$\lambda_1+\mu_1$	$1\ 0\ -1\ 0$ $(8)\ \mu_2$	$0\ -1\ 1\ 0$ $(2)\ \mu_4$	$0\ 1\ 0\ -1$ $(7)\ \mu_5+\mu_1$	0 0 0 1 $(4')\ \lambda_4$
XII	0 1 7 12 15	$-1\ 0\ 0\ 0$ 2 1 1 1	$\lambda_1+\mu_1$	$1\ 0\ 0\ -1$ $(9)\ \mu_3$	$0\ -1\ 0\ 1$ $(2)\ \mu_5$	$0\ 1\ -1\ 0$ $(6)\ \mu_4+\mu_1$	0 0 1 0 $(3')\ \lambda_3$

The independent regulators are expressed by the formulae

$$\lambda_1 = a_{11} + a_{13} + a_{14} + a_{34}, \lambda_2 = a_{22} + a_{23} + a_{24} + a_{34}, \lambda_3 = a_{31} + a_{32} + a_{33} + a_{34},$$

$$\lambda_4 = a_{41} + a_{42} + a_{43} + a_{44}, \mu_1 = a_{12} - a_{34}, \mu_2 = -a_{13} - a_{12},$$

$$\mu_3 = -a_{14} - a_{12}, \mu_4 = -a_{23} - a_{12}, \mu_5 = -a_{24} - a_{12}, \mu_6 = a_{34}.$$

The domain Δ'' of quaternary quadratic forms which defines the third type of primitive parallelohedra in four dimensions is determined by 10 independent inequalities

$$\lambda_1 \geq 0, \lambda_2 \geq 0, \lambda_3 \geq 0, \lambda_4 \geq 0,$$
$$\mu_1 \geq 0, \mu_2 \geq 0, \mu_3 \geq 0, \mu_4 \geq 0, \mu_5 \geq 0, \mu_6 \geq 0.$$

Each quaternary quadratic form belonging to the domain Δ'' can be written

$$f(x_1, x_2, x_3, x_4) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \lambda_4 x_4^2 + \mu_1 \omega + \mu_2 (x_1 - x_3)^2$$
$$+ \mu_3 (x_1 - x_4)^2 + \mu_4 (x_2 - x_3)^2 + \mu_5 (x_2 - x_4)^2 + \mu_6 (x_1 + x_2 - x_3 - x_4)^2,$$

the form ω being defined by the equality (4). The parallelohedra belonging to the IIIrd type are determined by 30 inequalities of the form (2) which are symbolically presented in the following table:

IIIrd type of parallelohedra

N	$l_1\ l_2 l_3\ l_4$	$f(l_1, l_2, l_3, l_4)$	$-l_1 - l_2 - l_3 - l_4$	N
1	1 0 0 0	$\lambda_1 + 2\mu_1 + \mu_2 + \mu_3 + \mu_6$	−1 0 0 0	$1'$
2	0 1 0 0	$\lambda_2 + 2\mu_1 + \mu_4 + \mu_5 + \mu_6$	0 −1 0 0	$2'$
3	0 0 1 0	$\lambda_3 + 2\mu_1 + \mu_2 + \mu_4 + \mu_6$	0 0 −1 0	$3'$
4	0 0 0 1	$\lambda_4 + 2\mu_1 + \mu_3 + \mu_5 + \mu_6$	0 0 0 −1	$4'$
5	1 −1 0 0	$\lambda_1 + \lambda_2 + 2\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5$	−1 1 0 0	$5'$
6	1 0 1 0	$\lambda_1 + \lambda_3 + 2\mu_1 + \mu_3 + \mu_4$	−1 0 −1 0	$6'$
7	1 0 0 1	$\lambda_1 + \lambda_4 + 2\mu_1 + \mu_2 + \mu_5$	−1 0 0 −1	$7'$
8	0 1 1 0	$\lambda_2 + \lambda_3 + 2\mu_1 + \mu_2 + \mu_5$	0 −1 −1 0	$8'$
9	0 1 0 1	$\lambda_2 + \lambda_4 + 2\mu_1 + \mu_3 + \mu_4$	0 −1 0 −1	$9'$
10	0 0 1 −1	$\lambda_3 + \lambda_4 + 4\mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5$	0 0 −1 1	$10'$
11	1 1 1 0	$\lambda_1 + \lambda_2 + \lambda_3 + 4\mu_1 + \mu_3 + \mu_5 + \mu_6$	−1 −1 −1 0	$11'$
12	1 1 0 1	$\lambda_1 + \lambda_2 + \lambda_4 + 4\mu_1 + \mu_2 + \mu_4 + \mu_6$	−1 −1 0 −1	$12'$
13	1 0 1 1	$\lambda_1 + \lambda_3 + \lambda_4 + 2\mu_1 + \mu_4 + \mu_5 + \mu_6$	−1 0 −1 −1	$13'$
14	0 1 1 1	$\lambda_2 + \lambda_3 + \lambda_4 + 2\mu_1 + \mu_2 + \mu_3 + \mu_6$	0 −1 −1 −1	$14'$
15	1 1 1 1	$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + 2\mu_1$	−1 −1 −1 −1	$15'$

117 We have determined three domains $\Delta, \Delta', \Delta''$ which characterise three types of primitive parallelohedra in four dimensions.

Theorem. The set (Δ) *of domains of quaternary quadratic forms is composed of three different classes which can be represented by the domain* $\Delta, \Delta', \Delta''$.

In my first mémoire cited, it has been demonstrated that the set (R) of domains of quaternary quadratic forms corresponding to the perfect quaternary quadratic forms is composed of two classes represented by the principal domain R and by a domain R, determined by the equalities

$$f(x_1, x_2, x_3, x_4) = \rho_1 x_1^2 + \rho_2 x_2^2 + \rho_3 x_3^2 + \rho_4 x_4^2 + \rho_5 (x_1 - x_3)^2 +$$
$$\rho_6 (x_1 - x_4)^2 + \rho_7 (x_1 - x_3)^2 + \rho_8 (x_2 - x_4)^2 + \rho_9 (x_3 - x_4)^2 +$$
$$\rho_{10} (x_1 + x_2 - x_3)^2 + \rho_{11} (x_1 + x_2 - x_4)^2 + \rho_{12} (x_1 + x_2 - x_3 - x_4)^2$$

where $\rho_1, \rho_2, \ldots, \rho_{12}$ are positive arbitrary parameters or zeros.

The domain R, corresponds to a perfect form

$$\varphi_1 = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1 x_3 + x_1 x_4 + x_2 x_3 + x_2 x_4 + x_3 x_4. \tag{5}$$

In the mémoire cited, it has been demonstrated that all the faces in 9 dimensions of domain R, are equivalent to two faces characterised: one by the quadratic form

$$x_1^2, x_2^2, x_3^2, x_4^2, (x_1 - x_3)^2, (x_1 - x_4)^2, (x_2 - x_3)^2, (x_2 - x_4)^2, (x_3 - x_4)^2$$

and the other by the quadratic form

$$x_1^2, x_2^2, x_3^2, x_4^2, (x_1 - x_3)^2, (x_1 - x_4)^2, (x_2 - x_3)^2, (x_2 - x_4)^2,$$
$$(x_1 + x_2 - x_3 - x_4)^2.$$

The first face verifies the equation

$$a_{12} = 0$$

and the second face verifies the equation

$$a_{12} - a_{34} = 0$$

The form ω determined by the formula (4) characterise the axis of the domain (R) which does not change when one transforms the domain R into itself.

One concludes that the domain R can be partitioned into groups all of which are equivalent to the two domains Δ' and Δ'' obtained. This results in that the principal domain Δ and the two domains Δ' and Δ'' can not be equivalent.

The theorem introduced is thus demonstrated.

By not considering as different the equivalent types of parallelohedra one can say that there are only three different types of primitive parallelohedra in the space in four dimensions.

By calling reduced the positive quadratic forms which belong to the domains Δ, Δ' and Δ'' one obtains a new method of reduction of quaternary positive quadratic forms which presents a modification of the method due to Mr. Charve. ‡

---

‡ See the introduction

In effect, following the method of Mr. Charve, one calls reduced the quaternary positive quadratic forms belonging to one of three simple domains R, R' and R''. The first two domain R and R' coincide with the domains Δ and Δ' and it is only the third domain R'' of Mr. Charve which differs from the domain Δ''. Any form belonging to the domain Δ'' is equivalent to a form belonging to the domain R'' and vice-versa.

118

By examining the two tables which contain the characteristics of faces of simplexes which define the 2^{nd} and the 3^{rd} type of primitive parallelohedra, one will observe that these characteristic coincide for the two types and are represented by the linear forms

$$\pm x_1, \pm x_2, \pm x_3, \pm x_4, \pm(x_1 - x_3), \pm(x_1 - x_4), \pm(x_2 - x_3), \pm(x_2 - x_4), \pm(x_3 - x_4),$$
$$\pm(x_1 + x_2 - x_3), \pm(x_1 + x_2 - x_4), \pm(x_1 + x_2 - x_3 - x_4).$$

It is remarkable that these linear forms define the set of representations of the minimum of the perfect form φ_1 determined by the equality (5).

By virtue of that which has previously been mentioned, one can affirm that the coincidence noticed appears as the characteristics of faces of all the primitive parallelohedra in $2, 3$ and 4 dimensions.

It would be interesting to find out whether this is only a coincidence or whether there really exists a relation between the two problems which seem to be different: between the problem of the uniform partition of the space with the help of congruent convex polyhedra and the study of perfect positive quadratic forms.

End of the second mémoire

[in German]

Immediately after the first sheet of this significant work was set, we received the grievous tidings that your author of the science has been taken away by Death. The editor had in the best power seen to it that this last work of he who so early departed for the other side was checked over with utmost care.

Marburg, 19th June 1909

K. Hensel

Table of materials

Introduction

First Part

Uniform partition of the analytical space in n dimensions with the help of translations of one convex polyhedron.

Section I
General properties of parallelohedra

Number

1–3 On the convex polyhedra in n dimensions
4 Set of domains in n dimensions corresponding to the various vertices of a convex polyhedron
5 Definition of the group of vectors
6–7 Translation of polyhedra
8 Definition of parallelohedra
9–11 Properties of the group of vectors of a parallelohedron
12–13 Properties of faces in $n-1$ dimensions of a parallelohedron
14–16 Faces congruent to different dimensions of a parallelohedron

Section II.
Fundamental properties of primitive parallelohedra

17 Definition of primitive parallelohedra
18 Edges of primitive parallelohedra of the set (R)
19 Canonical forms of equations which define a vertex of a primitive parallelohedron
20–21 Canonical form of inequalities which define a primitive parallelohedron
22–23 Set of simplexes corresponding to the various vertices of a primitive parallelohedron
24–28 On a function defined by the set of simplexes corresponding to the various vertices of a primitive parallelohedron
29–31 Canonical form of inequalities which define the set (R) of primitive parallelohedra
32–36 Generatrix function of the set (R) of primitive parallelohedra
37–38 Fundamental properties of the generatrix function $V(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n)$
39–40 Solution of the generatrix function $V(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n)$
41–42 Solution of the centre of primitive parallelohedra
43–44 Continuous group of linear transformations of primitive parallelohedra

Section III.
Solution of parallelohedra with the help of positive quadratic forms

45–46 Definition of the convex polyhedron corresponding to a positive quadratic form
47–48 Independent inequalities which define the convex polyhedron corresponding to a positive quadratic form
49 Set (R) of parallelohedra defined by a positive quadratic form
50 Algorithm for the study of the minimum of the form $\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i$ in the set E
51–54 Properties of systems of integers which characterise the faces in $n-1$ dimension of the parallelohedron corresponding to a positive quadratic form

Second Part

Domains of quadratic forms corresponding to the various types of primitive parallelohedra.

Section IV.

Various types of primitive parallelohedra

55 On the number of faces in $n-1$ dimensions of a primitive parallelohedron
56–59 Definition of the type of primitive parallelohedra
60–62 Definition of the set (L) of simplexes characterising a type of primitive parallelohedra
63–68 Properties of symbols S_ν and σ_ν $(\nu = 0, 1, 2, \ldots, n)$
69–72 Regulators and characteristics of edges of primitive parallelohedra
73–78 Fundamental transformation of the form $\sum\sum a_{ij}x_ix_j + 2\sum \alpha_i x_i - \sum\sum a_{ij}l_il_j - 2\sum \alpha_i l_i$.
79–84 Definition of quadratic forms with the help of corresponding regulators and characteristics

Section V.

Properties of the set (Δ) of domains of quadratic forms corresponding to the various types of primitive parallelohedra

85–86 Definition of the domain of quadratic forms corresponding to a type of primitive parallelohedra
87–90 Properties of independent regulators
91–95 Reconstruction of the set (L) of simplexes into another of set (L') simplexes
96 Algorithm for the study of domains of quadratic forms which are contiguous to a given domain by the faces in $\frac{n(n+1)}{2} - 1$ dimensions
97–99 Set (L) of domains of quadratic forms corresponding to the various types of primitive parallelohedra
100–101 On the primitive parallelohedra corresponding to the positive quadratic forms
102–104 Principal domain of the set (Δ)
111 Parallelohedra in two dimensions
112–113 Parallelohedra in three dimensions
114–118 Parallelohedra in four dimensions

End of the table of materials

§ E. Index

1-neighbour, 145.
2-homohedral tilings, 137.
2-neighbour, 145.
Abell Clusters, 19.
abundance of elements, 76.
adjointed substitution, 347.
AIESEC, 7.
Aitoff projection, 19.
alliteration, 330.
aluminium smeltering, 3.
America, 203.
Amsterdam, 204.
Andromeda galaxy, 151.
anti-particles, 193.
Archimedean polyhedra, 8.
assonance, 330.
assumption, 184.
Astronomy, 19.
asymmetric membrane, 18.
atomic cross-section, 19.
attrition mechanism, 211.
autostereogram, 43.
AVS, 319.
Ayudhya, 333.
bacteria, 114.
bamboo, 3.
Bangkok, 201.
Barbilonia, 68.
basis, 391.
basis of a group of vector, 390.
beta distribution, 25.
BibTeX, 40, 322.
big bang, 18.
big crunch, 18.
bimedians, 177.
binary positive quadratic form, 382.
binary principal perfect form, 367.
binary quadratic form, 68.
biology, 115.
blockage, 17.
bombarding, 199.
bread, 3.
Brussels, 204.
bubonic plague, 114.
Budapest, 7.
cake, 177.
caltrop, 31.
canonical equation, 405.
Canonical form, 395.
canonical inequalities, 396, 403.
cataclasite, 157.
centre of gravity, 183.
centre of mass, 183.
centroid, 177, 183.
characteristic area, 78.
characteristic length, 78.
characteristic of the edge, 430.
characteristic of the face, 430.
chicken pox, 114.
chiral, 159.
Christ, 333.
circular permutations of indices, 454.
cirrus, 200.
citation programme, 39, 322.
classes of equivalent domains, 358.
classes of equivalent forms, 343.
classes of perfect forms, 358.
Closed-Circuit Television, 151.
cluster size, 331.
coefficient of variation, 41.
coefficient of variation, 50.
Commonwealth, 5.
Community Action, 333.
compression, 99.
computer, 330.
conductivity of matter, 76.
congested, 201.
congruent, 424.
congruent faces of paparallelohedron, 393.
congruent polyhedra, 390.
congruent vertices, 464.
connective constant, 127.
contiguity condition, 167.
contiguous domain, 343.
contiguous parallelohedra, 407.
contiguous perfect forms, 343, 364.
continuum percolation, 3, 17, 128, 189, 211.
Control and Information Technology, 1.
convex polyhedron, 387.
cooperativity, 119.
coordination number, 65, 126, 150.
correlative, 424.
correlative domains, 352.
cosmic string, 151.
cosmology, 19.
Coulomb force, 162.
covering lattice, 45, 60, 67.
covering polygon, 185.
critical diameter, 171.
critical percolation probability, 3.
critical probability, 202.
critical temperature, 24.
crystal, 48.
crystallisation, 17.
crystalliser, 158.
cubic close-packed, 193.
cubic spline, 78.
Curie point, 120.
Daiy, 333.
dependent inequalities, 349.
determinant, 342.
determinant of a system, 392.
discoveries, 4.
distance between two points, 400.
domain, 348.
domain of quadratic forms, 441.
double equations, 370.
earthquake, 154.
Eastern Europe, 330.
edge of a domain, 349.
edge of polyhedron, 387.
Edges of primitive parallelohedra, 394, 429.
Edokkite, 333.
Edokko, 333.
Einstein, 19.
England, 332.
English, 330.
English Sweats, 114.
epidemic, 114.
equivalent form, 383.
European Union, 5.
even polygonal drawing, 121.
evolution, 114.
expression of a regulator, 455.
extreme, 342.
extreme positive quadratic form, 355.
face of a domain, 349.
face of polyhedron, 387.
face pole, 44.
faces of primitive parallelohedra, 428.
false friends, 330.
ferromagnet, 17, 25.
filtration, 20.
flexatube, 7.

flow, 47.
forest fire, 152.
forest fire station, 152.
forts of Kronstadt, 154.
fractal number distribution, 155.
free-flowing, 201.
Freiburg, 204.
frustration parameter, 25.
galaxies, 151.
gamma experiment, 24.
glue, 5.
God, 333.
great dodecadodecahedron, 12.
great dodecahedron, 12.
Great Kanto Earthquake, the, 154.
group of vectors, 389, 391.
hamiltonian, 25.
Hammer projection, 19.
Hammer-Aitoff, 19.
Hercules supercluster, 151.
Hermite method, 382.
hexagon, 8.
hexagonal close packing, 188.
hexagonal lattice packing, 188.
high speed impellers, 158.
homogeneous, 96.
honeycomb, 8.
honeycomb lattice, 60.
Hubble constant, 151.
Hubble time, 151.
hydrophobic and hydrophilic properties, 113.
icosidodecahedron, 11.
independent inequalities, 349.
independent vectors, 390.
infinite cluster, 16.
interference effect, 211.
interior points of a domain, 348.
interstitial velocity, 50.
invasion percolation, 113.
Ising model, , 17, 24, 120.
isotropic, 96.
Jesus, 333.
Journal of Statistical Physics, 201.
kagome lattice, 60.
Koch curve, 19.
Kozeny constant, 176.
Lanna, 333.
lattice, 60.
LaTEX, 39, 321, 322.
Lennard-Jones, 19.
Lennard-Jones function, 181.
Lennard-Jones potential, 182.
Leslie Lamport, 39, 321.
lightnings, 47.
Linear B, 2.
lobotomised, 39, 321.
Local Cluster, 19.
Local Supercluster, 19, 151.
long-range interaction, 182.
Lorentz transformation, 19.
M25, 203.
M60, 203.
M81, 151.
Macbeth, 330.
macros, 39, 321.
Manchester, 204.
manmac macro, 39, 321.
Matlab, 93, 319.
maximum maximorum, 342.
Me, 333.
membrane, 185.
mesh number, 50.
method of reduction, 367, 368.
method of reduction of positive quadratic forms, 386.
method of reduction of ternary positive quadratic forms, 386.
methods of reduction of quaternary positive quadratic forms, 386.
Milky Way galaxy, 151.
minimum, 422.
minimum circumradius, 167.
minimum of positive quadratic forms, 342.
Minkowski, 19.
mobile phone, 17.
modified second moment, 127.
Monte Carlo, 25.
multiple discoveries, 4.
nattou, 333.
neighbour, 7, 24.
network percolation, 17, 211.
New Zealand, 332.
non isometric crystal, 49.
non-Euclidean space, 19.
nonhomogeneous, 96.
nonisotropic, 96.
nonprimitive parallelohedron, 384.
novemdecillion, 330.
number of edges, 82.
number of faces, 49.
octagon, 8.
octahemioctahedron, 12.
oil reservoir, 113.
onomatopoeia, 330.
opposite form of a face, 380.
opposite vertices, 464.
orthocentre, 178.
parallel faces of a parallelohedron, 393.
parallelepiped, 338.
parallelepipedal systems, 335.
parallelogram, 336, 383.
parallelohedron, 390.
parallelohedron the polyhedron, 384.
partitioning of convex polyhedra, 445.
pentalogy, 332.
percolation, 1, 16, 113, 120, 142, 154, 155, 202.
percolation probabilities, 114, 145.
percolation thresholds, 204.
percolation within percolation, 155.
percolative process, 201.
perfect form, 372.
perfect positive quadratic form, 342, 346.
perfect quadratic forms, 365.
permeability, 113.
permeability coefficient, 176.
permutation of the coefficients, 382.
petroleum, 47.
phase transition, 24.
Pisces-Perseus supercluster, 151.
plain TEX, 39, 322. Platonic polyhedra, 8.
point, 348.
Poiseuille's equation, 176.
Poisson distribution, 23.
Poisson experiment, 24.
Poisson point process, 23.
Poisson process, 24.
Poisson's ratio, 159.
polycrystal, 48.
polygon, 7.
polyhedron, 43, 383.
porous media, 3, 17.
positive parallelohedra, 384.
positive quadratic form, 342, 365, 381.
positive ternary quadratic forms, 341.
poster presentation, 201.
primitive parallelohedron, 384, 394.
principal domain, 456.
principal perfect form, 365.
problem, 2.
projection of a point, 400.
puzzle, 65.
puzzle, 7.
quadratic equation, 68.

quadratic form, 68.
quadrilateral, 177.
Quantum Physics, 4.
quantum-mechanical effect, 211.
quasi-regular polyhedra, 8.
quasicrystal, 48.
quaternary perfect forms, 379.
quaternary positive quadratic form, 382.
quick hull, 319.
Rayleigh distribution, 95.
rbox, 320.
reduced, 343.
reduced binary positive quadratic forms, 383.
reduced parallelogram, 338.
reduced positive quadratic form, 364, 365.
reduced quadratic form, 365, 383.
reduced quaternary positive quadratic forms, 469.
reduction method of positive quadratic forms, 364, 451.
regeneration property, 24.
regular covering lattice, 45.
regular polyhedra, 8.
regulator, 442.
regulator of face, 385.
regulator of the edge, 385, 431.
regulators, 434.
renormalisation group, 5.
representations of the minimum, 342, 355, 365.
resistivity of matter, 76.
retina, 2.
rhombicuboctahedron, 13.
ring road, 203.
s-curve, 119.
Sakkayaphab, 4.
Sanskrit-Pinyin-Tiyapan system, 130.
Sculptor group, 151.
second moment, 23, 150.
sectioning, 74.
Seeberian method, 334.
semi-regular polyhedra, 8.
Siam, 333.
Siamese, 333.
simple domain, 397.
simple face, 394.
simplex, 210.
simplexes, 424.
simulation, 72.
singular perturbation, 331.
small ditriagonal icosidodecahedron, 11.
soap bubble, 18.
sodium stearate, 18.
solid-liquid transition, 25.
specific resistance, 177.
spontaneous magnetisation, 17, 24.
SPT, 130.
St.
Gabriel, 12.
St.
Petersburg, 203.
stand-still, 201.
standard normal distribution, 24.
standing wave equation, 211.
statistics, 150.
steepest gradient, 196.
Stephan-Boltmann constant, 186.
stereogram, 43.
stereographic net, 43.
stereographic projection, 42.
Sukhumvit Road, 201.
supercluster, 151.
superconductor, 25.
superficial area, 199.
superficial velocity, 50.
symmetries, 43.
symmetry, 193.
taxonomy, 11.
ternary positive quadratic form, 382.
ternary quadratic forms, 368.
tessellation, 142.
tetrahemihexahedron, 12.
Thailand, 333.
third moment, 150.
tile, 142.
tiling, 48, 60.
Tokyo, 154.
traffic, 154.
traffic condition, 201.
traffic congestion, 201.
transformations, 99.
translation, 330.
Translation of polyhedra, 390.
triangle, 7.
triangular lattice, 60.
true percolation theory, 20.
truncated cube, 13.
truncated octahedron, 12.
truncated tetrahedron, 12.
trust and distrust, 5.
Turtle geometry, 30.
TEX, 39, 321.
United States, 5.
universe, 151.
valence, 137.
variable structure control, 331.
variance, 23.
vector of a point, 389.
velocity flux, 50.
Ventris, Michael, 2.
vertex of polyhedron, 387.
Vertices of primitive parallelohedron, 464.
Virgo cluster, 19, 151.
virus, 114.
viscosity, 47.
Voronoi, 66, 74, 77, 93.
Voronoi graph, 25, 151.
Voronoi lattice, 113.
Voronoi percolation, 128.
Voronoi tessellation, 1, 75, 120, 151, 331.
walk limit, 127.
weighted mean, 23.
weighted variance, 23.
word games, 330.
work books, 1.
work notes, 1.
Wulff net, 43.
yeast, 3.
zeolite, 48, 157.